数理解析研究所講究録
985 巻 1997 年 151-166

NONLINEAR ERGODIC THEOREMS FOR FAMILIES OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

東京工業大学大学院 情報理工学研究科　厚芝 幸子 (Sachiko ATSUSHIBA)

1. INTRODUCTION

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillòn [2]: Let C be a nonempty closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means $S_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}T^kx$ converge weakly to some $y \in F(T)$. In Baillon's theorem, putting $y=Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT^n=T^nP=P$ for all positive integers n and $Px \in \overline{co}\,\{T^nx : n=1,2,\dots\}$ for each $x \in C$, where $\overline{co}\,A$ is the closure of the convex hull of A. Takahashi [30, 31] proved the existence of such retractions, "ergodic retractions", for noncommutative semigroups of nonexpansive mappings in a Hilbert space. Rodé [27] found a sequence of means on the semigroups, generalizing the Cesàro means on the positive integers, such that the corresponding sequence of mappings converges to an ergodic retraction onto the set of common fixed points. On the other hand, Miyadera and Kobayasi [22] proved nonlinear ergodic theorems for almost-orbits in the case when $S=\{t : 0 \le t < \infty\}$ and a Banach space E satisfies Opial's condition [24] or has a Fréchet differentiable norm. Hirano, Kido and Takahashi [14, 15] proved nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings in a uniformly convex Banach space with a Fréchet differentiable norm.

Recently, in a Hilbert space H, Wittmann [35] studied the following iteration scheme, first considered by Halpern [12]:

$$x_0 = x \in H \quad \text{and} \quad x_{n+1}=\alpha_{n+1}x+(1-\alpha_{n+1})Tx_n \tag{1}$$

for every $n \ge 0$, where a sequence $\{\alpha_n\}$ in $[0,1]$is chosen so that $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ (see also [25]). Wittmann proved that for any $x \in H$, the sequence $\{x_n\}$ converges strongly to the unique element $Px \in F(T)$, where P is the metric projection of H onto $F(T)$. Shimizu and Takahashi [29] introduced a new iteration scheme for a finite commutative family $\{T_i : i=1,2,\cdots n\}$ of nonexpansive mappings in a Hilbert space and proved that the iterates converge strongly to a common fixed point of the mappings $T_i, i=1,2,\cdots,n$. Further they considered an iteration scheme for a nonexpansive semigroup $\{S(t) : t \ge 0\}$ in a Hilbert space and they proved that the iterates converge strongly to a common fixed point of the mappings $S(t), t \geq 0$. Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself. Then, Takahashi and Kim [32] studied the following iteration scheme defined by a nonexpansive mapping T a Banach space:

$$x_1 \in C \quad \text{and} \quad x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n \tag{2}$$

for every $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. Such an iteration scheme was introduced by Ishikawa [16] (see also Mann [21] and Reich [26]). They proved that weak and strong convergence theorems for the iterates $\{x_n\}$ defined by (2).

Let S be a semigroup with identity and let T be a nonexpansive mapping of C into itself. Then $\mathcal{S} = \{T(s) : s \in S\}$ be a family of nonexpansive mappings of C a nonempty closed convex subset of a Banach space into itself satisfying $T(st) = T(s)T(t)$ for all $s, t \in S$ and $T(t)x$ is continuous in $t \in S$ for every $x \in C$, which is called a nonexpansive semigroup on C.

In this paper, we first get a nonlinear ergodic theorem for almost-orbits of commutative semigroups of nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition. We also consider the following iteration scheme:

$$x_1 = x \in C \quad \text{and} \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n \quad \text{for every} \quad n \geq 1, \tag{3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ and $\{\mu_n\}$ is a sequence of means defined on S. We provide weak and strong convergence theorems for the iterates $\{x_n\}$ for nonexpansive semigroup on C defined by (3), using ideas in the nonlinear ergodic theory (for instance, see [13, 15, 17, 29, 32, 35]).

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space and C is a nonempty subset of E unless other specified. We denote by E^* the dual space of E and also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. We write $x_n \rightharpoonup x$ (or $w\text{-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges weakly to x. Similarly $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) and $x_n \xrightarrow{w^*} x$ (or $w^*\text{-}\lim_{n\to\infty} x_n = x$) will symbolize strong convergence and w^*-convergence, respectively. We denote by $\mathbb{R}^+$ the set of all real nonnegative real numbers. For a subset A of E, coA (resp. $\overline{co}A$) means the convex hull of A (resp. the closure of convex hull of A).

Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. Then, for each $s \in S$ and $f \in B(S)$, we can define elements $r_s f \in B(S)$ and $l_s f \in B(S)$ by $(r_s f)(t) = f(ts)$ and $(l_s f)(t) = f(st)$ for all $t \in S$, respectively. We also denote by r_s^* and l_s^* the conjugate operators of r_s and l_s, respectively. Let D be a subspace of $B(S)$ and let μ be an element of D^*. Then, we denote by $\mu(f)$ the value of μ at $f \in D$. Sometimes, $\mu(f)$ will be also denoted by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$. When D contains constants on S, a linear functional μ on D is called a mean on D if $\|\mu\| = \mu(1) = 1$. We also know that μ is a mean on D if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in D$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for every $f \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean on S is also a mean on any subspace D of $B(S)$ containing constants on S. Further, let D be a subspace of $B(S)$ containing constants on S which is r_s-invariant i.e., $r_s D \subset D$ for each $s \in S$. Then, a mean μ on D is called right invariant if $\mu(r_s f) = \mu(f)$ for all $s \in S$ and $f \in D$. Similarly, we can define a left invariant mean on a l_s-invariant subspace of $B(S)$ containing constants on S. A right and left invariant mean is called an invariant mean. We also denote by $C(S)$ the set of all bounded continuous real valued functions on S.

Let S be a commutative semigroup with identity. In this case, $(S, \leq)$ is a directed system when the binary relation $\leq$ on S is defined by $a \leq b$ if and only if there is $c \in S$ with $a + c = b$.

