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高アスペクト比ファラデー水面波におけるパターン選択

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§1. Introduction

The waves parametrically excited by the vertical oscillation of a container are called Faraday waves. Recently it has been shown in various experiments at high aspect ratio that the waves form various patterns, such as, stripes, squares, hexagons and triangles, 8-fold and 12-fold quasipatterns[1][2][3][4]. Quasipatterns are those with long-range orientational order but with no spatial periodicity.

In the theoretical studies of Faraday waves, the irrotational motion of an incompressible inviscid fluid is often assumed. Moreover, the damping effect is included by the calculation of the dissipation function for the irrotational fluid motion. Benjamin & Ursell showed that the linearized hydrodynamic equations for an inviscid fluid can be written as the Mathieu's equation and waves are subharmonically excited[5]. Milner[6] developed the method of Ezerskii *et al.*[7], and showed that a nonlinear damping called the cubic damping is important to form patterns, using a Lyapunov functional. Moreover, he concluded that the square pattern is the most favorite for capillary waves of infinite depth. This conclusion is consistent with the experimental result by Tufillaro *et al.*[8]. Miles[9] first noted that the cubic forcing, which is the nonlinear parametric forcing, is comparable with the cubic damping. He *a priori* assumed that there are only two modes and investigated the stability of the stripes and squares in capillary-gravity waves[10]. Decent & Craik[11] calculated the effect of 3/2 harmonic modes, which affect the cubic forcing and were neglected by Miles[9]. They also obtained coefficient of six-wave interactions and examined a hysteresis, although they considered only the waves in a long rectangular tank. But, in their study, some higher-order terms such as a steady mode and the harmonic mode proportional to the focing amplitude were neglected, although these terms are necessary in the calculation of the cubic forcing.

In this paper, the pattern selection caused by the first instability in Faraday surface waves at high aspect ratio is examined with including all the necessary higher terms. In §2, using the weakly nonlinear approximation, the coefficients of the cubic damping and cubic forcing between two line modes intersecting at an arbitrary angle are obtained for capillary-gravity waves of infinite depth. Next, using a reduction theory, the quintic amplitude equations are derived, which have a Lyapunov functional. The quintic terms are comparable to the cubic damping and forcing. The most favorite pattern, defined by the minimum of this Lyapunov functional, is examined in §3. This pattern changes from squares to hexagons, 8-fold quasipatterns, 12-fold quasipatterns, squares and stripes, as a parameter characterizing the ratio of the gravitational effect increases, or equivalently, as the wavelength of the Faraday waves increases.

§2. Derivation of the quintic amplitude equations

2.1 The basic equations

Weakly nonlinear surface waves on an incompressible inviscid fluid are considered, which are excited by the vertical external sinusoidal forcing which gives the system an acceleration $f_0 \cos(2\omega t)$, where t is the time. Let (x, y) and z be the horizontal and vertical reference coordinates fixed to the virtual container which is laterally unbounded and infinitely deep. The free surface displacement and the bottom are denoted by $z = \eta(x, y, t)$ and $z = -\infty$, respectively. The flow is assumed to be irrotational with a velocity potential $\phi(x, y, z, t)$. Therefore the basic equations are shown as follows. From Bernoulli's theorem, the dynamics condition at the free surface is

$$\partial_t \phi + \frac{1}{2}(\nabla\phi)^2 + \{f_0 \cos(2\omega t) + g\}\eta - \frac{\sigma}{\rho}\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + (\nabla\eta)^2}}\right) = 0 \quad \text{at} \quad z = \eta, \qquad (2.1.1)$$

where g is the gravitational acceleration, σ is the surface tension coefficient, ρ is the density of the fluid, and $\nabla = (\partial_x, \partial_y, \partial_z)$. The continuity equation is written as

$$\nabla^2 \phi = 0 \qquad -\infty < z \leq \eta. \qquad (2.1.2)$$

The kinematic boundary condition at the free surface is

$$\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi \quad \text{at} \quad z = \eta. \qquad (2.1.3)$$

And the boundary condition at the bottom is

$$\partial_z \phi = 0 \quad \text{as} \quad z \to -\infty. \qquad (2.1.4)$$

Linearized version of eqs. (2.1.1) $\sim$ (2.1.4) was studied by Benjamin & Ursell [5]. They showed that the linearized hydrodynamic equations for an inviscid fluid can be written for every wavenumber k as the Mathieu's equation

$$\partial_{tt}\eta + \omega_0^2 \left(1 + \frac{f_0 k}{\omega_0^2}\cos(2\omega t)\right)\eta = 0, \qquad (2.1.5)$$

where $\omega_0^2 = gk + \sigma k^3/\rho$. In the nearly inviscid theory, a linear damping with coefficient $\hat{\alpha}$ is added to eq. (2.1.5). Thus, the relevant equation now becomes[14]

$$\partial_{tt}\eta + \hat{\alpha}\partial_t\eta + \omega_0^2 \left(1 + \frac{f_0 k}{\omega_0^2}\cos(2\omega t)\right)\eta = 0. \qquad (2.1.6)$$

The strongest parametric resonance of eq. (2.1.6) occurs when $\omega_0 = \omega$[13]. Therefore, we may expect that the wave with frequency ω_0 is excited.

