

On Association Schemes of Balanced Property with $m_1 = 4$

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Abstract

The work of classification of association schemes began in [2] is continued. The assumption of $m_1 = 4$ allows considering geometrical representation on the unit sphere $S^3 \subset R^4$. Using the balanced property [6, 7] of Terwilliger a classification of all possible local structures $\Gamma_1(x)$ is given.

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1 Introduction

The classification of association schemes is one of the most important tasks of Algebraic Combinatorics. In the present paper we continue the work began by Bannai [2] and consider the case $m_1 = 4$. (For definitions and basic properties of association schemes the reader is referred to the book of Bannai and Ito [1].)

We restrict our attention to primitive symmetric association schemes. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme. Let $A_i (0 \leq i \leq d)$ be the adjacency matrix with respect to the relation $R_i (0 \leq i \leq d)$ on X and let $\mathcal{A} = \langle A_0, A_1, \dots, A_d \rangle$ be the Bose-Mesner algebra of \mathcal{X} . Let $E_i (0 \leq i \leq d)$ be the primitive idempotents of \mathcal{A} . Let $k_i = p_{ii}^0$ be the subdegrees of \mathcal{X} and let $m_i (= q_{ii}^0 = \text{rank}(E_i))$ be the dual subdegrees of \mathcal{X} . Let $\Gamma_i(x) = \{w: \langle x, w \rangle \in R_i\}$ be the i -neighborhood of x .

By renumbering the relations if necessary, we may assume without loss of generality that $q_1(1) \geq q_1(i)$ for all $1 \leq i \leq d$. Let us set

$$E_1 = \frac{1}{|X|} \sum_{i=0}^d q_1(i) A_i.$$

Note that $q_1(0) = m_1 = 4$. We consider the spherical embedding of X in $S^3 = \{(x, y, z, v) \in \mathbf{R}^4: x^2 + y^2 + z^2 + v^2 = 1\}$, with respect to E_1 . That is, X is embedded in S^3 with its Gramm matrix G given by

$$G = \frac{|X|}{4} E_1.$$

By the primitivity of \mathcal{X} this embedding is injective. For the sake of simplicity, we identify elements of X with the vectors of \mathbf{R}^4 of the above embedding.

The concept of *balanced sets* was introduced by Terwilliger in [6, 7].

Definition 1.1 *Let X be set of vectors in S^{N-1} , and let $A = \{\alpha_0, \alpha_1, \dots, \alpha_d\}$ be the set of scalar products of elements of X . X is called balanced if*

- (i) *for all $x \in X$ and all i ($0 \leq i \leq d$), the vector $\sum_{y \in X, \langle x, y \rangle = \alpha_i} y$ is a scalar multiple of x*

(ii) for all $x, y \in X$ and all i, j ($0 \leq i, j \leq d$), the vector $\sum z \in X$, $z -$

$$\begin{aligned} \langle x, z \rangle &= \alpha_i, \\ \langle y, z \rangle &= \alpha_j \end{aligned}$$

$\sum w \in X$, w is a scalar multiple of $x - y$.

$$\begin{aligned} \langle x, w \rangle &= \alpha_j, \\ \langle y, w \rangle &= \alpha_i \end{aligned}$$

An association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is said to have the *balanced property* if its embedding with respect to E_1 is a balanced set. It was proved in [7] that a Q -polynomial association scheme has the balanced property. However, there are examples of association schemes of balanced property, which are not Q -polynomial.

A general method of classification of association schemes was introduced in [2]. That is, consider the embedding with respect to E_1 in S^{n-1} and suppose, that $\alpha_1 = q_1(0)$ is maximal amongst the α_i 's. Then an upper bound exists for k_1 by the so called *kissing problem* of spheres of equal radii. That is, the points of $\Gamma_1(x)$ are on a sphere in R^{n-1} whose radius is smaller than the minimum distance amongst points of X . Then one should try to characterize the possible geometric configurations on that sphere and "lift it up" to S^{n-1} . This was successfully done in [2] in the case of $m_1 = 3$, that is $n = 3$. The aim of this paper is to investigate $m_1 = 4$ case and determine all possible geometric configurations of $\Gamma_1(x)$ provided the association scheme is of the balanced property.