The following definition which was introduced by Takahashi [30] is crucial in the non-linear ergodic theory for abstract semigroups. Let u be a function of S into E such that the weak closure of $\{u(t) : t \in S\}$ is weakly compact and $\langle u(\cdot), y\rangle \in D$ for every $y \in E^*$. And let μ be an element of D^*. Then, there exists a unique element $u_\mu \in E$ such that $\langle u_\mu, y\rangle = \mu_s\langle u(s), y\rangle$ for all $y \in E^*$. If μ is a mean on D, then u_μ is contained in $\overline{co}\{u(t) : t \in S\}$ (for example, see [17, 18, 30]). Sometimes, u_μ will be denoted by $\int u(t)d\mu(t)$.

We say that E satisfies Opial's condition [24] if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x \in E$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{4}$$

holds for every $y \in E$ with $y \neq x$. In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [19]. It is known that all Hilbert spaces and $\ell^p (1 < p < \infty)$ satisfy Opial's condition. It is also known that every separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [9]). We also know that if E has a duality mapping which is weakly sequentially continuous at 0, then E satisfies Opial's condition (see [11]). However, the spaces L^p with $1 < p < \infty$ and $p \neq 2$ do not satisfy Opial's conditions (see also [24]). The norm of a Banach space E is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E, where $S_E = \{v \in E : \|v\| = 1\}$. It is said to be Fréchet differentiable if for each x in S_E, this limit is attained uniformly for y in S_E. With each $x \in E$, we associate the set $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$. Then, the multivalued operator $J : E \to E^*$ is called the duality mapping of E. If the norm of E is Gâteaux differentiable, the duality mapping is single-valued. A Banach space E is said to be strictly convex if $\frac{\|x + y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\| \text{ for } x, y \in E \text{ and } \lambda \in (0, 1),$$

then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by $\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \,\middle|\, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\}$. A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, there exists $\delta\left(\frac{\varepsilon}{r}\right) > 0$ such that $\left\| \frac{x+y}{2} \right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$ for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

3. Nonlinear Ergodic Theorem

Throughout this section, we assume that S is a commutative semigroup. Let C be a subset of a Banach space E. Then, a family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(s+t) = T(s)T(t)$ for all $s, t \in S$;

(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in S$, that is, $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$. If C is a bounded closed convex subset of a uniformly convex Banach space E, then we know that $F(\mathcal{S})$ is nonempty (for example, see [3]). A function $u : G \to C$ is called an almost-orbit of $\mathcal{S} = \{T(t) : t \in S\}$ if

$$\lim_s \sup_t \|u(t+s) - T(t)u(s)\| = 0$$

(see [22, 33]). We denote by $AO(\mathcal{S})$ the set of almost-orbits of $\mathcal{S} = \{T(t) : t \in S\}$.

Lemma 3.1 ([1]). Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let u be an almost-orbit of $\mathcal{S} = \{T(t) : t \in S\}$. Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0 \quad \text{for every } t \in S. \qquad (*)$$

Then, there exist $\{p_\alpha\}, \{q_\beta\} \subset S$ such that for any $z \in F(\mathcal{S})$,

$$\lim_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int u(q_\beta + t) d\lambda_\beta(t) - z \right\|. \qquad (5)$$

Proof. Define $\phi : S \to \mathbb{R}^+$ by $\phi(s) = \sup_t \|u(t+s) - T(t)u(s)\|$ for every $s \in S$ and let $\varepsilon > 0$. Then, for $\alpha \in I$ and $\beta \in J$, as in the proof of [14] or [23], there exist $p_\alpha, q_\beta \in S$ such that

$$\phi(w + p_\alpha) < \varepsilon, \; \phi(w + q_\beta) < \varepsilon,$$

$$\sup_{h \in S} \left\| \int T(h)u(w + p_\alpha + t) d\mu_\alpha(t) - T(h)\left(\int u(w + p_\alpha + t) d\mu_\alpha(t)\right) \right\| < \varepsilon$$

and

$$\sup_{h\in S}\left\|\int T(h)u(w+q_\beta+s)d\lambda_\beta(s)-T(h)\left(\int u(w+q_\beta+s)d\lambda_\beta(s)\right)\right\|<\varepsilon$$

for every $w\in S$. Fix $z\in F(\mathcal{S})$ and consider

$$\begin{aligned}
L &= \left\|\int u(p_\alpha+t)d\mu_\alpha(t)-z\right\|,\\
I_1 &= \left\|\int u(p_\alpha+t)d\mu_\alpha(t)-\iint u(p_\alpha+t+q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)\right\|,\\
I_2 &= \left\|\iint u(p_\alpha+t+q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)-z\right\|,\\
J_1^{(2)} &= \left\|\iint u(p_\alpha+t+q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)-\iint T(p_\alpha+t)u(q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)\right\|,\\
J_2^{(2)} &= \left\|\iint T(p_\alpha+t)u(q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)-\int T(p_\alpha+t)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)d\mu_\alpha(t)\right\|
\end{aligned}$$

and

$$J_3^{(2)}=\left\|\int T(p_\alpha+t)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)d\mu_\alpha(t)-z\right\|.$$

Then, we have $L\le I_1+I_2$ and $I_2\le J_1^{(2)}+J_2^{(2)}+J_3^{(2)}$. Suppose

$$\mu_\alpha=\sum_{i=1}^n a_i\delta_{t_i}\quad\left(a_i\ge 0,\ \sum_{i=1}^n a_i=1\right)\quad\text{and}\quad\lambda_\beta=\sum_{j=1}^m b_j\delta_{s_j}\quad\left(b_j\ge 0,\ \sum_{j=1}^m b_j=1\right).\tag{6}$$

