数理解析研究所講究録
992 巻 1997 年 1-4

A Map from the Lower-Half of the n-Cube onto the $(n-1)$-Cube which Preserves Intersecting Antichains

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Subject area: n-cube, antichain, intersecting antichain

Abstract

We prove that there is a 1-1 correspondence between the set of intersecting antichains in the lower-half of the n-cube and the set of intersecting antichains in the $(n-1)$-cube. This reduces the enumeration of intersecting antichains contained in the former set to that in the latter.

1. Introduction

Enumeration of intersecting antichains in the lower-half of the n-cube plays a central role in determining the number of so-called n-ary clique Boolean functions[?]. It is time consuming and is barely feasible already for $n=7$. In this note we present a bijection between the lower-half of the n-cube and the $(n-1)$-cube which preserves intersecting antichains and therefore reduces the enumeration of intersecting antichains in the former set to the latter set whereby the dimension of the space is reduced by 1. However, under current computer power the enumeration for $n=8$ seems still to be beyond the reach.

2. Definitions and Preliminaries

Let $E=\{0,1\}$ and n a positive integer. The cartesian power E^n is the *n-dimensional cube*. Let $\boldsymbol{a}=(a_1,\cdots,a_n)\in E^n$ and $\boldsymbol{b}=(b_1,\cdots,b_n)\in E^n$. We write $\boldsymbol{a}\preceq\boldsymbol{b}$ if $a_i\le b_i$ for all i, $1\le i\le n$. We call $A\subseteq E^n$ an *antichain* if $\boldsymbol{a}\prec\boldsymbol{a}'$ holds for no $\boldsymbol{a},\boldsymbol{a}'\in A$, i.e., no two elements from A are *comparable*; e.g., every A with $|A|\le 1$ is an antichain. Two vectors $\boldsymbol{a}=(a_1,\cdots,a_n)\in E^n$ and $\boldsymbol{b}=(b_1,\cdots,b_n)\in E^n$ are *intersecting* (at the i-th coordinate) provided $a_i=b_i=1$ for some $1\le i\le n$. A subset A of E^n is *intersecting* if all pairs $\boldsymbol{a},\boldsymbol{b}\in A$ are intersecting. Note that a singleton set $\{\boldsymbol{a}\}$ is intersecting unless $\boldsymbol{a}$ is the zero vector $\mathbf{o}=(0,\ldots,0)$. Also note that the empty set $\emptyset$ is an intersecting antichain.

For $\boldsymbol{a}=(a_1,\ldots,a_n)\in E^n$ set $w(\boldsymbol{a}):=a_1+\ldots+a_n$ (i.e., $w(\boldsymbol{a})$ is the number of 1s in $\boldsymbol{a}$). For $t=0,\ldots,n$ denote by $\mathcal{B}_t$ the t-th *layer* $w^{-1}(t)$ of the hypercube E^n.

Set $n':=\lfloor\frac{1}{2}n\rfloor$ and notice that $n'=\frac{1}{2}n$ if n is even and $n'=\frac{1}{2}(n-1)$ if n is odd. For even n the layer $\mathcal{B}_{n'}$ is called the *midlayer*. For even n and $i\in E$ set $C_i:=\{(a_1,\ldots,a_n)\in\mathcal{B}_{n'}:a_1=i\}$ and call C_1 and C_0 the *upper* and *lower halves* of the midlayer. For even n define the *upper* and *lower* *halves* of E^n as

$$L_n := \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{n'-1} \cup \mathcal{C}_0, \qquad U_n := \mathcal{C}_1 \cup \mathcal{B}_{n'+1} \cup \ldots \cup \mathcal{B}_n$$

while for odd n

$$L_n := \mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{n'}, \qquad U_n := \mathcal{B}_{n'+1} \cup \ldots \cup \mathcal{B}_n.$$

3. A map between L_n and E^{n-1}

As usual set $\bar{0} := 1$ and $\bar{1} := 0$. Define a map φ from L_n into E^{n-1} by setting

$$\varphi((a_1,\ldots,a_n)) = \begin{cases} (a_2,\ldots,a_n) & \text{if } a_1 = 0, \\ (\bar{a}_2,\ldots,\bar{a}_n) & \text{if } a_1 = 1. \end{cases} \tag{1}$$

Theorem 1. *The map φ is 1-1 and onto, and preserves intersecting antichains.*

Claim. The map φ is injective.