Section 2 contains general observations, the case-by-case analysis is contained in Section 3. In many cases the tedious proofs are omitted, or just a brief sketch is included. Finally, Section 4 contains some concluding remarks.

2 General observations

From now on, throughout the paper we assume that $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is an association scheme of balanced property such that $X \subset S^3$ and $(x, y) \in R_i \iff \langle x, y \rangle = \alpha_i$. The following Proposition is a direct consequence of the solution of the *kissing problem* in the three dimensional Euclidean space, see Leech [5].

Proposition 2.1 *By the above settings, $k_1 \leq 12$ holds.* ■

Let $\gamma_i(x) = \{y \in \mathcal{S}^3: \langle x, y \rangle = \alpha_i\}$. Then $\gamma_1(x)$ is an ordinary three dimensional sphere. On this sphere, the smallest distance between points of X is larger than 60° . Thus, applying Leech's ideas [5] it can be proved that the graph obtained by connecting points of X on $\gamma_1(x)$ whose distance is less than 90° , is planar.

Let us denote the scalar products of vectors of X on $\gamma_1(x)$ by $\alpha_{i_1} = \beta_1 > \alpha_{i_2} = \beta_2 > \dots > \alpha_{i_s} = \beta_s$. That is β_1 corresponds to the smallest distance occurring on $\gamma_1(x)$. Furthermore, let d_i be the distance corresponding to β_i , and let deg_i be the degree of the regular graph whose vertex set is $\Gamma_1(x)$ two vertices are connected iff their distance is d_i . We use the ambiguity in this notation, that d_i may denote angular or Euclidean distance, depending on the environment. Then, applying again Leech's ideas

$$k_1 \leq 7 + \sum_{d_i < 90^\circ} deg_i \quad (1)$$

follows. Indeed, there can be at most 6 points of X on a half sphere.

The balanced property is used for the following two Lemmas.

Lemma 2.2 *Let us assume that $deg_i = 1$ for some i . Then there exists an N positive integer such that d_i is the length of the shortest diagonal of a regular N -gon whose side is d_1 .*

Proof of Lemma 2.2 By the assumption, $p_{1j_i}^1 = 1$. Let $y \in \Gamma_1(x)$ and $\{z\} = \Gamma_{j_i}(x) \cap \Gamma_1(y)$ and $\{v\} = \Gamma_{j_i}(y) \cap \Gamma_1(x)$. By (ii) of the definition of balanced sets, the four points x, y, z, v are coplanar and form a symmetric trapezoid. Now considering the pair y, z playing the same role as x, y before, another coplanar point is obtained, say w . Continuing this process, the newly obtained point has to coincide with v after a while, otherwise we would get a shorter distance than d_1 on $\gamma_1(x)$. ■

Next case is degree 2.

Lemma 2.3 *Let us suppose that $deg_i = 2$ for some i . Then $i = 1$, furthermore, $j_i = 1$.*

Proof of Lemma 2.3 Let $\langle x, y \rangle = \alpha_1$ and suppose that $deg_i = 2$. We may assume without loss of generality that $x = (0, 0, a, b)$ and $y = (0, 0, a, -b)$, where $a^2 + b^2 = 1$ and $\alpha_1 = a^2 - b^2$. Let $z_k = (z_{k1}, z_{k2}, z_{k3}, z_{k4})$ $k = 1, 2$

be the elements of $\Gamma_{j_i}(y) \cap \Gamma_1(x)$, while $u_k = (u_{k1}, u_{k2}, u_{k3}, u_{k4})$ be those of $\Gamma_{j_i}(x) \cap \Gamma_1(y)$. If $i \neq 1$ or $j_i \neq 1$, then $z_k \neq u_k$. The distance relations can be expressed in the following set of equations.

