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Overlapping Domain Decomposition with Non-matching Grids

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Abstract — A macro-hybrid formulation based on overlapping domain decomposition is introduced and studied for a model elliptic partial differential equation. The problem is discretized by the mortar element method using non-matching grids on the interfaces between subdomains. An iterative method of an optimal order of arithmetical complexity is proposed for solving the arising algebraic systems in the case of regular quasiuniform hierarchical grids. Results of numerical experiments are presented.

1 INTRODUCTION

In this paper we consider two topics. In Section 2 we introduce a new macro-hybrid formulation for the Poisson equation with the Neumann boundary condition based on overlapping domain decomposition. An example of such formulation was originally given in [8]. The approach proposed here has many common points with the decentralization methods studied more than twenty years ago in [1, 10]. In these papers the authors used splittings of bilinear forms between different subdomains to decompose a variational problem.

The second important topic is presented in Section 4 where we consider an extension of results from [7, 8] to the case of overlapping subdomains. Here we present several results which mainly concern the construction of the interface preconditioner.

In Section 5 results of numerical experiments for a 2D test problem are given.

2 MACRO-HYBRID BASED ON OVERLAPPING DOMAIN DECOMPOSITION

Let us consider a model elliptic problem

$$\begin{aligned} -\Delta u + cu &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $f \in L_2(\Omega)$ is a given function, $c \equiv \text{const} \in (0; 1]$, $\partial\Omega$ is the boundary of a domain Ω and $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$. For the sake of simplicity we assume that

Ω is a polygon in $\mathbf{R}^2$, with $\operatorname{diam}(\Omega) \sim O(1)$, and all further subdomains of Ω are also polygons with diameters $O(1)$.

The classical weak formulation of (1) is: find $u \in H^1(\Omega)$ such that

$$\Phi(u) = \min_{v \in H^1(\Omega)} \Phi(v), \tag{2}$$

where

$$\Phi(v) = \int_\Omega \left[|\nabla v|^2 + cv^2 - 2fv \right] d\Omega. \tag{3}$$

Let Ω_1 and Ω_2 be two overlapping subdomains of Ω ($\Omega_1 \cap \Omega_2 \neq \emptyset$) such that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$. We assume that subdomains Ω_1 and Ω_2 are regularly shaped. An example of such a partitioning of Ω into two subdomains is given in Fig. 1.

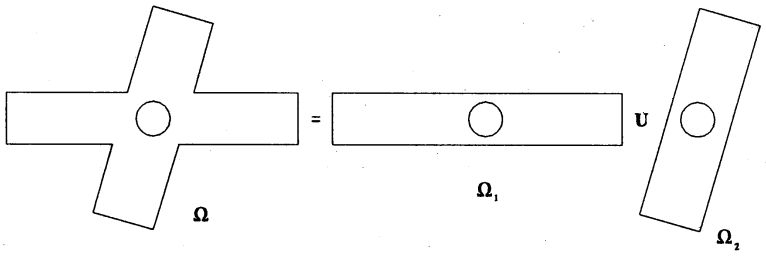

Figure 1: Overlapping domain decomposition

We denote the intersection of Ω_1 and Ω_2 by Ω_{12} and define two bilinear forms

$$a_k(u, v) = \int_{\Omega_k} [a_k \nabla v \cdot \nabla u + c_k uv] \, d\Omega, \qquad k = 1, 2, \tag{4}$$

two linear forms

$$l_k(v) = \int_{\Omega_k} f_k v \, d\Omega, \qquad k = 1, 2, \tag{5}$$

and two quadratic functionals

$$\psi_k(v) = a_k(v, v) - 2l_k(v), \qquad k = 1, 2. \tag{6}$$

The coefficients a_k, c_k and functions f_k are defined by

$$a_k = \begin{cases} 1 & \text{in } \Omega_k \setminus \Omega_{12} \\ q_k & \text{in } \Omega_{12} \end{cases} \qquad c_k = \begin{cases} c & \text{in } \Omega_k \setminus \Omega_{12} \\ q_k c & \text{in } \Omega_{12} \end{cases}$$

$$f_k = \begin{cases} f & \text{in } \Omega_k \setminus \Omega_{12} \\ q_k f & \text{in } \Omega_{12} \end{cases} \tag{7}$$

where q_k are positive constants, $k = 1, 2$ such that $q_1 + q_2 = 1$. It is important that

$$\psi_k(v) = q_k \Phi(v), \quad \forall v \in H^1(\Omega), \ \operatorname{supp} v \in \Omega_{12}, \ k = 1, 2. \tag{8}$$

