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THE ALPERIN AND DADE CONJECTURES FOR SOME FINITE GROUPS

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1. Alperin's Weight Conjecture

Let G be a finite group, p a prime, and $O_p(G)$ the largest normal p-subgroup of G. In addition, let B be a p-block, R a p-subgroup of G, φ an irreducible ordinary character of the factor group $N(R)/R$. Then a pair (R, φ) is called a *B-weight* (character version) if the p-defect of φ is 0 and if the block $B(\varphi)$ of the normalizer $N(R)$ containing φ induces the block B (in the sense of Brauer), where φ is also viewed as a charcater of $N(R)$ and the p-defect of φ is the largest integer a such that p^a divides $\frac{|G|}{\varphi(1)}$. A weight is always identified with its conjugates in G.

Alperin's Weight Conjecture (1987): *The number of B-weights equals the number of irreducible Brauer characters in the block B.*

In 1989, Knörr and Robinson translated the conjecture into one involving only ordinary irreducible characters.

A p-subgroup chain

$$C : Q_0 < Q_1 < \ldots < Q_n$$

of *length* $|C| = n$ is called a *normal* p-chain if each subgroup Q_i is a proper normal subgroup of Q_n for $1 \leq i \leq n-1$. Let $\mathcal{N}$ be the set of all normal p-chains. Then G acts on $\mathcal{N}$ by conjugation, and the stabilizer

$$N(C) = \cap_{i=1}^{n} N(Q_i)$$

of the chain C in G is called the normalizer of C. Denote by $\mathrm{k}(N(C), B)$ the number of irreducible characters ψ of $N(C)$ such that the block $B(\psi)$ of $N(C)$ containing ψ induces the given block B. Alperin's weight conjecture is equivalent to the following.

The Knörr-Robinson Form (1989): *Whenever G is a finite group and B is a p-block, we have*

$$\sum_{C}(-1)^{|C|}\,\mathrm{k}(N(C),B)=0,$$

where C runs over a set $\mathcal{N}/G$ of representatives for G-orbits in $\mathcal{N}$.

2. Dade's Ordinary Conjecture

A p-subgroup R of G is called a *radical* subgroup if R is the largest normal p-subgroup of its normalizer $N(R)$, that is, $R = O_p(N(R))$.

A p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_u$$

is called a *radical* p-chain if it satisfies the following two conditions

(a) $P_0 = O_p(G)$.

(b) P_k is a radical subgroup of the subgroup $\cap_{\ell=0}^{k-1} N(P_\ell)$

for each $1 \le k \le u$. Let $\mathcal{R} = \mathcal{R}(G)$ be the set of all radical p-chains of G.

Given a non-negative integer d, a p-block B of G and a radical p-chain C, let $\mathrm{k}(N(C),B,d)$ be the number of irreducible characters ψ of the normalizer $N(C)$ such that

$B(\psi)$ induces B and the defect of ψ is d.

Dade's Ordinary Conjecture (1990): *If $O_p(G) = 1$ and B is a block with non-trivial defect groups, then for any integer d,*

$$\sum_{C}(-1)^{|C|}\,\mathrm{k}(N(C),B,d)=0 \tag{2.1}$$

where C runs over a set $\mathcal{R}/G$ of representatives for the G-orbits in $\mathcal{R}$.

It was shown by Dade [D1] that

$$\sum_{C\in\mathcal{N}/G}(-1)^{|C|}\,\mathrm{k}(N(C),B,d)=\sum_{C\in\mathcal{R}/G}(-1)^{|C|}\,\mathrm{k}(N(C),B,d).$$

Thus Dade's ordinary conjecture implies the Knörr-Robinson form of Alperin's weight conjecture. It is also mentioned in Dade's paper [D1] that the ordinary conjecture is equivalent to the final conjecture if the group G has both trivial Schur multiplier $\mathrm{Mult}(G)$ and trivial outer automorphism group $\mathrm{Out}(G)$. These conditions are satisfied by the following 11 sporadic simple groups:

$$J_1,\ J_4,\ M_{11},\ M_{23},\ M_{24},\ Ly,$$
$$Co_2,\ Co_3,\ Fi_{23},\ Th,\ M.$$

3. Dade's Invariant Conjecture

Suppose the center $Z(G)$ of G is trivial. Then we can identify G with its inner automorphism group $\mathrm{Inn}(G)$. So the automorphism group $A = \mathrm{Aut}(G)$ acts naturally on each p-chain C, and moreover, the stabilizer $N_A(C)$ of C in A acts on each irreducible character ψ of $N_G(C)$. So $N_G(C)$ is a normal subgroup of the stabilizer $N_A(C,\psi)$ of ψ in $N_A(C)$. The factor group $N_A(C,\psi)/N_G(C)$ is isomorphic to the subgroup of an outer automorphism group $O = \mathrm{Out}(G)$ of G.