$$z_{13}a + z_{14}b = a^2 - b^2 \quad (2)$$

$$z_{23}a + z_{24}b = a^2 - b^2 \quad (3)$$

$$u_{13}a - u_{14}b = a^2 - b^2 \quad (4)$$

$$u_{23}a - u_{24}b = a^2 - b^2 \quad (5)$$

$$u_{13}a + u_{14}b = z_{13}a - z_{14}b \quad (6)$$

$$u_{23}a + u_{24}b = z_{23}a - z_{24}b \quad (7)$$

$$u_{13}a + u_{14}b = z_{23}a - z_{24}b \quad (8)$$

The (2)-(5) express the fact that $z_k \in \Gamma_1(x)$ and $u_k \in \Gamma_1(y)$, respectively. The (6)-(8) are for $\langle x, u_k \rangle = \langle y, z_l \rangle$ ($k = 1, 2$ $l = 1, 2$). From these equations it can be easily inferred that

$$u_{13} = z_{13} \quad (9)$$

$$u_{23} = z_{23} \quad (10)$$

$$u_{14} = -z_{14} \quad (11)$$

$$u_{24} = -z_{24}. \quad (12)$$

The balanced property implies that

$$z_{11} + z_{21} = u_{11} + u_{21} \quad (13)$$

$$z_{12} + z_{22} = u_{12} + u_{22}. \quad (14)$$

Furthermore, because z_k and u_k are unit vectors,

$$z_{11}^2 + z_{12}^2 = z_{21}^2 + z_{22}^2 = u_{11}^2 + u_{12}^2 = u_{21}^2 + u_{22}^2 \quad (15)$$

holds, as well. Thus, (z_{11}, z_{12}) , (z_{21}, z_{22}) and (u_{11}, u_{12}) , (u_{21}, u_{22}) are 2-dimensional vectors of same length, and their pairwise sum is the same, hence they are the same pair of vectors. Consequently, z_1 and u_1 , furthermore z_2 and u_2 are reflections of each other, respectively, to the hyperplane orthogonal to $(0, 0, 0, 1)$. However, this implies that z_1, z_2 and y cannot be in $\Gamma_1(x)$ at the same time, a contradiction. ■

3 Cases according to k_1

In this section several cases are considered according to possible values of k_1 . If the appropriate geometric embedding exists, then a deg_1 -regular planar graph of k_1 vertices exists. Many cases can be ruled out by Euler's formula. If the planar graph exists, then geometric considerations rule out many cases, i.e. the planar graph cannot be represented on S^2 by uniform length edges. As usual, f denotes the number of faces, v the number of vertices, e the number of edges of a planar graph, respectively. Similarly, f_i denotes the number of i -sided faces. The main formulas used are

$$v + f = e + 2 \quad (16)$$

$$\sum f_i = f \quad (17)$$

$$\sum i f_i = 2e \quad (18)$$

$$\sum (i - 3) f_i \geq 0 \quad (19)$$

For the sake of convenience, let F_i denote an i -sided face, thus a planar graph can be represented by the the symbol $f_3 * F_3 + f_4 * F_4 + \dots$

3.1 $k_1=12$

This is an extreme case. By (1) $\sum_{d_i < 90^\circ} deg_i \geq 5$ holds. Thus, connecting those points of $\Gamma_1(x)$ whose distance is less than 90° , we obtain a regular planar graph on 12 vertices of degree at least 5. Easy application of Euler's formula shows, that this graph has degree 5, indeed, and all its faces are triangles. Thus, it is the edge-graph of the icosahedron. By Lemma 2.2 $d_i < 90$ implies $deg_i \neq 1$ furthermore, if $j_i \neq 1$, then $deg_i \neq 2$ by Lemma 2.3. Thus, $\sum_{d_i < 90^\circ} deg_i = 5$ can only happen if either $j_1 = 1$ and $deg_1 = 2$ and $deg_2 = 3$ or $deg_1 = 5$. In the first case, or in the second case if $j_1 \neq 1$, we have a distance on $\gamma_1(x)$ that is very close to the shortest one, say the distance corresponding to α_i (either d_2 or d_1). Thus, on $\gamma_1(x)$ and $\gamma_i(x)$ together we have at least 15 points, which can be shown to be impossible by routine calculations.