To introduce and to analyze macro-hybrid formulations of elliptic problems we have to deal with interfaces between subdomains. To this end we introduce the following notation:

$$\begin{aligned} \Gamma_k &= (\partial\Omega_k \cap \Omega), \quad k = 1, 2, \\ \Gamma &= \Gamma_1 \cup \Gamma_2. \end{aligned} \tag{9}$$

Now we introduce the space $V = H^1(\Omega_1) \times H^1(\Omega_2)$, the space

$$W = \left\{ \bar{v} = (v_1, v_2) : \ \bar{v} \in V, \ \int_\Gamma (v_1 - v_2)\mu \, ds = 0, \ \forall \mu \in H^{-1/2}(\Gamma) \right\} \tag{10}$$

and the quadratic functional

$$\psi(\bar{v}) = \psi_1(v_1) + \psi_2(v_2), \quad \bar{v} \in V. \tag{11}$$

It can be shown (see, for instance [5]) that under the assumptions made the following macro-hybrid formulation of problem (1):

$$\bar{u} \in V : \ \psi(\bar{u}) = \min_{\bar{v} \in W} \psi(\bar{v}) \tag{12}$$

has a unique solution and is equivalent to problem (2). We understand the equivalence in the sense that

$$u(x) = u_k(x) \quad \forall x \in \Omega_k, \tag{13}$$

where u is the solution function to (2).

Problem (12) has also an equivalent formulation in terms of Lagrange multipliers. For instance, in the case of example in Fig. 1b it can be presented in the following form: find $(\bar{u}, \bar{\lambda}) \in V \times \Lambda$ such that

$$\begin{aligned} a_1(u_1, v_1) + \int_{\Gamma_1} \lambda_1 v_1 \, ds - \int_{\Gamma_2} \lambda_2 v_1 \, ds &= l_1(v_1), \\ a_2(u_2, v_2) - \int_{\Gamma_1} \lambda_1 v_2 \, ds + \int_{\Gamma_2} \lambda_2 v_2 \, ds &= l_2(v_2), \\ \int_{\Gamma_1} (u_1 - u_2)\mu_1 \, ds &= 0, \\ \int_{\Gamma_2} (u_1 - u_2)\mu_2 \, ds &= 0, \end{aligned} \tag{14}$$

$\forall (\bar{v}, \bar{\mu}) \in V \times \Lambda$. Here $\Lambda = \prod_{s=1}^{2} H^{-1/2}(\Gamma_s)$. It can be easily shown that

$$\lambda_1 = -q_1 \frac{\partial u_1}{\partial \mathbf{n_1}} \text{ on } \Gamma_1, \quad \lambda_2 = -q_2 \frac{\partial u_2}{\partial \mathbf{n_2}} \text{ on } \Gamma_2, \tag{15}$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are the outer normal vectors to $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Recall that $u_1 \equiv u$ in Ω_1 and $u_2 \equiv u$ in Ω_2.

In a compact form (14) can be presented [6, 5] by: find $(\bar{u}, \bar{\lambda}) \in V \times \Lambda$ such that

$$\begin{aligned} \hat{a}(\bar{u}, \bar{v}) + b(\bar{\lambda}, \bar{v}) &= \hat{l}(\bar{v}), \\ b(\bar{\mu}, \bar{u}) &= 0, \quad \forall (\bar{v}, \bar{\mu}) \in V \times \Lambda. \end{aligned} \tag{16}$$