Then, we have

$$\begin{aligned}
J_1^{(2)} &\le \sum_{i=1}^n\sum_{j=1}^m a_i\,b_j\,\|u(p_\alpha+t_i+q_\beta+s_j)-T(p_\alpha+t_i)u(q_\beta+s_j)\|\\
&\le \sum_{i=1}^n\sum_{j=1}^m(a_i\,b_j)\sup_h\|u(h+q_\beta+s_j)-T(h)u(q_\beta+s_j)\|=\sum_{j=1}^m b_j\phi(q_\beta+s_j)
\end{aligned}$$

and

$$\begin{aligned}
J_2^{(2)} &\le \sum_{i=1}^n a_i\left\|\int T(p_\alpha+t_i)u(q_\beta+s)d\lambda_\beta(s)-T(p_\alpha+t_i)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)\right\|\\
&\le \sup_{h\in S}\left\|\int T(h)u(q_\beta+s)d\lambda_\beta(s)-T(h)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)\right\|.
\end{aligned}$$

Since $z\in F(\mathcal{S})$, we obtain

$$J_3^{(2)}\le\sum_{i=1}^n a_i\left\|T(p_\alpha+t_i)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)-z\right\|\le\left\|\int u(q_\beta+s)d\lambda_\beta(s)-z\right\|.$$

Then, we have

$$I_2\le J_1^{(2)}+J_2^{(2)}+J_3^{(2)}<\varepsilon+\varepsilon+\left\|\int u(q_\beta+s)d\lambda_\beta(s)-z\right\|.$$

On the other hand, from (6), we obtain that

$$\begin{aligned} I_1 &= \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - \sum_{j=1}^m b_j \int u(p_\alpha + t + q_\beta + s_j) d\mu_\alpha(t) \right\| \\ &\le \sum_{j=1}^m b_j \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - \int u(p_\alpha + t) d(r^*_{q_\beta + s_j}\, \mu_\alpha)(t) \right\| \\ &\le \sum_{j=1}^m b_j \sup_{g \in S} \|u(g)\| \; \|\mu_\alpha - r^*_{q_\beta + s_j}\, \mu_\alpha\|. \end{aligned}$$

Therefore, from $\lim_\alpha I_1 = 0$, we have

$$\begin{aligned} &\limsup_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| = \limsup_\alpha L \\ \le\; &\limsup_\alpha (I_1 + I_2) \le 2\varepsilon + \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\|. \end{aligned}$$

Then, we have

$$\limsup_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| \le 2\varepsilon + \liminf_\beta \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\|.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| \le \liminf_\beta \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\|.$$

Similarly, we have

$$\limsup_\beta \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\| \le \liminf_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\|.$$

Therefore, we have

$$\lim_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int u(q_\beta + t) d\lambda_\beta(t) - z \right\|. \qquad \blacksquare$$

Remark 3.2 ([1])**.** In Lemma 3.1, take $\{p_\alpha'\}, \{q_\beta'\} \subset S$ such that $p_\alpha' \ge p_\alpha$ and $q_\beta' \ge q_\beta$. Then, we can see that

$$\lim_\alpha \left\| \int u(p_\alpha' + t) d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int u(q_\beta' + t) d\lambda_\beta(t) - z \right\|$$

for every $z \in F(\mathcal{S})$.

Using Lemma 3.1 and Remark 3.2, we can show the following lemma which is crucial to prove the main theorem (Theorem 3.4).

Lemma 3.3 ([1])**.** Let $E, C, \mathcal{S} = \{T(t) : t \in S\}$ and u be as in Lemma 3.1. Additionally, assume that E satisfies Opial's condition. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{for every} \quad t \in S. \qquad (*)$$

Then, $\int u(h+t)d\mu_\alpha(t)$ converges weakly to some $y \in F(\mathcal{S})$ uniformly in $h \in S$. Furthermore, such an element y of $F(\mathcal{S})$ is independent of $\{\mu_\alpha\}$ and for any invariant mean μ on D, $y = u_\mu = \int u(t)d\mu(t)$.

Proof. Let $\{\mu_\alpha : \alpha \in I\}$ and $\{\lambda_\beta : \beta \in J\}$ be nets of finite means on S such that

$$\lim_\alpha \|\mu_\alpha - r_t^*\mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^*\lambda_\beta\| = 0 \tag{$*$}$$

for every $t \in S$. As in the proof of [15, 23], for each $h \in S$, we have

$$\lim_\alpha \sup_p \left\| \int u(p+t)d\mu_\alpha(t) - T(h)\left(\int u(p+t)d\mu_\alpha(t)\right)\right\| = 0. \tag{7}$$

Further, we can take $\{p_\alpha\} \subset S$ such that for any $z \in F(\mathcal{S})$, $\lim_\alpha \left\|\int u(p_\alpha + t)d\mu_\alpha(t) - z\right\|$ exists. Let $\{\Phi_\alpha\} = \{\int u(p_\alpha + t)d\mu_\alpha(t) : \alpha \in I\}$. Then, we first prove that $\{\Phi_\alpha\}$ converges weakly to some $y \in F(\mathcal{S})$. Since E is uniformly convex and C is a bounded closed convex subset of E, $\{\Phi_\alpha\}$ must contain a subnet which converges weakly to a point in C. So, let $\{\Phi_{\alpha_\gamma}\}$ and $\{\Phi_{\alpha_\delta}\}$ be two subnets of $\{\Phi_\alpha\}$ such that $w\text{-}\lim_\gamma \Phi_{\alpha_\gamma} = v$ and $w\text{-}\lim_\delta \Phi_{\alpha_\delta} = v'$, where $w\text{-}\lim_\alpha x_\alpha = x$ means $x_\alpha \rightharpoonup x$. Then, from (7) and demiclosedness principle (see [4]), we have that v and v' are common fixed points of $T(t), t \in S$. Suppose $v \neq v'$. From Lemma 3.1 and Opial's condition, we obtain

$$\begin{aligned}
\lim_\alpha \|\Phi_\alpha - v\| = \lim_\gamma \|\Phi_{\alpha_\gamma} - v\| &< \lim_\gamma \|\Phi_{\alpha_\gamma} - v'\| \\
&= \lim_\delta \|\Phi_{\alpha_\delta} - v'\| \\
&< \lim_\delta \|\Phi_{\alpha_\delta} - v\| = \lim_\alpha \|\Phi_\alpha - v\|.
\end{aligned}$$