Now we expand eqs. (2.1.1) and (2.1.3) around $z = 0$. Moreover, we substitute

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \ \eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots, \ f_0 = \epsilon f, \ \partial_t \to \partial_t + \epsilon^2\partial_{\tau_1} + \epsilon^3\partial_{\tau_2} \qquad (2.1.7)$$

into them, where ϵ is an expansion parameter which implies the wave steepness. Also $\tau_1 = \epsilon^2 t$ and $\tau_2 = \epsilon^3 t$ are slow time variables. Equating the terms of $\mathcal{O}(\epsilon)$, we find

$$\begin{cases} \partial_t \phi_1 - \dfrac{\sigma}{\rho}\nabla^2 \eta_1 + g\eta_1 = 0, \\ \partial_t \eta_1 - \partial_z \phi_1 = 0. \end{cases} \tag{2.1.8}$$

Therefore, considering eqs.(2.1.2) and (2.1.4), we obtain the solutions

$$\begin{cases} \eta_1 = \sum_j \hat{a}_j(\tau_1, \tau_2) \exp \mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega t) + \text{c.c.}, \\ \phi_1 = -\dfrac{\mathrm{i}\omega}{k} \exp(kz) \sum_j \hat{a}_j(\tau_1, \tau_2) \exp \mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega t) + \text{c.c.}, \end{cases} \tag{2.1.9}$$

where the magnitude of the wavenumbervector $\boldsymbol{k}_j$ of each plane wave ($j = \pm 1, \pm 2, \cdots$) has the same value k determined from the linear dispersion relation $\omega^2 = kg + \sigma k^3/\rho$. Also $\boldsymbol{k}_{-j} = -\boldsymbol{k}_j$ is assumed and $\boldsymbol{x} = (x, y)$.

We can rewrite η as

$$\eta(x, y, t) = \sum_j \left(\hat{a}_j + \epsilon b_j + \epsilon^2 c_j\right) \exp \mathrm{i}\theta_j + \text{c.c.} + (\text{other terms}),$$

where $\theta_j = \boldsymbol{k}_j \cdot \boldsymbol{x} - \omega t$, and other terms contain the terms of $\mathcal{O}(\epsilon^3)$ and the terms proportional to $\exp \mathrm{i}l\theta_j$ ($l \neq 1$).

2.2 The third and fourth order balance equations including linear damping

Considering the solvability condition at each order of ϵ, we can obtain equations for $\hat{a}_j$, b_j and c_j. Combining these equations, we obtain the equations for $a_j = \hat{a}_j + \epsilon b_j + \epsilon^2 c_j$. Here we assumed that both the coefficient of linear damping α and forcing $\beta = \frac{k f_0}{4\omega}$ are $\mathcal{O}(\epsilon)$. Moreover, we assumed that $\mathrm{i}\beta a^*_{-j} + \alpha a_j = \mathcal{O}(\epsilon^3)$. The latter assumption is possible because the amplitude a_j of the excited waves is determined by the net effect of forcing and damping. The resultant third-order equations for a_j are given as

$$\partial_{\tau_1} a_j + \mathrm{i}\beta a^*_{-j} + \alpha a_j + \mathrm{i}\frac{\beta^2}{4\omega} a_j - \mathrm{i}k^2\omega \sum_l T^{(1)}_{lj} |a_l|^2 a_j - \mathrm{i}k^2\omega \sum_l T^{(2)}_{lj} a_l a_{-l} a^*_{-j} = 0, \tag{2.2.1}$$

where $T^{(i)}_{lj}$ ($i = 1, 2$) are given in Appendix as (A.1) and (A.2), which agrees the result of Milner[6]. Also, $T^{(i)}_{jj} = \frac{1}{2} \lim_{c \to 1} T^{(i)}_{lj}$, $T^{(1)}_{j-j} = \frac{1}{2} \lim_{c \to -1} T^{(1)}_{lj}$ ($i = 1, 2$). And the fourth-order equations are given as

$$\begin{aligned} &\partial_{\tau_2} a_j + \frac{\mathrm{i}\beta^2}{2\omega} a_j - \frac{\alpha\beta}{2\omega} a^*_{-j} \\ &\quad + \mathrm{i}\beta k^2 \sum_l P^{(1)}_{lj} a_l a_{-l} a_j - \mathrm{i}\beta k^2 \sum_l P^{(2)}_{lj} |a_l|^2 a^*_{-j} - \mathrm{i}\beta k^2 \sum_l P^{(3)}_{lj} a^*_l a^*_{-l} a_j = 0, \end{aligned} \tag{2.2.2}$$

where $P^{(i)}_{lj}$ ($i = 1, 2, 3$) is given in Appendix as (A.3), (A.4) and (A.5). Also, $P^{(i)}_{jj} = \frac{1}{2} \lim_{c \to 1} P^{(i)}_{lj}$, $P^{(2)}_{j-j} = \frac{1}{2} \lim_{c \to -1} P^{(2)}_{lj}$ ($i = 1, 2, 3$). The nonlinear terms in eqs. (2.2.1) and (2.2.2) imply the four-wave resonance, the cubic forcing and the nonlinear parametric forcing, respectively.