Here

$$V = \prod_{k=1}^{2} V_k, \qquad V_k = H^1(\Omega_k),\ k = 1,2,$$
$$\Lambda = \prod_{s=1}^{2} \Lambda_s, \qquad \Lambda_s = H^{-1/2}(\Gamma_s),\ s = 1,2, \tag{17}$$
$$\hat{a}(\bar{u}, \bar{v}) = \sum_{k=1}^{2} a_k(u, v), \quad \hat{l}(\bar{v}) = \sum_{k=1}^{2} l_k(v).$$

Remark If $\int_\Omega f\, d\Omega = 0$ and $c \ll 1$ then problem (1) can be considered as a singular perturbation of the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{18}$$

3 THE MORTAR ELEMENT METHOD AND ALGEBRAIC SYSTEMS

We consider the only case when Ω_{kh} are conforming triangular partitions of Ω_k, $k = 1,2$. Then V_{kh} are the standard piece-wide linear finite element subspaces of $V_k \equiv H^1(\Omega_k)$, $k = 1,2$. The finite element subspaces $\Lambda_{sh} \subset \Lambda \equiv H^{-\frac{1}{2}}(\Gamma_s)$, $s = 1,2$ are chosen using the mortar element technique ¿from [3, 2, 8].

The mortar finite element discretization of (16)–(17) is defined by: find $(\bar{u}_h, \bar{\lambda}_h) \subset V_h \times \Lambda_h$ such that

$$\begin{aligned} \hat{a}(\bar{u}_h, \bar{v}) + b(\bar{\lambda}_h, \bar{u}_h) &= \hat{l}(\bar{v}), \\ b(\bar{\mu}, \bar{u}_h) &= 0, \end{aligned} \tag{19}$$

$\forall (\bar{v}, \bar{\mu}) \in V_h \times \Lambda_h$ where $\Lambda_h = \prod_{s=1}^{2} \Lambda_{sh}$. Problem (19) leads to an algebraic system

$$\mathcal{A}x = y \tag{20}$$

with a saddle-point matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & O \end{pmatrix} \tag{21}$$

and vectors

$$x = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \qquad y = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{22}$$

Here A is a symmetric positive definite matrix and $\ker B^T = 0$. It follows immediately that $\det \mathcal{A} \neq 0$.

For further analysis we need a more detailed description of A and B in block forms. The simplest block representations of A and B^T are:

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \qquad B^T = \begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix} \tag{23}$$

Here the kth block corresponds to the degrees of freedom of the finite element space $V_{k,h}$, $k = 1, 2$.

For each subdomain Ω_k we partition degrees of freedom (grid nodes) into two groups. In the second group denoted by Γ we collect the degrees of freedom which correspond to the grid nodes belonging to Γ. All other degrees of freedom we collect in the first group denoted by I. These partitionings induce the following block representations:

$$A_k = \begin{pmatrix} A_{kI} & A_{kI\Gamma} \\ A_{k\Gamma I} & A_{k\Gamma} \end{pmatrix}, \qquad B_k^T = \begin{pmatrix} O \\ B_{k\Gamma}^T \end{pmatrix}. \tag{24}$$

Let $\mathcal{B}$ be a symmetric positive definite matrix and $\mathcal{H} = \mathcal{B}^{-1}$. Since $\mathcal{A} = \mathcal{A}^T$ the preconditioned Lanczos [11, 8] can be used to solve system (20). In this paper we also recommend the preconditioned conjugate method based on the $\mathcal{B}$-norm of minimal errors [11]:

$$\begin{aligned} \hat{p}_l &= \begin{cases} \mathcal{H}\xi^0, & l = 1, \\ \mathcal{H}\xi^{l-1} - \alpha_l \hat{p}_{l-1}, & l > 1, \end{cases} \\ p_l &= \mathcal{H}\mathcal{A}\hat{p}_l, \\ x^l &= x^{l-1} - \beta_l p_l, \end{aligned} \tag{25}$$

$$\alpha_l = \frac{(\xi^{l-1}, \mathcal{A}\hat{p}_{l-1})_{\mathcal{H}}}{(\mathcal{A}\hat{p}_{l-1}, \mathcal{A}\hat{p}_{l-1})_{\mathcal{H}}}, \quad \beta_l = \frac{(\xi^{l-1}, \hat{p}_l)}{(\mathcal{A}\hat{p}_l, \mathcal{A}\hat{p}_l)_{\mathcal{H}}},$$