Proof of the claim. Let $\boldsymbol{a} = (a_1,\ldots,a_n) \in L_n$ and $\boldsymbol{b} = (b_1,\ldots,b_n) \in L_n$ satisfy $\varphi(\boldsymbol{a}) = \varphi(\boldsymbol{b})$. We show that $a_1 = b_1$. Suppose to the contrary that $a_1 \neq b_1$. We can choose the notation so that $a_1 = 0$ and $b_1 = 1$. Then $b_1 = \bar{a}_1$ and from $\varphi(\boldsymbol{a}) = \varphi(\boldsymbol{b})$ and from (1) we obtain $b_i = \bar{a}_i$ for all $i = 2,\ldots,n$. As $\boldsymbol{a},\boldsymbol{b} \in L_n$ clearly $w(\boldsymbol{a}), w(\boldsymbol{b}) \leq n'$ and

$$\begin{aligned} n' \geq w(\boldsymbol{b}) &= b_1 + \ldots + b_n = \bar{a}_1 + \ldots + \bar{a}_n \\ &= 1 - a_1 + \ldots + 1 - a_n \\ &= n - w(\boldsymbol{a}) \geq n - n'. \end{aligned} \tag{2}$$

Thus $n' \geq n - n'$ and hence n is even, $n = 2n'$ and we have equality in (2) proving $w(\boldsymbol{a}) = w(\boldsymbol{b}) = n'$. However, now $\boldsymbol{b} \in C_1$ in contradiction to $\boldsymbol{b} \in C_0$. Thus $a_1 = b_1$ and from $\varphi(\boldsymbol{a}) = \varphi(\boldsymbol{b})$ and (1) we obtain $\boldsymbol{a} = \boldsymbol{b}$ proving the claim. □

Proof. We show that φ induces a bijection from the set of intersecting antichains in L_n onto the set of intersecting antichains in E^{n-1}.

Let W be an intersecting antichain in L_n. For $i = 0, 1$ set $W_i = \{(a_1,\ldots,a_n) \in W : a_1 = i\}$. From the definition of φ clearly φ is an order preserving map on W_0 and so $\varphi(W_0)$ is an antichain. It is immediate that $\varphi(W_0)$ is intersecting. Similarly φ is an order reversing map on W_1 and so $\varphi(W_1)$ is also an antichain in E^{n-1}. To show that $\varphi(W_1)$ is intersecting let $(1,a_2,\ldots,a_n)$, $(1,b_2,\ldots,b_n) \in W_1$. We prove that $a_i = b_i = 0$ for some $1 < i \leq n$. Indeed, if not then $a_j + b_j \geq 1$ for all $j = 2,\ldots,n$ and $n + 1 \leq w(\boldsymbol{a}) + w(\boldsymbol{b})$. As $\boldsymbol{a},\boldsymbol{b} \in W_1 \subseteq L_n$ we get $w(\boldsymbol{a}), w(\boldsymbol{b}) \leq n'$ leading to the contradiction $n + 1 \leq w(\boldsymbol{a}) + w(\boldsymbol{b}) \leq 2n' \leq n$. Thus $a_i = b_i = 0$ for some $1 < i \leq n$, hence both $\varphi(a)$ and $\varphi(b)$ have the i-th coordinate $\bar{a}_i = \bar{b}_i = 1$ and $\varphi(W_1)$ is intersecting.

We show that the images of every $\boldsymbol{a} = (0,a_2,\ldots,a_n) \in W_0$ and of each $\boldsymbol{b} = (1,b_2,\ldots,b_n) \in W_1$ are incomparable. As $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the intersecting set W we have $a_i = b_i = 1$ for some $1 < i \leq n$ and therefore $a_i = 1 > 0 = \bar{b}_i$. Suppose to the contrary that $a_k \geq \bar{b}_k$ for all $k = 2,\ldots,n$. Then $a_k + b_k \geq 1$ for all k and $w(\boldsymbol{a}) + w(\boldsymbol{b}) \geq 1 + n - 1 = n$. From $\boldsymbol{a},\boldsymbol{b} \in L_n$ we see that $n' \geq w(\boldsymbol{a})$ and $n' \geq w(\boldsymbol{b})$. Consequently

$$2n' \geq w(\boldsymbol{a}) + w(\boldsymbol{b}) \geq n. \tag{3}$$

Thus n is even, we have equality in (3) and so $w(\boldsymbol{a}) = w(\boldsymbol{b}) = n'$. However, this leads to the contradiction $\boldsymbol{b} \in C_0 \cap W_1 = \emptyset$. Thus $\varphi(a)$ and $\varphi(b)$ are incomparable. Finally we show that the images of $\boldsymbol{a}$ and $\boldsymbol{b}$ are intersecting. As $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the antichain W, clearly $a_j = 1 > 0 = b_j$ for some $1 < j \leq n$. Now $\varphi(\boldsymbol{a}) \wedge \varphi(\boldsymbol{b}) \neq \mathbf{o}$ due to $a_j = 1 = \bar{b}_j$. We have proved that φ maps intersecting antichains in L_n into intersecting antichains in E^{n-1}.