For convenience, let us analyze the equations which are yielded by summing eq. (2.2.1) and eq. (2.2.2), and by rewriting $\partial_{\tau_1} + \partial_{\tau_2}$ as $\frac{d}{d\tau}$.

2.3 Damping

In this subsection, we calculate the linear and cubic damping, following Milner[6] and Miles[9]. It is assumed that the system is laterally unbounded and of infinite depth, and that the fluid is uncontaminated. Under these conditions, damping may be caused only by the dissipation in the bulk [6][9][13]. Now the energy dissipation of an incompressible fluid is expressed as

$$\frac{d}{d\tau} < E_{\mathrm{mech}} >=< \mathcal{D} >, \tag{2.3.1}$$

where E_{mech} is the mechanical energy and $\mathcal{D}$ is the dissipation function. The operator $< \cdot >$ denotes the averaging over the time period $2\pi/\omega$.

Using the velocity potential ϕ, E_{mech} and $\mathcal{D}$ for this system are written as

$$\left\{\begin{array}{ll} E_{\mathrm{mech}} &= \dfrac{1}{2}\displaystyle\int d\boldsymbol{x}\int_{-\infty}^{\eta}(\nabla\phi)^2 dz + \int d\boldsymbol{x}\int_0^{\eta}(g+f_0\cos(2\omega t))z\,dz \\ & \quad +\dfrac{\sigma}{\rho}\displaystyle\int d\boldsymbol{x}\left[\sqrt{1+(\nabla\eta)^2}-1\right], \\ \mathcal{D} &= -2\nu\displaystyle\int_{-\infty}^{\eta} dz\int d\boldsymbol{x}\left(\sum_{j,l=1}^{3}\frac{\partial^2\phi}{\partial x_j\,\partial x_l}\right)^2, \end{array}\right. \tag{2.3.2}$$

where ν is the kinematic viscosity, and $(x_1, x_2, x_3) = (x, y, z)$.

Expanding eq. (2.3.2) around $z = 0$, and using (2.1.7), we obtain

$$\left\{\begin{array}{ll} < E_{\mathrm{mech}} > &= \epsilon^2\left(2\dfrac{\omega^2}{k}+\epsilon^2\dfrac{25}{128}\dfrac{kf^2}{\omega^2}\right)\displaystyle\sum_j |a_j|^2 + \epsilon^4 k\omega^2\sum_j\sum_i h_{ij}^{(1)}|a_j|^2|a_i|^2 \\ & \quad +\epsilon^4 k\omega^2\displaystyle\sum_j\sum_i h_{ij}^{(2)} a_i a_{-i} a_{-j}^* a_j^* + \epsilon^3\frac{f}{2}\sum_j (a_j^* a_{-j}^* + \text{c.c.}) + \cdots, \\ < \mathcal{D} > &= -\epsilon^2\left(8\nu k\omega^2 + \epsilon^2\dfrac{33}{32}\nu\dfrac{k^2 f}{\omega}\right)\displaystyle\sum_j |a_j|^2 - 2\epsilon^4\nu k^3\omega^2\sum_j\sum_i d_{ij}^{(1)}|a_j|^2|a_i|^2 \\ & \quad -2\epsilon^4\nu k^3\omega^2\displaystyle\sum_j\sum_i d_{ij}^{(2)} a_i a_{-i} a_{-j}^* a_j^* - 2\epsilon^3\nu k^2 f\sum_j (a_j^* a_{-j}^* + \text{c.c.}) + \cdots, \end{array}\right. \tag{2.3.3}$$

where $h_{lj}^{(i)}$ and $d_{lj}^{(i)}$ $(i = 1, 2)$ are given in Appendix as (A.6), (A.7), (A.8) and (A.9). Also, $h_{jj}^{(i)} = \dfrac{1}{2}\lim\limits_{c\to 1} h_{lj}^{(i)}$, $h_{j-j}^{(1)} = \dfrac{1}{2}\lim\limits_{c\to -1} h_{lj}^{(1)}$, $d_{jj}^{(i)} = \dfrac{1}{2}\lim\limits_{c\to 1} d_{lj}^{(i)}$, $d_{j-j}^{(1)} = \dfrac{1}{2}\lim\limits_{c\to -1} d_{lj}^{(1)}$ $(i = 1, 2)$.