where $\xi^l = \mathcal{A}x^l - y$ are the residual vectors, $l = 1, 2, \ldots$ Assume that the eigenvalues of $\mathcal{H}\mathcal{A}$ belong to the union of segments $[d_1; d_2]$ and $[d_3; d_4]$ with $d_1 \le d_2 < 0 < d_3 \le d_4$. Then the convergence estimate

$$\|x^l - x\|_{\mathcal{H}} \le 2q^l \|x^0 - x\|_{\mathcal{H}}, \quad l \ge 1, \tag{26}$$

holds [11] where $q = \dfrac{\hat{d} - \check{d}}{\hat{d} + \check{d}}$, $\hat{d} = \max\{d_4; |d_1|\}$, and $\check{d} = \min\{d_3; |d_2|\}$.

4 BLOCK DIAGONAL PRECONDITIONER

We propose a preconditioner $\mathcal{H}$ as a block diagonal matrix:

$$\mathcal{H} = \begin{pmatrix} H_A & O \\ O & H_\lambda \end{pmatrix} \tag{27}$$

where H_A is also a block diagonal matrix:

$$H_A = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}. \tag{28}$$

All blocks are symmetric positive definite matrices. H_k are said to be the subdomain preconditioners, and H_λ is said to be the interface preconditioner.

If matrices H_k are spectrally equivalent to the matrices A_k^{-1} with constants independent of the value of the coefficient c, and if a matrix H_λ is spectrally equivalent to the matrix S_λ^{-1} with S_λ given by

$$S_\lambda = BA^{-1}B^T \equiv \sum_{k=1}^{2} B_{k\Gamma} S_{k\Gamma}^{-1} B_{k\Gamma}^T \tag{29}$$

with the constants independent of the value of c then the values of $\hat{d}, \check{d}$ in (26) are positive constants [7] also independent of c. Here

$$S_{k\Gamma} = A_{k\Gamma} - A_{k\Gamma I} A_{kI}^{-1} A_{kI\Gamma} \tag{30}$$

are the Schur complements. Our aim is to construct a preconditioner $\mathcal{H}$ spectrally equivalent [7] to the matrix $\mathcal{A}^{-1}$ with constants independent of c.

4.1 Subdomain Preconditioners

Let us define matrices $\overset{\circ}{A}_k$ and M_k by:

$$\begin{aligned} (\overset{\circ}{A}_k v, w) &= \int_{\Omega_k} \nabla v_h \cdot \nabla w_h d\Omega, \\ (M_k v, w) &= \int_{\Omega_k} v_h w_h d\Omega \end{aligned} \tag{31}$$

$\forall v_h, w_h \in V_{kh}$, $k = 1, 2$. Thus, matrices $\overset{\circ}{A}_k$ are the stiffness matrices for the operator $-\Delta$ with the Neumann boundary conditions, and M_k are the corresponding mass matrices. It can be easily shown [8] that

$$A_k^{-1} \sim \left(\overset{\circ}{A}_k + M_k\right)^{-1} + \frac{1}{c} P_k \tag{32}$$

where P_k is the M_k-orthogonal projector onto $\ker \overset{\circ}{A}_k$ and the sign "$\sim$" denotes the spectral equivalence. Moreover, the constants of the spectral equivalence in (32) are independent of the value of c.

Suppose that a matrix $\overset{\circ}{H}_k$ is spectrally equivalent to the matrix $\left(\overset{\circ}{A}_k + M_k\right)^{-1}$. Then the matrix

$$H_k = \overset{\circ}{H}_k + \frac{1}{c} P_k \tag{33}$$

is spectrally equivalent to matrix A_k^{-1} with constants independent of the value of c.

We have plenty of choices for $\overset{\circ}{H}_k$, $k = 1, 2$.