This is a contradiction. So, we have that $v = v'$, which implies that $\{\Phi_\alpha\}$ converges weakly to some $y \in F(\mathcal{S})$. Next we prove that $\{\int u(h+t)d\mu_\alpha(t)\}$ converges weakly to y uniformly in h. In the above argument, take $\{p_\alpha'\} \subset S$ such that $p_\alpha' \geq p_\alpha$ for each $\alpha \in I$. Then, repeating the above argument, we see that $\{\Phi_\alpha'\} = \{\int u(p_\alpha' + t)d\mu_\alpha(t) : \alpha \in I\}$ converges weakly to some $y' \in F(\mathcal{S})$. We show $y = y'$. From Lemma 3.1 and Remark 3.2, we know that

$$\lim_\alpha \left\|\int u(p_\alpha' + t)d\mu_\alpha(t) - z\right\| = \lim_\alpha \left\|\int u(p_\alpha + t)d\mu_\alpha(t) - z\right\| \tag{8}$$

for every $z \in F(\mathcal{S})$. Since y and y' are common fixed points of $T(t), t \in S$, from (8) and Opial's condition, we see that $y = y' \in F(\mathcal{S})$. Since $\{p_\alpha'\}$ is any subset in S such that $p_\alpha' \geq p_\alpha$ for each $\alpha \in I$, we have that $w\text{-}\lim_\alpha \int u(h + p_\alpha + t)d\mu_\alpha(t) = y$ uniformly in $h \in S$. Let $x^* \in E^*$ and $\varepsilon > 0$. Then, there exists α_0 such that

$$\left|\int \langle u(h + p_\alpha + s), x^*\rangle\, d\mu_\alpha(s) - \langle y, x^*\rangle\right| < \frac{\varepsilon}{2} \tag{9}$$

for every $\alpha \geq \alpha_0$ and $h \in S$. Suppose

$$\mu_{\alpha_0} = \sum_{k=1}^m b_k \delta_{s_k} \quad (b_k \geq 0,\ \sum_{k=1}^m b_k = 1). \tag{10}$$

Put $\mu_0 = \mu_{\alpha_0}$ and $p_0 = p_{\alpha_0}$. From (9), we have that

$$\begin{aligned}
&\left|\iint \langle u(h+t+p_0+s), x^*\rangle \, d\mu_0(s)d\lambda_\beta(t) - \langle y, x^*\rangle\right| \\
= &\left|\int \left\langle \int u(h+t+p_0+s)d\mu_0(s), x^*\right\rangle d\lambda_\beta(t) - \int \langle y, x^*\rangle \, d\lambda_\beta(t)\right| \\
\leq &\int \left|\left\langle \int u(h+t+p_0+s)d\mu_0(s) - y, x^*\right\rangle\right| d\lambda_\beta(t) < \frac{\varepsilon}{2}
\end{aligned}$$

for every $h \in S$ and $\beta \in J$. Since $\{\lambda_\beta\}$ satisfies $(*)$, there exists β_1 such that

$$\left\|\lambda_\beta - r^*_{p_0+s_k}\lambda_\beta\right\| < \frac{\varepsilon}{2\max\{1, M\,\|x^*\|\}}$$

for all $k = 1, 2, \dots, m$ and $\beta \geq \beta_1$, where $M = \sup_{g \in S}\|u(g)\|$. Then, it follows that

$$\begin{aligned}
&\left|\int \langle u(h+t), x^*\rangle \, d\lambda_\beta(t) - \iint \langle u(h+t+p_0+s), x^*\rangle \, d\mu_0(s)d\lambda_\beta(t)\right| \\
= &\left|\int \langle u(h+t), x^*\rangle \, d\lambda_\beta(t) - \int \left\langle \sum_{k=1}^{m} b_k u(h+t+p_0+s_k), x^*\right\rangle d\lambda_\beta(t)\right| \\
\leq &\sum_{k=1}^{m} b_k \left|\int \langle u(h+t), x^*\rangle \, d\lambda_\beta(t) - \int \langle u(h+t), x^*\rangle \, d(r^*_{p_0+s_k}\lambda_\beta)(t)\right| \\
\leq &\sum_{k=1}^{m} b_k\, M\,\|x^*\|\,\|\lambda_\beta - r^*_{p_0+s_k}\lambda_\beta\| < \frac{\varepsilon}{2}
\end{aligned}$$

for every $\beta \geq \beta_1$ and $h \in S$. Therefore,

$$\begin{aligned}
&\left|\int \langle u(h+t), x^*\rangle \, d\lambda_\beta(t) - \langle y, x^*\rangle\right| \\
\leq &\left|\int \langle u(h+t), x^*\rangle \, d\lambda_\beta(t) - \iint \langle u(h+t+p_0+s), x^*\rangle \, d\mu_0(s)d\lambda_\beta(t)\right| \\
&+\left|\iint \langle u(h+t+p_0+s), x^*\rangle \, d\mu_0(s)d\lambda_\beta(t) - \langle y, x^*\rangle\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for every $\beta \geq \beta_1$ and $h \in S$. Hence, $w\text{-}\lim_\beta \int u(h+t)d\lambda_\beta(t) = y$ uniformly in $h \in S$. Since $\{\lambda_\beta\}$ is an arbitrary net of finite means on S such that $\lim_\beta\|\lambda_\beta - r^*_t\lambda_\beta\| = 0$ for every $t \in S$, we have that such an element y of $F(\mathcal{S})$ is independent of $\{\lambda_\beta\}$ and $\{\mu_\alpha\}$.