One may then assume the following form as the damping terms in the amplitude equations,

$$\frac{da_j}{d\tau} = -\delta a_j - \gamma_f a_{-j}^* - \epsilon^2\nu k^4\sum_l \gamma_{lj}^{(1)}|a_l|^2 a_j - \epsilon^2\nu k^4\sum_l \gamma_{lj}^{(2)} a_l a_{-l} a_{-j}^* + \cdots, \tag{2.3.4}$$

where $\gamma_{lj}^{(i)}$ is the cubic damping. Substituting eqs. (2.3.3) and (2.3.4) into eq. (2.3.1) and comparing the coefficients, we obtain

$$\delta = \alpha\left\{1+\frac{1}{2}\left(\frac{\beta}{\omega}\right)^2\right\}, \quad \alpha = 2\nu k^2, \quad \gamma_{lj}^{(i)} = \frac{d_{lj}^{(i)}}{2} - 2h_{lj}^{(i)}\ (i=1,2), \quad \gamma_f = 0,$$

where $\gamma_{jj}^{(i)} = \frac{1}{2}\lim_{c\to 1}\gamma_{lj}^{(i)}$, $\gamma_{j-j}^{(1)} = \frac{1}{2}\lim_{c\to -1}\gamma_{lj}^{(1)}$ $(i=1,2)$. The relations among $\gamma_{lj}^{(i)}, d_{lj}^{(i)}$, and $h_{lj}^{(i)}$ are similar to those in Milner[6], but the values of $d_{lj}^{(i)}, h_{lj}^{(i)}$ are different from the values by him and Miles[9]. Moreover, our linear-damping coefficient δ includes the fourth-order term $\frac{\alpha}{2}(\frac{\beta}{\omega})^2$ caused by forcing, which was neglected by Milner and Miles.

2.4 The amplitude equations including linear and nonlinear damping

Appending the fourth-order damping into the equations yielded by combining eqs. (2.2.1) and (2.2.2), we obtain

$$\begin{aligned}
&\frac{da_j}{d\tau} + \mathrm{i}\beta a_{-j}^* + \delta a_j + \frac{3\mathrm{i}\beta^2}{4\omega}a_j - \frac{\alpha\beta}{2\omega}a_{-j}^* - \mathrm{i}k^2\omega\sum_l T_{lj}^{(1)}|a_l|^2 a_j - \mathrm{i}k^2\omega\sum_l T_{lj}^{(2)}a_l a_{-l}a_{-j}^* \\
&+ \frac{\alpha}{2}k^2\sum_l \gamma_{lj}^{(1)}|a_l|^2 a_j + \frac{\alpha}{2}k^2\sum_l \gamma_{lj}^{(2)}a_l a_{-l}a_{-j}^* \\
&+ \mathrm{i}\beta k^2\sum_l P_{lj}^{(1)}a_l a_{-l}a_j - \mathrm{i}\beta k^2\sum_l P_{lj}^{(2)}|a_l|^2 a_{-j}^* - \mathrm{i}\beta k^2\sum_l P_{lj}^{(3)}a_l^* a_{-l}^* a_j = 0.
\end{aligned} \tag{2.4.1}$$

2.5 Reduction

Following Milner[6] and Umeki[12], we reduce eq. (2.4.1) to simpler amplitude equations, which have a Lyapunov functional. Linearizing eq. (2.4.1), we find

$$\frac{da_j}{d\tau} + \delta a_j + \mathrm{i}\beta a_{-j}^* + \mathrm{i}\frac{3\beta^2}{4\omega}a_j - \frac{\alpha\beta}{2\omega}a_{-j}^* = 0. \tag{2.5.1}$$

Assuming that a_j evolves as $\exp(-\lambda\tau)$, $(\lambda \in \boldsymbol{C}^1)$, we analyze the linear stability of the null solution i.e. $(a_j, a_{-j}^*) = (0,0)$, in order to find the transformation of the variables (a_j, a_{-j}^*) into (A_j, B_j), where A_j is neutral or slightly unstable mode and B_j is stable mode. From eq. (2.5.1),

$$\begin{pmatrix} -\lambda+\delta+\mathrm{i}\frac{3\beta^2}{4\omega} & \mathrm{i}\beta-\frac{\alpha\beta}{2\omega} \\ -\mathrm{i}\beta-\frac{\alpha\beta}{2\omega} & -\lambda^*+\delta-\mathrm{i}\frac{3\beta^2}{4\omega} \end{pmatrix}\begin{pmatrix} a_j \\ a_{-j}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.5.2}$$