To show that this map of antichain is surjective let W' be an intersecting antichain in E^{n-1}. Set

$$W_0 := \{(0,a_1,\ldots,a_{n-1}) : \boldsymbol{a} = (a_1,\ldots,a_{n-1}) \in W', \; w(a) \leq n'\}$$

$$W_1 := \{(1,\bar{a}_1,\ldots,\bar{a}_{n-1}) : \boldsymbol{a} = (a_1,\ldots,a_{n-1}) \in W', \; w(a) > n'\}.$$

Clearly $W_0 \subseteq L_n$. For $\boldsymbol{c} = (1,\bar{a}_1,\ldots,\bar{a}_n) \in W_1$ we have $w(\boldsymbol{c}) = 1 + n - 1 - w(\boldsymbol{a}) = n - w(\boldsymbol{a}) < n - n'$ and so $w(\boldsymbol{c}) \leq n'$ if n is odd and $w(\boldsymbol{c}) < n'$ if n is even. This shows that $\boldsymbol{c} \in L_n$ and $W_1 \subseteq L_n$. As above it can be verified that W_0 and W_1 are intersecting antchains. Consider $\boldsymbol{a} = (0,a_1,\ldots,a_{n-1}) \in W_0$ and $\boldsymbol{b} = (1,\bar{b}_1,\ldots,\bar{b}_{n-1}) \in W_1$. As $(a_1,\ldots,a_{n-1})$ and $(b_1,\ldots,b_{n-1})$ belong to the intersecting antichain W', clearly $a_i = b_i = 1$ and $a_j = 1$, $b_j = 0$ for some $1 \leq i,j \leq n-1$. Now $a_j = \bar{b}_j = 1$ and $a_i = 1$, $\bar{b}_i = 0$. Thus $W_0 \cup W_1$ is an intersecting antichain and $\varphi(W_0 \cup W_1) = W'$. In particular, for every $\boldsymbol{b} \in E^{n-1} \setminus \{\mathbf{o}\}$ the antichain $\{\boldsymbol{b}\} \in E^{n-1}$ is the image of some antichain $\{\boldsymbol{a}\} \in L_n$. Moreover, as $\varphi((0,\ldots,0)) = (0,\ldots,0)$ clearly φ is surjective. This concludes the proof. □

Example. Let $W = \{01001, 01010, 01100, 11000\}$ (where "abcde" stands for (a,b,c,d,e)). It can be checked that W is an intersecting antichain in L_5 whose image is the intersecting antichain in E^4: $\varphi(W) = \{1001, 1010, 1100, 0111\}$. □

To show an application first we state a result from [PMNR97].

Theorem 2. $\binom{n-1}{\lfloor\frac{n}{2}\rfloor-1}$ *is the maximum size of an intersecting antichain in* L_n.

As a corollary of the above two theorems we have the following.

Corollary 3. $\binom{n}{\lfloor\frac{n-1}{2}\rfloor}$ *is the maximum size of an intersecting antichain in* E^n.

Proof. Replace n by $n+1$ in the formula given in Theorem 2. □

The maximum size m of an antichain in L_n seems to be not yet considered in the literature. We show that m coincides with the maximum size of intersecting antichain in E^n.

Theorem 4. $\binom{n}{\lfloor\frac{n-1}{2}\rfloor}$ *is the maximum size of an antichain in* L_n.