Finally, we prove that for any invariant mean μ on D, $y = u_\mu$. Since the set of all finite means is weak*-dense in the set of all means and as in the proof of [8, Theorem 1 in Section 5], we see that for any invariant mean μ on D, there exists a net $\{\mu_\beta\}$ of finite means on S such that $\lim_\beta\|\mu_\beta - r^*_s\mu_\beta\| = 0$ for every $s \in S$ and μ_β converges to μ in the weak* topology. Then, we have $w\text{-}\lim_\beta \int u(t)d\mu_\beta(t) = \int u(t)d\mu(t) = u_\mu$. On the other hand, we obtain $\int u(t)d\mu_\beta(t) \rightharpoonup y$. Hence, we obtain that $y = u_\mu$. ∎

Let D be a subspace of $B(S)$ containing constants and invariant under every $r_s, s \in S$. Then, according to Hirano, Kido and Takahashi [15], a net $\{\mu_\alpha : \alpha \in I\}$ of continuous linear functionals on D is called strongly regular if it satisfies the following conditions :

(a) $\sup_\alpha \|\mu_\alpha\| < +\infty$;

(b) $\lim_\alpha \mu_\alpha(1) = 1$;

(c) $\lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0$ for every $s \in S$.

Using Lemma 3.3, we can prove the following main theorem.

Theorem 3.4 ([1]). Let E be a uniformly convex Banach space which satisfies Opial's condition, let C be a nonempty bounded closed convex subset of E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let u be an almost-orbit of $\mathcal{S} = \{T(t) : t \in S\}$ and let D be a subspace of $B(S)$ containing constants and invariant under every r_s, $s \in S$. Suppose that for each $x^* \in E^*$, the function $t \to \langle u(t), x^* \rangle$ is in D. If $\{\lambda_\alpha\}$ is a strongly regular net of continuous linear functionals on D, then $\int u(h+t)d\lambda_\alpha(t)$ converges weakly to some $y \in F(\mathcal{S})$ uniformly in $h \in S$. Further, such an element y of $F(\mathcal{S})$ is independent of $\{\lambda_\alpha\}$ and for any invariant mean μ on D, $y = u_\mu = \int u(t)d\mu(t)$. In this case, putting $Qu = w\text{-}\lim_\alpha \int u(t)d\lambda_\alpha(t)$ for each $u \in AO(\mathcal{S})$, then Q is a mapping of $AO(\mathcal{S})$ onto $F(\mathcal{S})$ such that it satisfies the following conditions (i),(ii) and (iii):

(i) Q is nonexpansive in the sense that $\|Qu - Qv\| \leq \sup_{t\in S}\|u(t) - v(t)\| = \|u - v\|_\infty$ for every $u, v \in AO(\mathcal{S})$;

(ii) $QT(t)u = T(t)Qu = Qu$ for every $t \in S$ and $u \in AO(\mathcal{S})$;

(iii) $Qu \in \bigcap_{s\in S} \overline{co}\{u(t) : t \geq s\}$ for every $u \in AO(\mathcal{S})$.

The following result is a generalization of Hirano [13, Theorems 3.1 and 3.2].

Corollary 3.5 ([1]). Let $E, C, \mathcal{S} = \{T(t) : t \in S\}$ and u be as in Theorem 3.4. Then, $\{u(t) : t \in S\}$ is weakly convergent if and only if $u(s+t) - u(t) \rightharpoonup 0$ for every $s \in S$. In this case, the limit point of $\{u(t)\}$ is a common fixed point of $T(t), t \in S$.

4. Convergence Theorems of iterations

Throughout this section, we assume that S is a semigroup and C is a nonempty closed convex subset of a Banach space. A family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(st) = T(s)T(t)$ for all $s, t \in S$;

(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$;

(iii) $T(s)x$ is continuous in s for all $x \in C$.

Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$ and suppose that the weak closure of $\{T(t)x : t \in S\}$ is weakly compact for $x \in C$. Let D be a subspace of $B(S)$ which D contains constants and for any $x \in C$ and $x^* \in E^*$, $\langle T(\cdot)x, x^* \rangle \in D$. Now consider the following iteration scheme :

$$x_1 = x \in C \quad \text{and} \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n}x_n \tag{11}$$

for every $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$ and $\{\mu_n\}$ is a sequence of means on D. For any mean μ on D and $x \in C$, there exists a unique element $T_\mu x$ in C such that $\langle T_\mu x, y\rangle = \mu_s\langle T(s)x, y\rangle$ for all $x \in C$ and $y \in E^*$ (see [30, 15]). We know that T_μ is a nonexpansive mapping of C into itself. Then putting $T_n x = \alpha_n x + (1-\alpha_n)T_{\mu_n}x$ for every $x \in C$, the mapping T_n of C into itself is also nonexpansive. Further, we have $F(\mathcal{S}) \subset F(T_{\mu_n}) \subset F(T_n)$ for every $n \geq 1$ and hence $F(\mathcal{S}) \subset \bigcap_{n=1}^{\infty} F(T_n)$. The iterates $\{x_n\}$ defined by (11) can be written as

$$x_{n+1} = T_n T_{n-1} \cdots T_1 x_1. \tag{12}$$

Putting $S_n = T_n T_{n-1} \cdots T_1$, x_{n+1} will be also denoted by $x_{n+1} = S_n x_1$.

Motivated by (12), we obtain the following lemma (see also [13, 32, 20]).

Lemma 4.1. Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let $\{T_1, T_2, T_3, \ldots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and set $S_n = T_n T_{n-1} \cdots T_1$ for every $n \geq 1$. Then, the set $\bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 4.2 ([28]). Let E be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for every $n \geq 1$, and $a \geq 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 4.3. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $x^* \in E^*$, the function $t \to \langle T(t)x, x^*\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n} x_n$ for every $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $x_{n_i} \rightharpoonup y_0$ implies $y_0 \in F(\mathcal{S})$.

