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ALGEBRAIC STRUCTURE OF NULL DESIGNS

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ABSTRACT. Null designs are defined as the elements of the kernel of the incidence matrices of k-subsets and t-subsets of an n-set. It has been known that the set of null designs is the direct sum of the Specht modules of certain types as a group representation of the symmetric group. The same is true for the q-analogue of null designs if we use irreducible unipotent representations of the general linear groups over a finite field

A bijection between two known bases of the module of null designs of the Boolean algebras ($q = 1$) is constructed.

1. Introduction

Let B_n^q denote the subspace lattice of an n-dimensional vector space over the finite field $\mathbb{F}_q$ (if $q = 1$ then the subset lattice of an n-set $[n] \equiv \{1, 2, \ldots, n\}$), for a positive integer n and a prime power q.

For $0 \leq i \leq n$, let

$$X_i \equiv \{x \in B_n^q \,:\, rank(x) = i\}$$

and for a given field K and a finite set X, let $K[X]$ be the K-vector space of the formal sums $\sum_{\substack{x \in X \\ c_x \in K}} c_x x$.

We will deal with only a field K of characteristic zero for the purpose of this paper.

For $0 \leq i \leq j \leq n$, we define two K-linear maps
$d_{ji} : K[X_j] \to K[X_i]$ and $u_{ij} : K[X_i] \to K[X_j]$ by

$$d_{ji}(x) = \sum_{\substack{y \leq x \\ y \in X_i}} y \quad \text{for} \quad x \in X_j \quad \text{and}$$

$$u_{ij}(y) = \sum_{\substack{y \leq x \\ x \in X_j}} x \quad \text{for} \quad y \in X_i \ .$$

Note that d_{ij} and u_{ji} just represent the incidence matrix between X_j and X_i.

If we take K as the underlying field then, for given integers $0 \leq t \leq k \leq n - k$, the set of *null* (t, k)*-designs* is defined by the K-vector space

$$N_{k,t}^q \equiv Ker(d_{k,t}).$$

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The following is a well known theorem which will be playing a key role in the proof of the main theorem. $\binom{n}{m}_q$ is the number of m-dimensional subspace of an n-dimensional space over $\mathbb{F}_q$, which is defined by $\frac{[n][n-1]\dots[n-m+1]}{[m][m-1]\dots[1]}$, where $[i] = 1 + q + \cdots + q^{i-1}$.

Theorem 1.1 [5]. *For $0 \le i \le j \le n-i-1$, the $\binom{n}{i}_q$ by $\binom{n}{j}_q$ incidence matrix $A_{ij} = (a_{xy})$, defined by*

$$a_{xy} = \begin{cases} 1 & \text{if } y \le x \\ 0 & \text{otherwise} \end{cases}$$

has the full rank $\binom{n}{j}_q$ over a field of characteristic zero. Hence, d_{ji} is a surjection and u_{ij} is an injection. ∎

In the next section, we summarize the known theorems about the ordinary representations of the symmetric group and the general linear group over $\mathbb{F}_q$. Then, in the third section, we state a theorem which express $N^q_{t,k}$ as a representation of the symmetric group on n letters or the general linear group over a finite field. Finally, a construction of a bijection between two known bases of $N^1_{t,k}$ is given.

2. Group Representations

Obviously, $N^q_{t,k}$ is a representation of the symmetric group S_n of n letters if $q = 1$, and it is a representation of the general linear group over $\mathbb{F}_q$, $GL_n(q)$, if $q \neq 1$.

Remember that we only deal with a field of characteristic 0 as the underlying field of group representations.

To investigate the structure of $N^q_{t,k}$ as group representations, we summarize the main theorems we will need about the representations of S_n and $GL_n(q)$. For the detailed definition and the proof of the theorems we refer to [3] , [4] and [6].