So, if $k_1 = 12$, then $\Gamma_1(x)$ is an icosahedron, with edge length equal to the shortest distance among points of X . Considering the graph with vertex set X , edge set determined by the shortest distance, a locally icosahedron regular graph is obtained, such graphs are characterized by Blokhuis et. al [3].

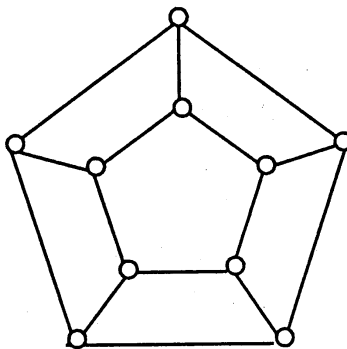


Figure 1

3.2 $k_1 = 11$

In this case all deg_i 's are even. $\sum_{d_i < 90^\circ} deg_i \geq 6$ contradicts to Euler Formula. If $deg_1 = 2$, then by (1) $d_2 < 90^\circ$ holds, so $\sum_{d_i < 90^\circ} deg_i \geq 6$ follows, a contradiction. Thus, $deg_1 = 4$ and $\sum_{i > 1} deg_i = 6$. This could only occur as $3 + 3$, $5 + 1$, $4 + 1 + 1$, $3 + 1 + 1 + 1$, because of Lemma 2.3. However, in each case an odd degree should exist, a contradiction.

3.3 $k_1 = 10$

If $deg_1 < 3$ then by (1) $\sum_{d_i < 90^\circ} deg_i \geq 5$, which contradicts to Euler Formula. If $deg_1 = 3$ and $d_2 \geq 90^\circ$, then the only way to obtain 10 points on $\gamma_1(x)$ is if two points of distance d_1 do not have common neighbour of distance d_1 from each. Thus, considering the planar graph determined by distance d_1 , $f_3 = 0$. On the other hand, $v = 10$, $e = 15$, $f = 7$, $\sum (i - 4)f_i = 2$. This allows two possibilities: $5 * F_4 + 2 * F_5$ or $6 * F_4 + 1 * F_6$. The second possibility is easily seen to be unrealizable. Also, it is not hard to see that the only realization (as a planar graph) of the first possibility is the graph of Figure 1. Routine, but very tedious case-by-case analysis shows that the only way to geometrically represent this graph is if the two pentagons are in parallel planes. But in this case $deg_i = 2$ for some $i > 1$, a contradiction.

If $deg_1 = 4$, then by Euler Formula we have the following possibilities.

$1 * F_7 + 11 * F_3$: clearly impossible.

$1 * F_6 + 1 * F_4 + 10 * F_3$: If there is a point surrounded by triangles only, then geometrically it means a square pyramid with uniform edge lengths. That

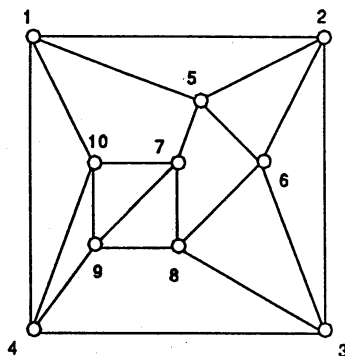


Figure 2

already determines the sphere, and easy to see, that there is not enough room left for the other five points. If there is no such vertex, then each one is incident to the quadrangle or the hexagon, i.e. these two faces have no common vertex, which is impossible to draw.

$1 * F_5 + 2 * F_4 + 9 * F_3$: Again, either we get a point surrounded by triangles, or cannot finish the drawing of the graph.

$2 * F_5 + 10 * F_3$: If there is no vertex surrounded by triangles, then the only possibility is that the two pentagons are disjoint, which leads to the graph obtained from the icosahedron by removing an antipodal point pair. The only geometric representation of this graph having no i with $deg_i = 2$ is 10 points of an icosahedron, so that the missing two points are antipodal. The same geometric consideration can be applied as in the $\Gamma_1(x) = \text{icosahedron}$ case to show that $j_1 = 1$. Then the graph on X whose edges are the pairs of distance d_1 is locally P_{10} graph, where P_{10} is the graph obtained from the icosahedron by removing an antipodal point pair. This can be eliminated by the method of Blokhuis et. al. [3].