Proof (*Sketch*). For $x \in C$ and $f \in F(\mathcal{S})$, put $r = \|x - f\|$ and set $X = \{u \in E : \|u - f\| \leq r\} \cap C$. Then X is a nonempty bounded closed convex subset of C which is $T(t)$-invariant for every $t \in S$ and contains $x_1 = x$. So, without loss of generality, we may assume that C is bounded. It follows from the definition of $\{x_n\}$ that $x_{n+1} - T_{\mu_n}x_n = \alpha_n(x_n - T_{\mu_n}x_n)$. Putting $M = 2\sup_{z\in C}\|z\|$, then $\|x_{n+1} - T_{\mu_n}x_n\| = \alpha_n\|x_n - T_{\mu_n}x_n\| \leq M\alpha_n$.

So we obtain, for each $t \in S$,

$$\begin{aligned}
& \|T(t)x_n - x_n\| \\
\leq\ & \|T(t)x_n - T(t)T_{\mu_{n-1}}x_{n-1}\| + \|T(t)T_{\mu_{n-1}}x_{n-1} - T_{\mu_{n-1}}x_{n-1}\| + \|T_{\mu_{n-1}}x_{n-1} - x_n\| \\
\leq\ & 2\|T_{\mu_{n-1}}x_{n-1} - x_n\| + \|T(t)T_{\mu_{n-1}}x_{n-1} - T_{\mu_{n-1}}x_{n-1}\| \\
=\ & 2\alpha_{n-1}\|x_{n-1} - T_{\mu_{n-1}}x_{n-1}\| + \|T(t)T_{\mu_{n-1}}x_{n-1} - T_{\mu_{n-1}}x_{n-1}\| \\
\leq\ & 2\alpha_{n-1}M + \|T(t)T_{\mu_{n-1}}x_{n-1} - T_{\mu_{n-1}}x_{n-1}\|. \qquad (13)
\end{aligned}$$

Using [7], we can prove that

$$\lim_{n\to\infty} \sup_{y\in C} \|T(t)T_{\mu_n}y - T_{\mu_n}y\| = 0 \quad \text{uniformly in } t \in S. \qquad (14)$$

Assume $x_{n_i} \rightharpoonup y_0$. Then, since $0 \leq \alpha_n \leq a < 1$, we have $\liminf_{i\to\infty} \alpha_{n_i} = 0$ or $1 > a \geq \liminf_{i\to\infty} \alpha_{n_i} > 0$. If $\liminf_{i\to\infty} \alpha_{n_i} = 0$, then there exists a subsequence $\{\alpha_{n_{i_j}}\}$ of $\{\alpha_{n_i}\}$ such that $\alpha_{n_{i_j}} \to 0$ as $j \to \infty$. So, from (13) and (14),

$$\lim_{j\to\infty} \|T(t)x_{n_{i_j}} - x_{n_{i_j}}\| = 0. \qquad (15)$$

Then, from (15) and the demiclosedness principle (see [4]), we have that y_0 is a common fixed points of $T(t), t \in S$. In the case when $1 > a \geq \liminf_{i\to\infty} \alpha_{n_i} > 0$, let w be a common fixed point of $T(t), t \in S$. Then, we see that $\{\|x_n - w\|\}$ is a decreasing sequence and hence $\lim_{n\to\infty} \|x_n - w\|$ exists. Put $c = \lim_{n\to\infty} \|x_n - w\|$. Since $\|T_{\mu_n}x_n - w\| \leq \|x_n - w\|$, we have

$$\limsup_{n\to\infty} \|T_{\mu_n}x_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c. \qquad (16)$$

Further, we have

$$\begin{aligned}
\lim_{n\to\infty} \|\alpha_n(T_{\mu_n}x_n - w) + (1-\alpha_n)(x_n - w)\| &= \lim_{n\to\infty} \|x_{n+1} - w\| \\
&= c. \qquad (17)
\end{aligned}$$

So, from (16) , (17) and Lemma 4.2, we have

$$\lim_{i\to\infty} \|T_{\mu_{n_i}}x_{n_i} - x_{n_i}\| = 0. \qquad (18)$$

Since

$$\begin{aligned}
\|T(t)x_n - x_n\| &\leq \|T(t)x_n - T(t)T_{\mu_n}x_n\| + \|T(t)T_{\mu_n}x_n - T_{\mu_n}x_n\| + \|T_{\mu_n}x_n - x_n\| \\
&\leq 2\|T_{\mu_n}x_n - x_n\| + \|T(t)T_{\mu_n}x_n - T_{\mu_n}x_n\|,
\end{aligned}$$

from (14) and (18), we have

$$\lim_{i\to\infty} \|T(t)x_{n_i} - x_{n_i}\| = 0. \qquad (19)$$

Therefore, from (19) and the demiclosedness principle (see [4]), we obtain that y_0 is a common fixed point of $T(t), t \in S$. ∎

Now we can prove a weak convergence theorem for a nonexpansive semigroup in a Banach space.

Theorem 4.4. Let E be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Let C be a nonempty closed convex subset of E and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $x^* \in E^*$, the function $t \to \langle T(t)x, x^* \rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^* \mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n$ for every $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $T(t), t \in S$.

Proof. We assume that E satisfies Opial's condition. Let w be a common fixed point of $T(t), t \in S$. Then, as in the proof of Lemma 4.3, $\lim_{n\to\infty} \|x_n - w\|$ exists. As in the proof of Lemma 4.3, we may assume that C is bounded. And since E is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$. Then, from Lemma 4.3 we have that z_1 and z_2 are common fixed points of $T(t), t \in S$. Next, we show $z_1 = z_2$. If not, from Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - z_2\| = \lim_{n\to\infty} \|x_n - z_2\| = \lim_{j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{j\to\infty} \|x_{n_j} - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is a contradiction. Hence, we obtain $x_n \rightharpoonup y_0 \in F(\mathcal{S})$.