Given a radical p-chain C, a p-block $B \in \mathrm{Blk}(G)$, a non-negative integer d, and a subgroup U of $O = \mathrm{Out}(G)$, let $\mathrm{k}(N(C), B, d, U)$ be the number of irreducible characters ψ of $N_G(C)$ such that the block of $N(C)$ containing ψ induces the block B, the defect of ψ is d, and $N_A(C,\psi)/N_G(C) = U$. The Dade invariant conjecture is stated as follows:

Dade's Invariant Conjecture [D3]: *If $Z(G) = O_p(G) = 1$ and B is a p-block of G with non-trivial defect group, then for any integer $d \geq 0$ and any subgroup $U \leq \mathrm{Out}(G)$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d, U) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

Dade's invariant conjecture is equivalent to his final conjecture whenever G has trivial Schur multiplier $\mathrm{Mult}(G)$ and an outer automorphism group all of whose Sylow subgroups are cyclic. A lot of finite simple groups satisfy these conditions, for example,

$$He,\ HN,\ R_1(q),\ R_2(q),\ {}^3D_4(q),$$
$$G_2(q)\,(\text{with } q \neq 3,4),\ F_4(q)\,(\text{with } q \neq 4), E_8(q).$$

4. Current Works

1. Alperin's weight conjecture has been verified for the following groups and blocks:

Blacks:

(a) Cyclic and tame blocks (by Dade, Uno).

(b) Abelian defect blocks with small inertial index (by Puig and Usami).

(c) Abelian defect principal 2-blocks (by Fong and Harris).

(d) Abelian defect unipotent blocks of a finite reductive group (by Broué, Malle and Michel).

Groups:

(a) p-solvable groups (by Okuyama, Isaacs, Navarro, Gres, Barker).

(b) Groups of Lie type in the defining characteristic (by Alperin, Cabanes, and reproved by Lehrer and Thévenaz).

(c) S_n (by Alperin and Fong).

(d) Classical groups in non-defining characteristics (by Alperin, Fong, Conder and An). In this case, the numbers of irreducible Brauer characters for blocks of symplectic and even-dimensional orthogonal groups are unknown (when $p \neq 2$).

(e) $Sz(2^{2n+1})$, ${}^2G_2(q^2)$, ${}^2F_4(q^2)$, $G_2(q)$, ${}^3D_4(q)$ (by Dade, An).

(f) M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, He, Co_3, J_1 (by Dade, Conder, An).

(g) The covering groups of S_n and A_n ($p \neq 2$) (by Michler and Olsson).

(h) Wrath product groups $G \wr S_n$ provided the conjecture holds for that finite group G (by Ewert)

2. Dade's final conjecture has been verified for the following cases:

(a) 10 sporadic simple groups:

M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, He, J_1, J_2, J_3, Co_3 (by Dade, Huang, Kotlica, Schwartz, Conder, An).

(b) $L_2(q)$, $L_3(q)$ $(p|q)$, $Sz(q)$, ${}^2G_2(3^{2n+1})$ $(p \neq 3)$, $G_2(q)$ ($p \neq 3$ and $p \nmid q \neq 4$), the Tits group (by Dade, An).

(c) Cyclic blocks (by Dade).

3. The invariant conjecture has been verified for all tame blocks, and for the group McL ($p \neq 2$) (by Uno, Murray).

4. The ordinary conjecture has been verified for the following cases:
(a) $\mathrm{GL}_n(q)$ $(p|q)$, ${}^2F_4(2^{2n+1})$ $(p \neq 2)$, $G_2(q)$ $(p \not| q)$ (by Olsson, Uno, An).
(b) S_n (by Olsson and Uno when p odd, An when $p = 2$).
(c) Ru (by Dade).
(d) Unipotent abelian defect blocks (by Broué, Malle and Michel).
(e) Abelian defect principal 2-blocks (by Fong and Harris).
(f) All abelian defect blocks with small inertia index (by Usami).

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