$4 * F_4 + 8 * F_3$: Again either there exists a vertex surrounded by triangles or the graph of Figure 2 is obtained. Consider the following transformation $\phi = (8, 1)(7, 5)(10, 6)(4, 3)(9, 2)$. It is not hard to see that ϕ preserves distances amongst points of $\Gamma_1(x)$, i.e. it can be extended to an orthogonal transformation of $\gamma_1(x)$ satisfying the property $\phi^2 = Id$. Thus ϕ is either a reflection to plane or 180° degree rotation around a line or reflection to a point. Now, if ϕ is not reflection to a plane, then all $(j, \phi(j))$ pairs should be antipodal, but $(3, \phi(3))$ is the shortest distance, a contradiction. So ϕ is

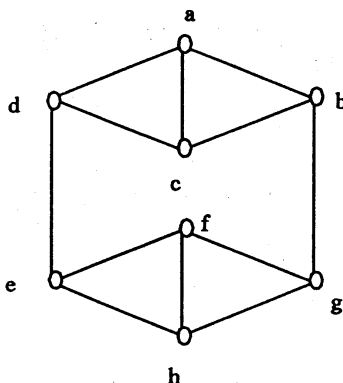


Figure 3

reflection to a plane. In that case, lines 1-8, 7-5, 10-6, 4-3, and 9-2 are all parallel. However, the orientation is different, a contradiction.

3.4 $k_1 = 9$

All deg_i 's must be even, and they sum up to 8. There cannot be two of them equal to 2, so the only possibility is $deg_1 = deg_2 = 4$. However, two-distance set on 9 points does not exist.

3.5 $k_1 = 8$

If $deg_1 = 2$, then $deg_2 = 5$ or $deg_2 = 4$ and $deg_3 = 1$. In both cases easy geometric considerations yield contradictions.

If $deg_1 = 3$, then Euler Formula yield $f = 6$ and $\sum(i - 3)f_i = 6$. It is immediate to rule out the $f_8 > 0$ and $f_7 > 0$ cases. If $f_6 > 0$, then the graph shown on Figure 3 is possible only. In the geometric representation of the graph of Figure 3 the two tetrahedrons $abcd$ and $efgh$ are isomorphic. By (i) of balanced property, these two tetrahedrons must be in antipodal position. However, that would result in $deg_i = 2$ for some $i > 1$.

If $f_5 > 0$, then there is only one possible graph, shown on Figure 4. Again, using (i) of balanced property, the bisector plane of the angle formed by two normal vectors of the planes of the triangular faces must contain the two vertices not incident with the triangles. This results in again $deg_i = 2$ for

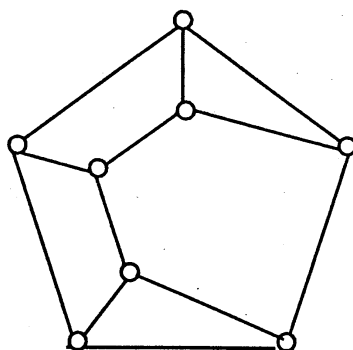


Figure 4

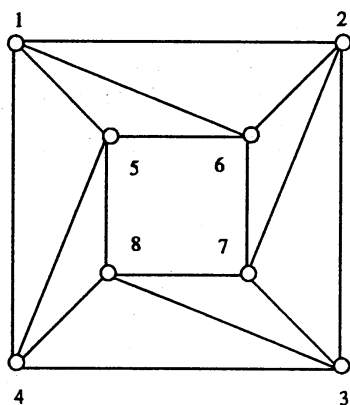


Figure 5

some $i > 1$.

If $f_8 = f_7 = f_6 = f_5 = 0$, then the graph is the edge graph of the cube, and the only possible balanced geometric realization is the cube itself.