From the condition that the determinant of the matrix of eq. (2.5.2) must vanish, we obtain $\mathrm{Re}\lambda = \delta \pm \sqrt{\beta^2 - \Delta}$ where $\Delta = \left(\mathrm{Im}\lambda - \frac{3\beta^2}{4\omega} - \frac{\alpha\beta}{2\omega}\right)\left(\mathrm{Im}\lambda - \frac{3\beta^2}{4\omega} + \frac{\alpha\beta}{2\omega}\right)$. The effect of Δ is neglected, because we consider the standing waves. Therefore, we obtain the eigenvalues

$$\lambda_\pm = \delta \pm \beta + \mathrm{i}\left(\frac{3\beta^2}{4\omega} \pm \frac{\alpha\beta}{2\omega}\right). \tag{2.5.3}$$

After truncating higher-order terms, we obtain the eigenvectors as $(a_j, a_{-j}^*) = (1, -\mathrm{i})$ for the eigenvalue λ_+ of the stable mode, and $(a_j, a_{-j}^*) = (1, \mathrm{i})$ for λ_- of the neutral or slightly unstable mode for the forcing strength β just above the threshold δ. The neutral and stable modes may thus be written as

$$(a_j(\tau), a^*_{-j}(\tau)) = A_j(\tau)(1, \mathrm{i}) + B_j(\tau)(1, -\mathrm{i}), \tag{2.5.4}$$

where A_j and B_j are C^1-value functions. It may be expected that $A_j = \mathcal{O}(\epsilon)$ and $B_j = \mathcal{O}(\epsilon^2)$, because A_j and B_j are amplitudes of the slightly unstable mode and the stable mode, respectively. Therefore, transforming the variables (a_j, a^*_{-j}) into (A_j, B_j) and truncating the terms of $\mathcal{O}(\epsilon^5)$ in eq. (2.4.1) , we obtain

$$\begin{aligned}
\frac{dA_j}{d\tau} &- (\beta - \delta) A_j - \mathrm{i}\frac{3\beta^2}{4\omega} B_j - \mathrm{i}\frac{\alpha\beta}{2\omega} B_j \\
&- \mathrm{i}k^2\omega \sum_{l>0} \mathcal{T}^{(2)}_{lj} (B_l A^*_l - B^*_l A_l) A_j + \mathrm{i}k^2\omega \sum_{l>0} \mathcal{T}^{(2)}_{lj} |A_l|^2 B_j \\
&- \mathrm{i}k^2\omega \sum_{l>0} \mathcal{T}^{(1)}_{lj} |A_l|^2 B_j + k^2 \sum_{l>0} \left(\beta \mathcal{P}_{lj} + \frac{\alpha}{2} \Gamma_{lj} \right) |A_l|^2 A_j = 0, \qquad (2.5.5) \\
\frac{dB_j}{d\tau} &+ (\beta + \delta) B_j + \mathrm{i}\frac{3\beta^2}{4\omega} A_j - \mathrm{i}\frac{\alpha\beta}{2\omega} A_j - \mathrm{i}k^2\omega \sum_{l>0} \mathcal{T}_{lj} |A_l|^2 A_j = 0, \qquad (2.5.6)
\end{aligned}$$

where $\mathcal{P}_{lj} = P^{(1)}_{lj} + P^{(1)}_{-lj} + P^{(2)}_{lj} + P^{(2)}_{-lj} + P^{(3)}_{lj} + P^{(3)}_{-lj}$, $\Gamma_{lj} = \gamma^{(1)}_{lj} + \gamma^{(2)}_{-lj} + \gamma^{(2)}_{lj} + \gamma^{(2)}_{-lj}$, $\mathcal{T}^{(1)}_{lj} = T^{(1)}_{lj} + T^{(1)}_{-lj}$, $\mathcal{T}^{(2)}_{lj} = T^{(2)}_{lj} + T^{(2)}_{-lj}$, $\mathcal{T}_{lj} = \mathcal{T}^{(1)}_{lj} + \mathcal{T}^{(2)}_{lj}$, $\mathcal{T}_{jj} = \frac{1}{2} \lim_{c\to 1} \mathcal{T}_{lj}$, $\mathcal{P}_{jj} = \frac{1}{2} \lim_{c\to 1} \mathcal{P}_{lj}$, $\Gamma_{jj} = \frac{1}{2} \lim_{c\to 1} \Gamma_{lj}$. Γ_{lj}, $\mathcal{P}_{lj}$ and $\mathcal{T}_{lj}$ are functions of G and c, and are shown in Figs.1, 2 and 3. Here $G = kg/\omega^2$ expresses the ratio of the gravity effect for the gravity-capillary waves, implying capillary waves for $G = 0$ and gravity waves for $G = 1$. And $c = c_{lj} = \hat{\boldsymbol{k}}_l \cdot \hat{\boldsymbol{k}}_j$, where $\hat{\boldsymbol{k}}_j$ is the unit wavevector of the j th wave.