The (q)-Specht modules are defined for each partition $\lambda = (\lambda_1, \dots, \lambda_h)$ of n. Remember that the *diagram* $[\lambda]$ is the set of ordered pairs (a, b), $1 \le a \le h$, $1 \le b \le \lambda_a$ and a *tableau of type* λ is an array of integers obtained by replacing the nodes in $[\lambda]$ by the numbers $1, 2, \dots, n$. *Tabloids* are the tableaux with forgotten columns, i.e. we think each row of a tabloid as a set and we use $\{T\}$ for the tabloid obtained from tableau T. Let V be an n-dimensional vector space over $\mathbb{F}_q$ ($V = [n]$, if $q = 1$). *Flags* of type λ are the sequences of subspaces (subsets, if $q = 1$) of V

$$\langle 0 \rangle = V_0 \subset V_1 \subset \cdots \subset V_n = V \quad \text{, where}$$

$$Dim(V_i/V_{i-1}) = \lambda_i \quad (|V_i - V_{i-1}| = \lambda_i \text{ if } q = 1) \quad \text{for } 1 \le i \le n\ .$$

M^q_λ is the *permutation representation* of the flags of type λ, hence $M^q_{(n-i,i)}$ is the permutation representation of i-dimensional spaces of B^q_n, if we only read V_1 of the given flag i.e.

$$M^q_{(n-i,i)} = K[X_i]\,.$$

For each partition λ, an irreducible submodule of M^q_λ, called (q)-*Specht module*, exists, and they are all non-isomorphic. We are only interested in the two part partitions $\lambda = (n-i,i)$, $2i \leq n$, so we introduce one way to describe the (q)-Specht module for $\lambda = (n-i,i)$.

Theorem 2.1 (Kernel Intersection Theorem, [3, p72], [4, p76]).

$$S^q_{(n-i,i)} = \bigcap_{j=0}^{i-1} Ker\, d_{ij} \qquad . \qquad \blacksquare$$

Remark on Theorem 2.1. For $Ker\, d_{ij}$'s to be a $KGL_n(q)$-module (or KS_n-module), we expect d_{ij}'s to be module homomorphisms. It, however, is easy enough to check.

Theorem 2.2.

$$Dim S^q_{(n-i,i)} = \binom{n}{i}_q - \binom{n}{i-1}_q \,. \qquad \blacksquare$$

Theorem 2.3 (Young's Rule).

$$M^q_{(n-k,k)} \cong \bigoplus_{i=0}^{k} S^q_{(n-i,i)} \,. \qquad \blacksquare$$

For a given tableau T, let C_T be the subgroup of of S_n consisted with the column stabilizers of T, then a generator of $S^1_{(n-i,i)}$ is given by

$$e_T \equiv \kappa_T\{T\} \quad \text{,where} \quad \kappa_T = \sum_{\pi \in C_T} (sgn\pi)\pi \,.$$

Remember that a tableau T is called *standard* if each row and each column of T form increasing sequences.

The following theorem gives us a very natural basis of the Specht modules.

Theorem 2.4.

$$\{e_T \,:\, T \textit{ is a standard } (n-i,i)\textit{-tableau}\,\}$$

is a basis for $S^1_{(n-i,i)}$. $\blacksquare$

3. $N_{t,k}^q$ as a group representation

Since $N_{t,k}^q$ is a submodule of $K[X_k] = M_{(n-k,k)}^q$, by Theorem 2.3, $N_{t,k}^q$ must be a direct sum of $S_{(n-i,i)}^q$'s. The following theorem shows how $N_{t,k}^q$ is decomposed.

Theorem 3.1. *As $KGL_n(q)$-modules (or as KS_n-modules if $q = 1$),*

$$N_{t,k}^q \cong \bigoplus_{i=t+1}^{k} S_{(n-i,i)}^q \,.$$

Sketch of the proof. For each $t+1 \leq i \leq k$, we can embed $S_{(n-i,i)}^q$ into $M_{(n-k,k)}^q$ through u_{ik} since u_{ik} is a monomorphism. Then, we can show that $u_{ik}(S_{(n-i,i)}^q)$ is a submodule of $N_{t,k}^q$ by doing some calculation (either by direct way or by the help of Möbius functions). Now, Theorem 2.2 finishes the proof by comparing the dimensions. ∎

The following theorem, due to R. L. Graham, S. -Y. R. Li and W. -C. W. Li , gives a very nice basis of $N_{t,k}^1$. For convenience, we use a square free polynomial $x_{i_1}x_{i_2}\dots x_{i_k}$ to represent a k-subset $\{i_1, i_2, \dots, i_k\}$ of $[n]$.