If $deg_1 = 4$, then the only possible planar graph is shown on Figure 5. Since the points are on a sphere, tetrahedrons (4835) and (2367) are isomorphic, hence distances 35 and 36 are equal. This implies by Lemma 2.3 that $deg_2 = 3$. Thus, any two points not joined by a d_1 distance edge are of distance d_2 . In particular, tetrahedrons (4578) and (4378) are isomorphic, which implies distances 47 and 43 are equal, i.e. $d_1 = d_2$, a contradiction.

$deg_1 \geq 5$ is clearly impossible geometrically.

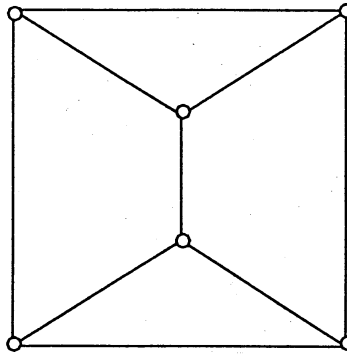


Figure 6

3.6 $k_1 = 7$

There exists no a -regular planar graph on 7 vertices if $a \geq 3$. Thus $deg_1 = 2$ and $deg_2 = 4$. Let $a, b \in \Gamma_1(x)$ be of distance d_1 . Then they have at least three common neighbours of distance d_2 that possible only if a and b are antipodal, a contradiction.

3.7 $k_1 = 6$

If $deg_1 = 4$ then $deg_2 = 1$. The graph is the edge graph of the octahedron, and the only geometrical representation is the octahedron itself, because it is an antipodal 2-distance set of 6 points, which must be a tight spherical 3-design, see [4].

If $deg_1 = 3$, then the only possible planar graph is shown on Figure 6. By (i) of balanced property, the two triangles must lay in paralell planes so that their centers are mirror images of each other to the center of the sphere. Elementary geometric considerations yield that either $deg_2 = 2$ or $deg_2 = 1$ and $deg_3 = 1$ but one of d_2 or d_3 is less than $\sqrt{d_1}$ that contradicts to Lemma 2.2.

If $deg_1 = 2$, then the graph of the shortest distance on $\Gamma_1(x)$ is either a union of two triangles or a hexagon. The other degrees must be $deg_2 = deg_3 = deg_4 = 1$ in both cases. Using Lemma 2.2 elementary geometric calculations show that the required polyhedra do not exist.

3.8 $k_1 = 5$

This case is impossible because all deg_i 's are even and their sum is 4, but then either $deg_1 = 4$, which is clearly impossible, or $deg_1 = deg_2 = 2$ that contradicts to Lemma 2.3.

3.9 $k_1 = 4$

If $deg_1 = 3$, then $\Gamma_1(x)$ is the tetrahedron.

If $deg_1 = 2$ then $deg_2 = 1$ and $\Gamma_1(x)$ is a square on a great circle of $\gamma_1(x)$ by (i) of balanced property. This implies using Lemma 2.3 that $d_1 = 90^\circ$ on S^3 . However, then the configuration is antipodal, that is \mathcal{X} is imprimitive.

3.10 $k_1 = 3$

In this case $\Gamma_1(x)$ is a regular triangle on a great circle of $\gamma_1(x)$ by (i) of balanced property. This implies using Lemma 2.3 that $d_1 = 120^\circ$ on S^3 . It is easy to see that this case results in an ordinary tetrahedron, that is $m_1 = 3$, a contradiction.

3.11 The Main Theorem

According to the above analysis, the following is true.

Theorem 3.1 *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme of balanced property with $m_1 = 4$. Then in the geometric representation of \mathcal{X} the neighborhood $\Gamma_1(x)$ of a point $x \in X$ is one of the following regular polyhedra: Icosahedron, Cube, Octahedron or Tetrahedron.*

4 Some Remarks

The case of Icoshedron is completely settled by Blokhuis et. al. in [3]. The regular polytopes provide examples for the other cases. If the edges of the four-dimensional polytope X are only those of the shortest distance, then the dual polytope X^* is regular faced polytope. Those are well known to be classified, which classification gives results for our case by duality.

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