We neglect the time derivative of B_j in eq. (2.5.6), since it is of higher order than other terms. Therefore, we obtain

$$B_j = \frac{\mathrm{i}}{\alpha + \beta} \left(k^2\omega \sum_{l>0} \mathcal{T}_{lj} |A_l|^2 - \frac{3\beta^2}{4\omega} + \frac{\alpha\beta}{2\omega} \right) A_j, \tag{2.5.7}$$

after truncating the higher-order terms. Thus, $B_j = \mathcal{O}(\epsilon^2)$ and it is consistent with the assumption. Umeki[12] assumed that the forcing amplitude is $\mathcal{O}(\epsilon^2)$, and that the stable mode becomes $\mathcal{O}(\epsilon)$, which is inconsistent with his assumption, although this does not affect his results. Therefore, we cannot separate A_j and B_j when forcing amplitude is $\mathcal{O}(\epsilon^2)$. Substituting (2.5.7) into (2.5.5), and transforming $(kA_j, \frac{\alpha}{\omega}, \frac{\beta}{\omega}, \omega\tau)$ into dimensionless parameters $(A_j, \alpha, \beta, \tau)$, we obtain the simpler amplitude equations

$$\frac{dA_j}{d\tau} = \mu A_j - \sum_l \Phi_{lj} |A_l|^2 A_j - \sum_{n>0} \sum_{l>0} \Psi_{jln} |A_n|^2 |A_l|^2 A_j, \tag{2.5.8}$$

where $\mu = \beta - \alpha \left(1 + \frac{1}{2}\beta^2 \left\{1 - \frac{1}{8}\frac{(3\beta - 2\alpha)^2}{\alpha(\alpha+\beta)}\right\}\right)$, and

$$\Phi_{lj} = \frac{\alpha}{2}\Gamma_{lj} + \frac{\beta(3\beta - 2\alpha)}{2(\alpha + \beta)} \mathcal{T}_{lj} + \beta \mathcal{P}_{lj}, \quad \Psi_{jln} = \frac{1}{\alpha + \beta} \mathcal{T}_{nl} \mathcal{T}_{lj}. \tag{2.5.9}$$