Now we assume that E has a Fréchet differentiable norm. As in the proof of Lemma 4.3, we may assume that C is bounded. So, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y_0$. Then, from Lemma 4.3, we obtain $y_0 \in F(\mathcal{S})$. We know $F(\mathcal{S}) \subset F(T_n)$ for every $n \geq 1$ and hence $\bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap F(\mathcal{S}) \subset \bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n)$. From [32], we have $\omega_w(S_n x) = \bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\}$, where $\omega_w(S_n x)$ is the set of all weak limit points of subsequences of the sequence $\{S_n x : n = 1, 2, \dots\}$. So, from Lemma 4.1, we have $\{y_0\} = \bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \geq n\} \cap F(\mathcal{S})$. Hence, we obtain $x_n \rightharpoonup y_0 \in F(\mathcal{S})$. ∎

Lemma 4.5. Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Let T be a nonexpansive mapping of C into E. Then, we have

$$\lim_{n\to\infty} (x_n - Tx_n) = y \quad \text{and} \quad x_n \rightharpoonup x \quad \text{imply} \quad x - Tx = y.$$

In particular,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \quad \text{and} \quad x_n \rightharpoonup x \quad \text{imply} \quad x \in F(T)$$

Lemma 4.6. Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Let $0 < b \le t_n \le c < 1$ for every $n \ge 1$ and $a \ge 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of C such that $\limsup_{n\to\infty}\|x_n\| \le a$, $\limsup_{n\to\infty}\|y_n\| \le a$ and $\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 4.7. Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\bigcup_{t\in S} T(t)\big(C\big) \subset K \subset C$ for some compact subset K of C. Let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $x^* \in E^*$, the function $t \to \langle T(t)x, x^*\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty}\|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n$ for every $n \ge 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $x_{n_i} \to y_0$ implies $y_0 \in F(\mathcal{S})$.

Proof (*Sketch*). From Mazur's theorem [10], $\overline{co}\left(\{x_1\} \cup \bigcup_{t\in S} T(t)\,(C)\right)$ is a compact subset of C containing $\{x_n\}$. Putting $M = 2\sup_{z\in C}\|z\|$, then as in the proof of Lemma 4.3, we obtain

$$\|T(t)x_n - x_n\| \le 2\alpha_{n-1}M + \|T(t)T_{\mu_{n-1}}x_{n-1} - T_{\mu_{n-1}}x_{n-1}\|. \tag{20}$$

From the strictly convexity of E and compactness of C, we can prove that

$$\lim_{n\to\infty}\sup_{y\in C}\|T(t)T_{\mu_n}y - T_{\mu_n}y\| = 0 \quad \text{uniformly in } t \in S. \tag{21}$$

From compactness, $\{x_n\}$ must contain a subsequence which converges to a point in C. Assume $x_{n_i} \to y_0$. Then, since $0 \le \alpha_n \le a < 1$, we have $\liminf_{i\to\infty}\alpha_{n_i} = 0$ or $1 > a \ge \liminf_{i\to\infty}\alpha_{n_i} > 0$. If $\liminf_{i\to\infty}\alpha_{n_i} = 0$, then there exists a subsequence $\{\alpha_{n_{i_j}}\}$ of $\{\alpha_{n_i}\}$ such that $\alpha_{n_{i_j}} \to 0$ as $j \to \infty$. So, from (20) and (21)

$$\lim_{j\to\infty}\|T(t)x_{n_{i_j}} - x_{n_{i_j}}\| = 0. \tag{22}$$

Then, from (22) and Lemma 4.5, we have that y_0 is a common fixed point of $T(t), t \in S$. In the case when $1 > a \ge \liminf_{i\to\infty}\alpha_{n_i} > 0$, let w be a common fixed point of $T(t), t \in S$. Then, as in the proof of Lemma 4.3, $\lim_{n\to\infty}\|x_n - w\|$ exists. Put $c = \lim_{n\to\infty}\|x_n - w\|$. Since $\|T_{\mu_n}x_n - w\| \le \|x_n - w\|$, we have

$$\limsup_{n\to\infty}\|T_{\mu_n}x_n - w\| \le \limsup_{n\to\infty}\|x_n - w\| = c. \tag{23}$$

Further, we have

$$\begin{aligned}\lim_{n\to\infty}\|\alpha_n(T_{\mu_n}x_n - w) + (1-\alpha_n)(x_n - w)\| &= \lim_{n\to\infty}\|x_{n+1} - w\| \\ &= c. \end{aligned}\tag{24}$$

So, from (23), (24) and Lemma 4.6, we have

$$\lim_{i\to\infty} \|T_{\mu_{n_i}} x_{n_i} - x_{n_i}\| = 0. \tag{25}$$

Since

$$\begin{aligned} \|T(t)x_n - x_n\| &\leq \|T(t)x_n - T(t)T_{\mu_n}x_n\| + \|T(t)T_{\mu_n}x_n - T_{\mu_n}x_n\| + \|T_{\mu_n}x_n - x_n\| \\ &\leq 2\|T_{\mu_n}x_n - x_n\| + \|T(t)T_{\mu_n}x_n - T_{\mu_n}x_n\|, \end{aligned}$$

from (21) and (25), we have

$$\lim_{i\to\infty} \|T(t)x_{n_i} - x_{n_i}\| = 0. \tag{26}$$

Therefore, from (26) and Lemma 4.5, we have that y_0 is a common fixed point of $T(t), t \in S$. ∎

Now we can prove a strong convergence theorem for a nonexpansive semigroup ins a Banach space.

Theorem 4.8. Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\bigcup_{t\in S} T(t)\big(C\big) \subset K \subset C$ for some compact subset K of C. Let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $x^* \in E^*$, the function $t \to \langle T(t)x, x^*\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n$ for every $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges strongly to a common fixed point of $T(t), t \in S$.