4.2 Interface Preconditioner

We can easily shown [8] that

$$S_{\Gamma k}^{-1} \sim \tilde{S}_k^{-1} + \frac{1}{c} P_{\Gamma k} \tag{34}$$

where $\tilde{S}_k^{-1}$ is the Schur complement for the matrix $\overset{\circ}{A}_k + M_k$ and $P_{\Gamma k}$ is the $M_{\Gamma k}$ orthogonal projector onto $\ker S_{\Gamma k}$ in the case $c = 0$. Moreover, the constants of equivalence in (34) are independent of the value of c. Here $M_{\Gamma k}$ is the interface mass matrix defined by:

$$(M_{\Gamma k} v, w) = \int_{\Gamma_k} v_h w_h ds \quad \forall v_h, w_h \in V_{k\Gamma h} \tag{35}$$

where $V_{k\Gamma h}$ is the trace of V_{kh} into Γ_k, $k = 1, \ldots, m$.

Let the matrices

$$\overset{\circ}{H}_k = \begin{pmatrix} \overset{\circ}{H}_{kI} & \overset{\circ}{H}_{kI\Gamma} \\ \overset{\circ}{H}_{k\Gamma I} & \overset{\circ}{H}_{k\Gamma} \end{pmatrix} \tag{36}$$

be spectrally equivalent to the matrices $\left(\overset{\circ}{A}_k + M_k\right)^{-1}$, $k = 1, 2$. We can also prove that the matrix

$$\hat{S}_\lambda = \sum_{k=1}^{2} B_{k\Gamma} (\overset{\circ}{H}_{k\Gamma} + \frac{1}{c} P_{\Gamma k}) B_{k\Gamma}^T \tag{37}$$

is spectrally equivalent to S_λ with constants independent of the value of c.

To construct the interface preconditioner H_λ we shall use the preconditioned Chebyshev iterative procedure [4, 7]. Let $\hat{H}_\lambda$ be a symmetric positive defined matrix and $\nu_\lambda = \lambda_{\max}/\lambda_{\min}$ where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximal and minimal eigenvalues of $\hat{H}_\lambda \hat{S}_\lambda$, respectively. Then for any $t_\lambda \sim \sqrt{\nu_\lambda}$ the matrix

$$H_\lambda = \left[I_\lambda - \prod_{t=1}^{t_\lambda} \left(I_\lambda - \alpha_t \hat{H}_\lambda \hat{S}_\lambda \right) \right] \hat{S}_\lambda^{-1} \tag{38}$$

is spectrally equivalent to the matrix S_λ^{-1}.

Let $\hat{B}_\lambda$ be a symmetric positive definite matrix such that $1 \in [\mu_{\min}; \mu_{\max}]$ where $\mu_{\min}$ and $\mu_{\max}$ are the minimal and maximal eigenvalues of the matrix $\hat{B}_\lambda^{-1} \sum_{k=1}^{2} B_{k\Gamma} \overset{\circ}{H}_{k\Gamma} B_{k\Gamma}^T$, respectively. Then for the choice $\hat{H}_\lambda = \hat{R}_\lambda^{-1}$ where

$$\hat{R}_\lambda = \hat{B}_\lambda + \frac{1}{c} \sum_{k=1}^{2} B_{k\Gamma} P_{k\Gamma} B_{k\Gamma}^T, \tag{39}$$

the estimate

$$\nu_\lambda \leq \hat{\nu}_\lambda \equiv \mu_{\max}/\mu_{\min} \tag{40}$$

holds.

A solution algorithm for a system

$$\hat{R}_\lambda z = g$$

is presented in [8, 9]. It includes a so called "coarse grid" problem based on the projectors $P_{k\Gamma}$, $k = 1, 2$.

4.3 Arithmetical Complexity for Hierarchical grids

Assume that grids Ω_{kh} are regular, quasiuniform and hierarchical with the average grid step size $h \sim \sqrt[p]{N}$ where N is the dimension of matrix $\mathcal{A}$.

In this case we can use various V-cycle multilevel preconditioners to define matrix $\overset{\circ}{H}_k$ in (33). These preconditioners are spectrally equivalent to the matrices $(\overset{\circ}{A}_k + M_k)^{-1}$, $k = 1, 2$ and have the optimal order of arithmetical complexity [12, 13], i. e. the multiplication with such a preconditioner by a vector costs $O(N)$ arithmetical operations.