Proof. From [Spe28] for n odd the largest antichain in E^n is of size $\binom{n}{n'}$. Thus let n be even and let A be an antichain in L_n of size m. Again from [Spe28] we obtain that $A \subseteq \mathcal{B}_{n'-1} \cup C_0$. We show that A coincides with the antichain $M = C_0 \cup \{(a_1,\dots,a_n) \in \mathcal{B}_{n'-1} \ : a_1 = 1\}$. For $i = 0, 1$ set

$$A_i \ := \ \{(a_1,\dots,a_n) \in A \cap \mathcal{B}_{n'-1} : \ a_1 = i\}.$$

Also set

$$D \ := \ \{\boldsymbol{d} \in C_0 : \ \boldsymbol{d} \succ \boldsymbol{a} \text{ for some } \boldsymbol{a} \in A_0\}$$

and

$$\alpha \ := \ |\{(\boldsymbol{a},\boldsymbol{d}) : \boldsymbol{a} \in A_0, \boldsymbol{d} \in D, \boldsymbol{a} \prec \boldsymbol{d}\}|.$$

For a fixed $\boldsymbol{a} = (0, a_2, \dots, a_n) \in A_0$ we have $w(\boldsymbol{a}) = n' - 1$ and there are exactly n' elements $\boldsymbol{d} \in D$ with $\boldsymbol{d} \succ \boldsymbol{a}$. Similarly for each fixed $\boldsymbol{d} = (0, d_2, \dots, d_n) \in D$ due to $w(\boldsymbol{d}) = n'$ there are exactly n' elements $\boldsymbol{b} \in \mathcal{B}_{n'-1}$ with $\boldsymbol{b} \prec \boldsymbol{d}$. Together $n'|A_0| \leq \alpha \leq n'|D|$ and so $|A_0| \leq |D|$. Now A being an antichain , D is disjoint from $A \cap C_0$, and $D \cup (A \cap C_0) \cup A_1$ is an antichain in L_n. Then $D \cup (A \cap C_0) \cup A_1$ is a subset of the antichain M and from maximality A coincides with M. Thus

$$\begin{aligned} m &= \binom{n-1}{n'} + \binom{n-1}{n'-2} = \binom{n-1}{n-1-n'} + \binom{n-1}{n'-2} \\ &= \binom{n-1}{n'-1} + \binom{n-1}{n'-2} = \binom{n}{n'-1}. \end{aligned}$$

□

Lastly we note that the numbers of antichains in L_n and E^{n-1} do not coincide (as the proof indicates, the condition of intersecting is necessary for 1-1 correspondence). This can be seen in Table 1 and Table 2, where we list the numbers of antichains in L_n and in E^n. The numbers of intersecting antichains in L_n (and hence in E_{n-1}) is given in [PMNR97] for up to $n = 7$.

References

[EKR61] Erdös P., Ko C., Rado R., Intersection theorems for systems of finite sets, Quart. J. Math. Oxford, vol.12, pp. 313-320, 1961.

[Fra88] Frankl P., Intersection and containment problems without size restrictions, *Algebraic, extremal and metric combinatorics 1986* (Deza M-M, Frankl P., Rosenberg I.G. eds), London Mathematical Society Lecture Notes Series 131, Cambridge University Press, pp. 62-111, 1988.

[PMNR97] Pogosyan G., Miyakawa M., Nozaki A. and Rosenberg I.G., On the number of clique Boolean functions, to appear.

[Spe28] Sperner, E., Ein Satz über Untermengen einer Endlichen Menge, *Math. Z*, **27**(1928) 544-548.

[StM88] Stojmenović I. and Miyakawa M., Application of a subset generating algorithm to base enumeration, knapsack and minimal covering problems, *The Computer Journal*, **31** (1988) 1, 65-70.

Table 1. Numbers $b(r,n)$ of size r antichains in L_n; the maximum size is $m = \binom{n}{\lfloor \frac{n-1}{2} \rfloor}$.

$r \backslash n$	1	2	3	4	5	6
1	1	2	4	8	16	32
2			3	15	85	375
3			1	11	235	2365
4				2	355	8895
5					338	21941
6					240	38065
7					125	48020
8					45	44470
9					10	30090
10					1	14646
11						5087
12						1275
13						235
14						30
15						2

Table 2. Numbers $a(r,n)$ of size r antichains in E_n; the maximum size is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.
The numbers given in total correspond to the Dedekind numbers of n-ary monotone functions.

$r \backslash n$	1	2	3	4	5	6
0	1	1	1	1	1	1
1	2	4	8	16	32	64
2		1	9	55	285	1351
3			2	64	1090	14000
4				25	2020	82115
5				6	2146	304752
6				1	1380	759457
7					490	1308270
8					115	1613250
9					20	1484230
10					2	1067771
11						635044
12						326990
13						147440
14						57675
15						19238
16						5325
17						1170
18						190
19						20
20						1
total	3	6	20	168	7581	7828354