Proof. From Mazur's theorem [10], $\overline{co}\left(\{x_1\} \cup \bigcup_{t\in S} T(t)(C)\right)$ is a compact subset of C containing $\{x_n\}$. Then, there exist a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ and a point $y_0 \in C$ such that

$$x_{n_i} \to y_0 \in C. \tag{27}$$

So, from Lemma 4.7, we obtain $T(t)y_0 = y_0$ for every $t \in S$. Then, since there exists $\lim_{n\to\infty} \|x_n - y_0\|$ and from (27) $\lim_{n\to\infty} \|x_n - y_0\| = \lim_{i\to\infty} \|x_{n_i} - y_0\| = 0$. Therefore, $\{x_n\}$ converges strongly to a common fixed point of $T(t), t \in S$. ∎

We have also a strong convergence theorem which is connected with results of [20, 33, 34].

Theorem 4.9. Let C be a nonempty closed convex subset of a uniformly convex Banach space E, let S be a semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $x^* \in E^*$, the function $t \to \langle T(t)x, x^*\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$

for every $s \in S$. Let P be the metric projection of C onto $F(\mathcal{S})$. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n$ for all $n \geq 1$, where $\alpha_n \in [0,1]$. Then, the $\lim_{n\to\infty} Px_n$ exists and $\lim_{n\to\infty} Px_n = z_0$, where z_0 is a unique element of $F(\mathcal{S})$ such that $\lim_{n\to\infty}\|x_n - z_0\| = \min\{\lim_{n\to\infty}\|x_n - w\| : w \in F(\mathcal{S})\}$.

We have also the following theorem which is connected with Theorems 4.4 and 4.9.

Theorem 4.10. Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and invariant under every $l_s, s \in S$. Suppose that for each $x \in C$ and $x^* \in E^*$, the function $t \to \langle T(t)x, x^*\rangle$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty}\|\mu_n - l_s^*\mu_n\| = 0$ for every $s \in S$. Let P be the metric projection of C onto $F(\mathcal{S})$. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n$ for all $n \geq 1$, where $\alpha_n \in [0,a]$ for some a with $0 < a < 1$. Then, $\{x_n\}$ converges weakly to an element z_0 of $F(\mathcal{S})$, where $z_0 = \lim_{n\to\infty} Px_n$ and $\lim_{n\to\infty}\|x_n - z_0\| = \min\{\lim_{n\to\infty}\|x_n - w\| : w \in F(\mathcal{S})\}$.

References

1. S. Atsushiba and W. Takahashi, *Nonlinear Ergodic Theorems in a Banach Space Satisfying Opial's Condition*, Tokyo J. Math. to apper.
2. J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), 1511-1514.
3. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041-1044.
4. F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc, Sympos. Pure Math. **18**, Amer. Math. Soc. Providence, R.I. 1976.
5. R. E. Bruck, *On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set,* Israel J. Math. **29** (1978), 1-16.
6. R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 107-116.
7. R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1981), 304-314.
8. M. M. Day, *Amenable semigroups*, Illionois J. Math. **1** (1957), 509-544.
9. D. Van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London. Math. Soc. **25** (1982), 139-144.
10. N. Dunford and J.T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.
11. J. P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific. J. Math. **40** (1972), 565-573.
12. B. Halpern, *Fixed points of nonexpanding maps* , Bull. Amer. Math. Soc. **73** (1967), 957-961.
13. N. Hirano, *A proof of the mean ergodic theorem for nonexpansive mappings in Banach space*, Proc. Amer. Math. Soc. **78** (1980), 361-365.
14. N. Hirano, K. Kido and W. Takahashi, *Asymptotic behavior of commutative semigroups of nonexpansive mappings in Banach Spaces*, Nonlinear Anal. **10** (1986), 229-249.

15. N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269-1281.
16. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.
17. K. Kido and W. Takahashi, *Mean ergodic theorems for semigroups of linear continuous operators in Banach spaces*, J. Math. Anal. Appl. **103** (1984), 387–394.
18. K. Kido and W. Takahashi, *Means on commutative semigroups and nonlinear erngodic theorems*, J. Math. Anal. Appl. **111** (1985), 585–605.
19. A. T. Lau, *Semigroup of Nonexpansive Mappings on a Hilbert Space*, J. Math. Anal. Appl. **105** (1985), 514-522.
20. A.T. Lau and W. Takahashi, *Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings*, Pacific J. Math. **126** (1987), 277-294.
21. W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.
22. I. Miyadera and K. Kobayasi, *On the asymptotic behavior of almost-orbits of nonlinear contraction semigroups in Banach spaces*, Nonlinear Anal. **6** (1982), 349-365.
23. H. Oka, *Nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings*, Nonlinear Anal. **18** (1992), 619-635.
24. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597.
25. S. Reich, *Some problems and results in fixed point theory* , Contemp. Math. **21** (1983), 179-187.
26. S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274-276.
27. G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85e** (1982), 172-178.
28. J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153-159.
29. T. Shimizu and W. Takahashi *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. (to apper).
30. W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253-256.
31. W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in Hilbert space*, Proc. Amer. Math.Soc. **97** (1986), 55–58.
32. W. Takahashi and G. E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Topol. Methods Nonlinear Anal. (to apper)
33. W. Takahashi and J. Y. Park, *On the asymptotic behavior of almost-orbits of commutative semigroups in Banach spaces*, Nonlinear and Convex Analysis (B. Lin and S. Simons, eds.) Marcel Dekker, Inc. 1987, pp. 271-293.
34. W. Takahashi and P. J. Zhang, *Asymptotic behavior of almost-orbits of semigroups of Lipschitzian mappings in Banach spaces*, Kodai. Math. J. **11** (1988), 129–140.
35. R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., **58** (1992), 486-491.

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-okayama, Meguroku, Tokyo 152, Japan

E-mail address: atsusiba@is.titech.ac.jp