Our choice $\overset{\circ}{H}_{k\Gamma}$ in (37) as the corresponding blocks of V-cycle multilevel preconditioner (BPX or MDS-type) is based on two observations. The first one is obvious: spectral equivalence of $\overset{\circ}{H}_{k\Gamma}$ and $\tilde{S}_{\Gamma k}^{-1}$ follows directly from the spectral equivalence of H_k and $\left(\overset{\circ}{A}_k + M_k\right)^{-1}$, $k = 1, 2$. The second observation is rather technical and concerns implementation algorithms for V-cycle multilevel preconditioners: multiplication of $\overset{\circ}{H}_{k\Gamma}$ by a vector can be implemented with $O(h^{-1})$ arithmetical operations. The latter observation has at least one very important consequence: the corresponding matrix $\hat{S}_\lambda$ can be multiplied by a vector with $O(h^{-1})$ arithmetical operations, i.e. multiplication with $\hat{S}_\lambda$ has the optimal order of arithmetical complexity.

It remains to choose preconditioner $\hat{R}_\lambda$, and we do not need an optimal preconditioner because the dimension of S_λ is much smaller than the dimension of A.

In paper [7] we proposed to choose $\hat{B}_\lambda$ being equal to a scalar matrix which is a spectrally equivalent to the matrix $\sum_{k=1}^{2} B_{k\Gamma} M_{k\Gamma}^{-1} B_{k\Gamma}^T$. With this choice, obviously

$$\nu_\lambda \leq \text{const} \cdot h^{-2}$$

where the constant is independent of h and c, and the multiplication B_λ^{-1} by a vector can be implemented with $O(h^{-1})$ arithmetical operations.

On the basis of the latter facts we conclude that t_ν should be proportional to h^{-1}, and arithmetical complexity of the corresponding preconditioner H_λ in (38) is of the order $O(h^{-1})$. In some particular cases we can prove [4, 7] that $t_\nu \sim h^{-1/2}$ and consequently the arithmetical complexity of H_λ is of the order $O(h^{-3/2})$.

5 NUMERICAL EXPERIMENT

The numerical experiments have been performed for the test case given in Fig. 2.

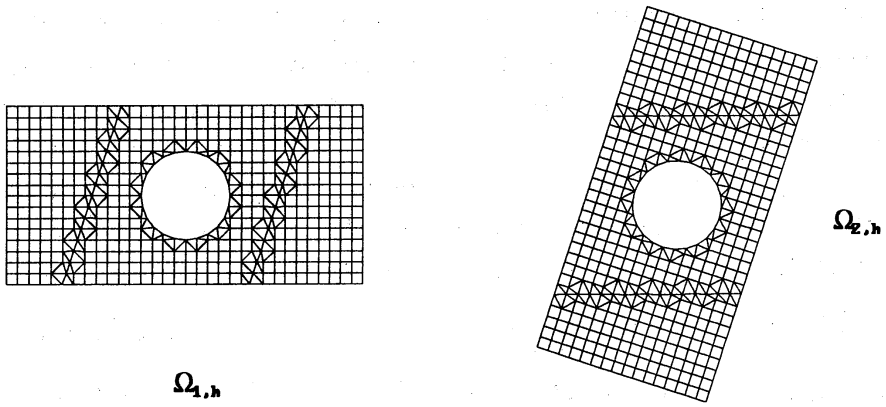

Figure 2: Cartesian locally fitted grids in Ω_1 and Ω_2

In the subdomains Ω_1 and Ω_2 we use rectangular cartesian grids which are fitted to the interface boundary which consists of four straight segments. These grids are given in Fig. 2.

Table 1: Results of numerical experiments

Cartesian grids in Ω_1	Cartesian grids in Ω_2	Number of Chebyshev iterations	Number of Lanczos iterations
16×8	16×8	14	44
32×16	32×16	23	52
64×32	64×32	32	52
128×64	128×64	45	54
256×128	256×128	63	55

Remark For numerical experiments the subdomain BPX preconditioners were used in combination with the fictitious domain technique, because grids Ω_{kh}, $k = 1, 2$ aren't hierarchical. The procedure of coupling is described in [7].

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