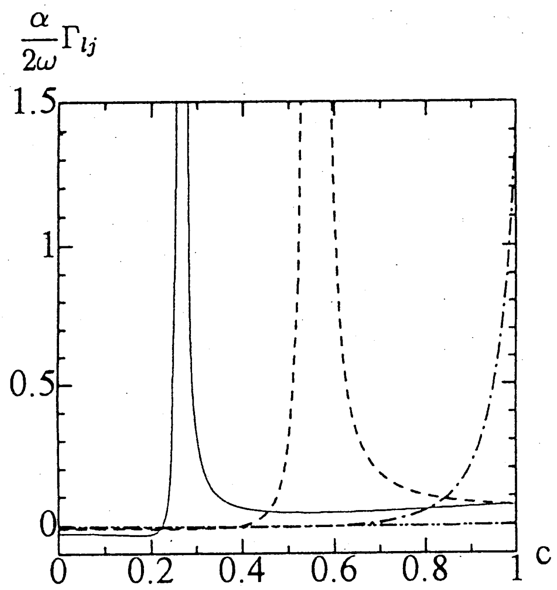

Fig.1 the coefficient of the cubic damping

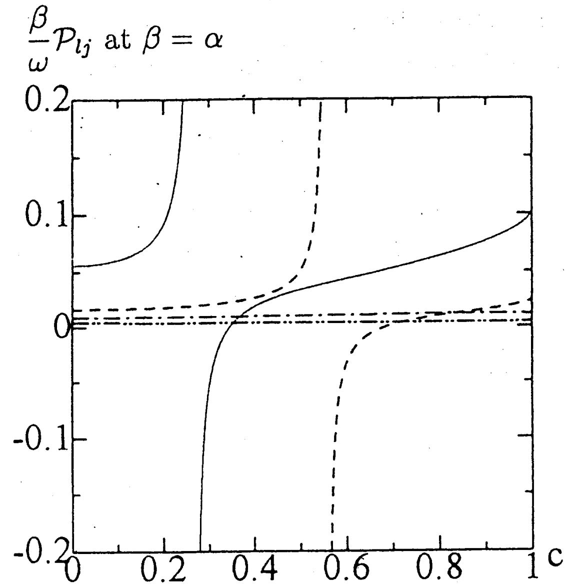

Fig.2 the coefficient of the cubic forcing

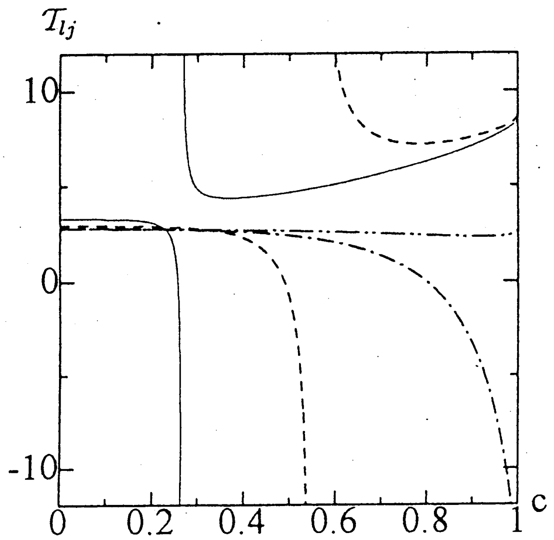

Fig.3 the coefficient of the four-wave interaction

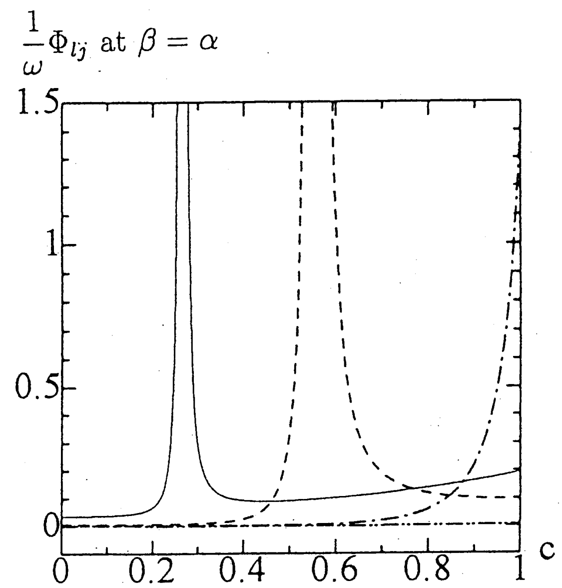

Fig.4 the coefficient of the cubic term

Solid curve, $G = 0.01$; dashed curve, $G = 0.4$; dot-dashed curve, $G = 0.7$; dot-dot-dashed curve $G = 0.9$.

Φ_{lj} is shown as Fig.4. We note that all the nonlinear terms in eq. (2.5.8) are of $\mathcal{O}(\epsilon^4)$ because $\Phi_{lj} = \mathcal{O}(\epsilon)$ and $\Psi_{jln} = \mathcal{O}(\epsilon^{-1})$. Now, rewriting $\frac{dA(\tau)}{d\tau}$ as $(\partial_{\tau_1} + \partial_{\tau_2})A(\tau_1, \tau_2)$, and assuming that $\mu = \mathcal{O}(\epsilon^3)$, we obtain the following balance equations in the third and fourth order:

$$\begin{cases} \partial_{\tau_1} A_j = 0 \\ \partial_{\tau_2} A_j = \mu A_j - \sum_l \Phi_{lj}|A_l|^2 A_j - \sum_{n>0}\sum_{l>0} \Psi_{jln}|A_n|^2|A_l|^2 A_j. \end{cases} \tag{2.5.10}$$

Therefore, $A_j(\tau_1, \tau_2) = A_j(\tau_2)$. We note that τ_1 and τ_2 imply the time scale of the phase evolution of A_j and the amplitude evolution of A_j, respectively. The merit of this reduction is to have a Lyapunov functional. On the other hand, the demerit is that the phase evolution becomes arbitrary. Rewriting τ_2 as τ, we obtain the following fourth-order quintic amplitude equations which can be expressed in a gradient form with the Lyapunov functional $\mathcal{F}$

$$\begin{cases} \dfrac{dA_j}{d\tau} = -\dfrac{\partial \mathcal{F}}{\partial A_j^*}, \\ \mathcal{F} = -\dfrac{1}{2}\mu \sum_{j>0} |A_j|^2 + \dfrac{1}{4}\sum_{j>0}\sum_{l>0} \Phi_{jl}|A_l|^2|A_j|^2 + \dfrac{1}{6}\sum_{j>0}\sum_{l>0}\sum_{n>0} \Psi_{jln}|A_n|^2|A_l|^2|A_j|^2. \end{cases} \tag{2.5.11}$$

§3. The most favorite pattern

In this section, we examine the most favorite pattern in capillary-gravity waves which minimizes the Lyapunov functional. We note that $\mathcal{F}$ decreases monotonically with τ in the wider sense. Therefore, it is assumed that the minimum of $\mathcal{F}$ is the most favorite[1]. If all the amplitudes are assumed to be equal, i.e. $A_j = A$ for $^\forall j \in \boldsymbol{N}$, we obtain from eq.(2.5.11)

$$\bar{\mathcal{F}} = -\frac{1}{2}\mu N|A|^2 + \frac{1}{4}N\bar{\Phi}|A|^4 + \frac{1}{6}N\bar{\Psi}|A|^6, \tag{3.1.1}$$

where $\bar{\Phi} = \sum_{l>0}^{N} \Phi_{lj}$ and $\bar{\Psi} = \sum_{n>0}^{N}\sum_{l>0}^{N} \Psi_{jln}$ are independent of j. N is the number of excited standing waves. The functional has the local minimum

$$\bar{\mathcal{F}}_{\text{local min}} = \frac{1}{4}N\bar{\Phi}\bar{\Psi}^{-1}\left(\mu + \frac{1}{6}\bar{\Phi}^2\bar{\Psi}^{-1}\right) - \frac{1}{6}N\bar{\Psi}^{-1}\sqrt{\bar{\Phi}^2 + 4\mu\bar{\Psi}}\left(\mu + \frac{1}{4}\bar{\Phi}^2\bar{\Psi}^{-1}\right), \tag{3.1.2}$$

for

$$|A|^2 = \frac{-\bar{\Phi} + \sqrt{\bar{\Phi}^2 + 4\mu\bar{\Psi}}}{2\bar{\Psi}}. \tag{3.1.3}$$

