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Complex dynamics on $\mathbf{P}^n$ and Kobayashi Metric

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1. Let f be a holomorphic map from the n dimensional complex projective space $\mathbf{P}^n$ to itself. In what follows we assume that f is of degree ≥ 2, i.e., it is not constant nor a projective transformation. We study the complex dynamics defined by the iterates of f.

As in the case of dimension 1, the Fatou set Ω for f is defined by

$$\Omega = \{p \in \mathbf{P}^n | f^j \ (j = 1, 2, \ldots) \text{ is a normal family on a neighborhood of } p\}$$

This set Ω is an open set $\neq \mathbf{P}^n$. It may be empty (see [FS1], [U1],[U3]). The Fatou set Ω and hence every Fatou component are pseudoconvex and Kobayashi hyperbolic ([FS3],[U2].

In this note we will discuss Fatou maps, which generalizes the concept of Fatou set. As an application, we give a result concerning the classification of Fatou components.

2. Let $\mathcal{X} = \mathbf{C}^{n+1} - \{0\}$ and $\pi : \mathcal{X} \to \mathbf{P}^n$ be the natural projection. For a holomorphic map $f : \mathbf{P}^n \to \mathbf{P}^n$, there exists a map $F : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ such that $\pi \circ F|\mathcal{X} = f \circ \pi$. Here F is defined by an $n+1$ tuple of homogeneous polynomials $(f_0(x), \ldots, f_n(x))$ of degree d. We define the degree of f by $\deg f = d$. We define the Green function h on $\mathbf{C}^{n+1}$ by

$$h(x) = \lim_{j\to\infty} \frac{1}{d^j} \log \|F^j(x)\|$$

This Green function $h(x)$ is plurisubharmonic on $\mathbf{C}^{n+1}$.

Now let

$$\mathcal{H} = \{x \in \mathbf{C}^{n+1} | h \text{ is pluriharmonic in a neighborhood of } x\}$$

Then the Fatou set Ω can be characterized using this set $\mathcal{H}$:

$$\mathcal{H} = \pi^{-1}(\Omega).$$

Now we will define a generalization of the concept of Fatou set.

Definition A holomorphic map φ from a complex manifold Z into $\mathbf{P}^n$ is said to be a Fatou map for f if the sequence of the maps

$$f^j \circ \varphi : Z \to \mathbf{P}^n \quad (j = 0, 1, 2, \ldots)$$

constitutes a normal family.

Remark An open set U in $\mathbf{P}^n$ is contained in the Fatou set Ω if and only if the inclusion map $U \to \mathbf{P}^n$ is a Fatou map.

Suppose that $\varphi : Z \to \mathbf{P}^n$ is a holomorphic map. A holomorphic map $\Phi : Z \to \mathcal{X}$ is said to be a lift of φ if $\pi \circ \Phi = \varphi$. We note that, for any point $a \in Z$, there exists a neighborhood V of a such that $\varphi|V$ has a holomorphic lift.

We can characterize Fatou maps in terms of the Green function h.

Theorem 1. For a holomorphic map $\varphi : Z \to \mathbf{P}^n$, the following properties are equivalent to one another:

(1) φ is a Fatou map for f.

(2) The sequence $\{f^j \circ \varphi\}$ contains a subsequence that is uniformly convergent on compact sets.

(3) If V is an open set in Z and $\Phi_V : V \to \mathcal{X}$ is a holomorphic lift of $\varphi|V$, then $h \circ \Phi_V$ is a pluriharmonic function on V.

(4) For any point $a \in Z$, there exist an open set V containing a and a holomorphic lift Φ_V of $\varphi|V$ such that $h \circ \Phi_V$ is identically zero.

This theorem can be proved in the same way as Proposition 2.1 and Theorem 2.2 in [U2].

We fix a distance ρ determined by a Riemannian metric on $\mathbf{P}^n$. For a complex manifold Z, we denote by d_Z the Kobayashi pseudodistance on Z. Using Theorem 1, we can prove the following theorem.

Theorem 2 For a holomorphic map $f : \mathbf{P}^n \to \mathbf{P}^n$ of degree ≥ 2, there exists a constant $C > 0$ with the following property: If $\varphi : Z \to \mathbf{P}^n$ is a Fatou map for f, then the inequality

$$\rho(\varphi(a_1), \varphi(a_2)) \leq C\, d_Z(a_1, a_2)$$

holds for any $a_1, a_2 \in Z$.

We note that the constant C can be determined only by the distance ρ and the map f, independently of Z and φ.

Corollary 1 If $\varphi : Z \to \mathbf{P}^n$ is an injective Fatou map, then Z is Kobayashi hyperbolic.

Corollary 2 Let Z be a complex manifold and let $\mathcal{S}_{Z,f}$ denote the set of all Fatou maps $\varphi : Z \to \mathbf{P}^n$. Then $\mathcal{S}_{Z,f}$ is compact with respect to the topology of uniform convergence on compact sets.

We denote by Δ the unit disk $\{\zeta \in \mathbf{C} |\ |\zeta| < 1\}$ and by $\Delta^* = \Delta - \{0\}$ the punctured unit disk.

Theorem 3. Let $\varphi : \Delta^* \to \mathbf{P}^n$ be a Fatou map for f. Then φ can be extended to a Fatou map $\hat{\varphi} : \Delta \to \mathbf{P}^n$ for f.

This theorem can be regarded as an analogue of the Kwack theorem (see for example [K]): Let M be a Kobayashi hyperbolic complex manifold and $\varphi : \Delta^* \to M$ a holomorphic map. Then φ can be extended to a holomorphic map $\hat{\varphi} : \Delta \to M$. Theorem 3 can be proved in the same manner as the Kwack theorem.

3. A connected component of the Fatou set Ω is said to be a Fatou component. A Fatou component for f is called recurrent if there exists a point $p \in U$ such that a sequence $\{f^j(p)\}$ contains a subsequence convergent to a point in U. If U is recurrent, then it is invariant under f^k for some integer $k \geq 1$.

In the case of dimension 2, the following theorem is proved in [FS4].

Theorem (Fornaess - Sibony) Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be a holomorphic map of degree ≥ 2. Then an invariant and recurrent Fatou component for is of one of the following three types:

(1) U contains an attracting fixed point and U is its immediate attracting basin.

(2) There exists a complex 1-dimensional closed submanifold S of U with the following properties: (a) S is biholomorphic to either a disk Δ, a punctured disk Δ^* or an annulus; (b) $\{f^j|U\}$ contains a subsequence that is convergent to a holomorphic map $\varphi : U \to S$ such that $\varphi|S$ is the identity map.

(3) U is a rotation domain, i.e., the sequence $\{f^j|U\}$ contains a subsequence that converges to the identity map of U uniformly on compact sets.

Concerning this theorem we can show the following fact:

Theorem 4. In the case (2) of the theorem of Fornaess-Sibony, the submanifold S is not biholomorphic to the punctured disk.

This can be proved by using Theorem 3 as follows:

In the situation of case (2) of the theorem, suppose that φ is a biholomorphic map Δ^* of onto S. Then φ is a Fatou map for f. By Theorem 3, the map φ can be extended to a Fatou map $\hat{\varphi} : \Delta \to \mathbf{P}^2$. By the following lemma, the image $\hat{\varphi}(0)$ is contained in the Fatou set Ω. This contradicts the fact that S is a closed submanifold of the Fatou component U.

Lemma Let $\varphi : \Delta \to \mathbf{P}^n$ be a Fatou map for f. If $\varphi(\Delta^*)$ is contained in the Fatou set Ω, then $\varphi(\Delta)$ is contained in Ω.

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