Theorem 3.4 [1]. *For $0 \leq t \leq k \leq n-t-1$, let $S_{t,k,n}$ consist of those $\sigma \in S_n$ which satisfy:*

a. $\sigma(1) < \sigma(3) < \cdots < \sigma(2t+1)$,

b. $\sigma(2) < \sigma(4) < \cdots < \sigma(2t+2)$,

c. $\sigma(2i-1) < \sigma(2i)$, $1 \leq i \leq t+1$,

d. $\sigma(2t+1) < \sigma(2t+3) < \sigma(2t+4) < \cdots < \sigma(k+t+1)$,

e. $\sigma(2t+1) < \sigma(k+t+2) < \sigma(k+t+3) < \cdots < \sigma(n)$, *and*

f. If $2t+3 \leq i \leq k+t+1 < j \leq n$ *and* $\sigma(i) < \sigma(2t+2)$ *then* $\sigma(i) < \sigma(j)$.

Then $\{\sigma(\omega) : \sigma \in S_{t,k,n}\}$ is a basis of $N_{t,k}^1$, where

$$\omega = (x_1 - x_2)(x_3 - x_4)\cdots(x_{2t+1} - x_{2t+2})x_{2t+3}\cdots x_{k+t+1}$$

and permutations in S_n act on ω by permuting the indices of x_i.

4. A bijection between two bases of $N_{t,k}^1$

In this section, we construct a bijection between $S_{t,k,n}$ (see Theorem 3.4) and

$$ST_{t,k,n} \equiv \bigcup_{i=t+1}^{k} \{T : T \text{ is standard of shape } (n-i,i)\},$$

which give two bases of $N_{t,k}^1$ (See Theorem 2.4, 3.1 and 3.4).

Construction.

For convenience, given $\sigma \in S_{t,k,n}$, we use the tableau

$$\begin{array}{lllllllllll} Tab(\sigma) = \sigma(1) & \sigma(3) & \cdots & \sigma(2t+1) & \sigma(2t+3) & \cdots & \sigma(k+t+1) & \sigma(k+t+2) & \cdots & \sigma(n) \\ \sigma(2) & \sigma(4) & \cdots & \sigma(2t+2) & & & & & & \end{array}$$

to represent σ.

A mapping ϕ from $S_{t,k,n}$ to $ST_{t,k,n}$.

1. If $Tab(\sigma)$, $\sigma \in S_{t,k,n}$ is standard, then

$$\phi(\sigma) = Tab(\sigma).$$

2. If $Tab(\sigma)$, $\sigma \in S_{t,k,n}$ is not standard, then
   2a. Find the smallest i_1 such that $2t+3 \leq i_1 \leq k+t+1$ and $\sigma(k+t+2) < \sigma(i_1)$.
   2b. Push down $\sigma(i_1)$ to the second row of $Tab(\sigma)$ (put it at the right end of the second row) and put $\sigma(k+t+2)$ at the position where $\sigma(i_1)$ was, then slide $\sigma(k+t+3), \ldots, \sigma(n)$ to the left by one position. Call the new tableau T_1.
3. T_1 is of shape $(n-t-2, t+2)$ and by the definition of $S_{t,k,n}$'s, $T_1 = Tab(\sigma_1)$ for $\sigma_1 \in S_{t+1,k,n}$. Apply 1 and 2 to σ_1.

A mapping ψ from $ST_{t,k,n}$ to $S_{t,k,n}$.

1. If $T \in ST_{t,k,n}$ is of shape $(n-t-1, t+1)$, then $\psi(T) = \sigma$, where $Tab(\sigma) = T$.
2. If $T \in ST_{t,k,n}$ is of shape $(n-i, i)$, $i > t+1$, then repeat the following until having a tableau of shape $(n-t-1, t+1)$.
   2a. Find the right end number n_1 of the second row of T. Then, find the largest number l such that $l < n_1$ in the first k columns of the first row of T. Now, insert l between the k^{th} and $(k+1)^{th}$ numbers of the first row and put n_1 at the position where l was.

Remark. It is a routine work to prove that ϕ and ψ are inverses each other.

Example.

If

$$\begin{array}{l} Tab(\sigma) = 1\ 3\ 6\ 8\ 5\ 7 \\ \qquad\qquad\ \ 2\ 4 \end{array}$$

for $\sigma \in S_{1,4,8}$, then

$$\phi(\sigma) = \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{array} \in ST_{1,4,8}$$

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