We consider only the case with $N = 1$ (stripes), 2 (squares), 3 (hexagons), 4 (8-fold quasipatterns), 6 (12-fold quasipatterns). We note that $\bar{\mathcal{F}}_{\text{local min}}$ is negative for these patterns. The most favorite pattern is examined for capillary-gravity water waves, using the values $\nu = 0.01\,\text{cm}^2/\text{s}$, $\rho = 1.00\,\text{g/cm}^3$, $\sigma = 73\,\text{dyn/cm}$, $g = 980\,\text{cm/s}^2$. The result is shown in Fig.5. Here β_c is the critical value which satisfies the condition $\mu = 0$. The most favorite pattern changes from squares to hexagons, 8-fold quasipatterns, 12-fold

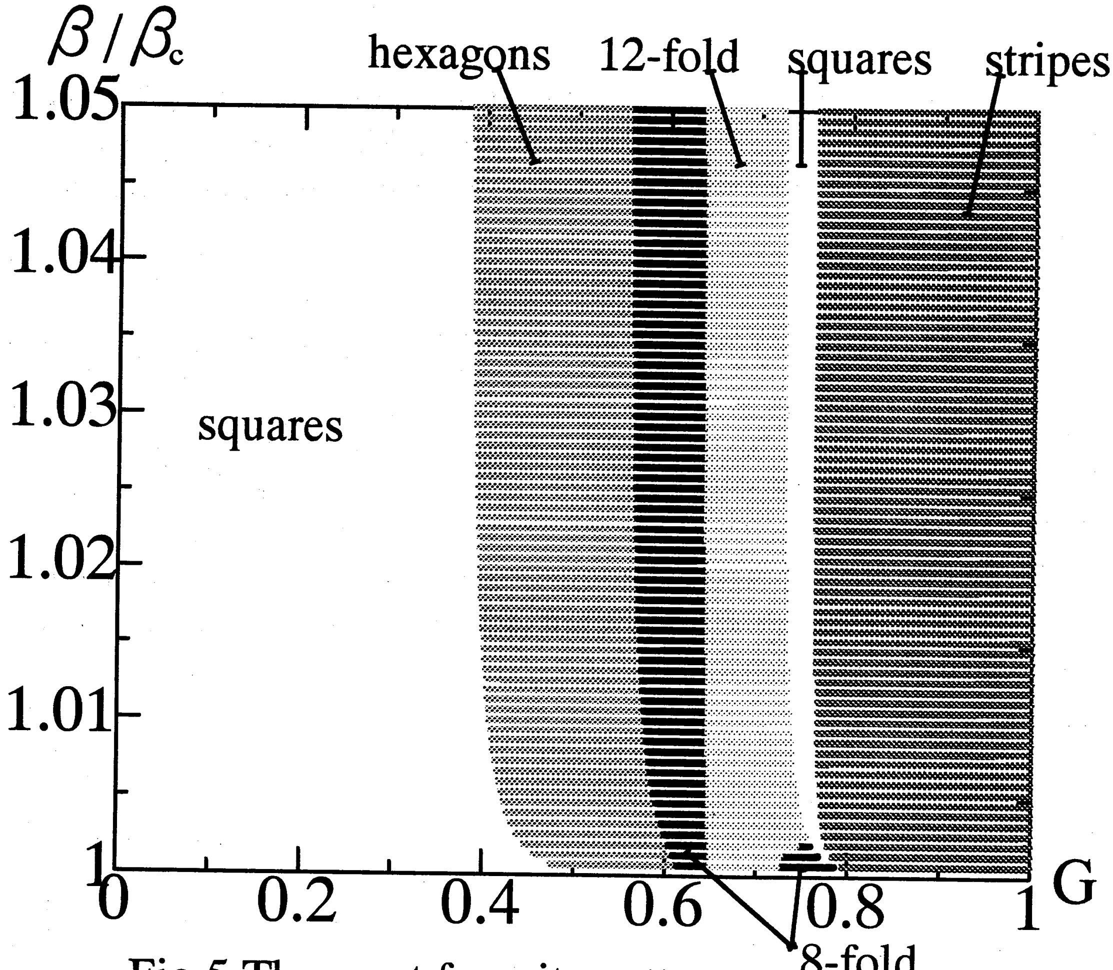

Fig.5 The most favorite pattern

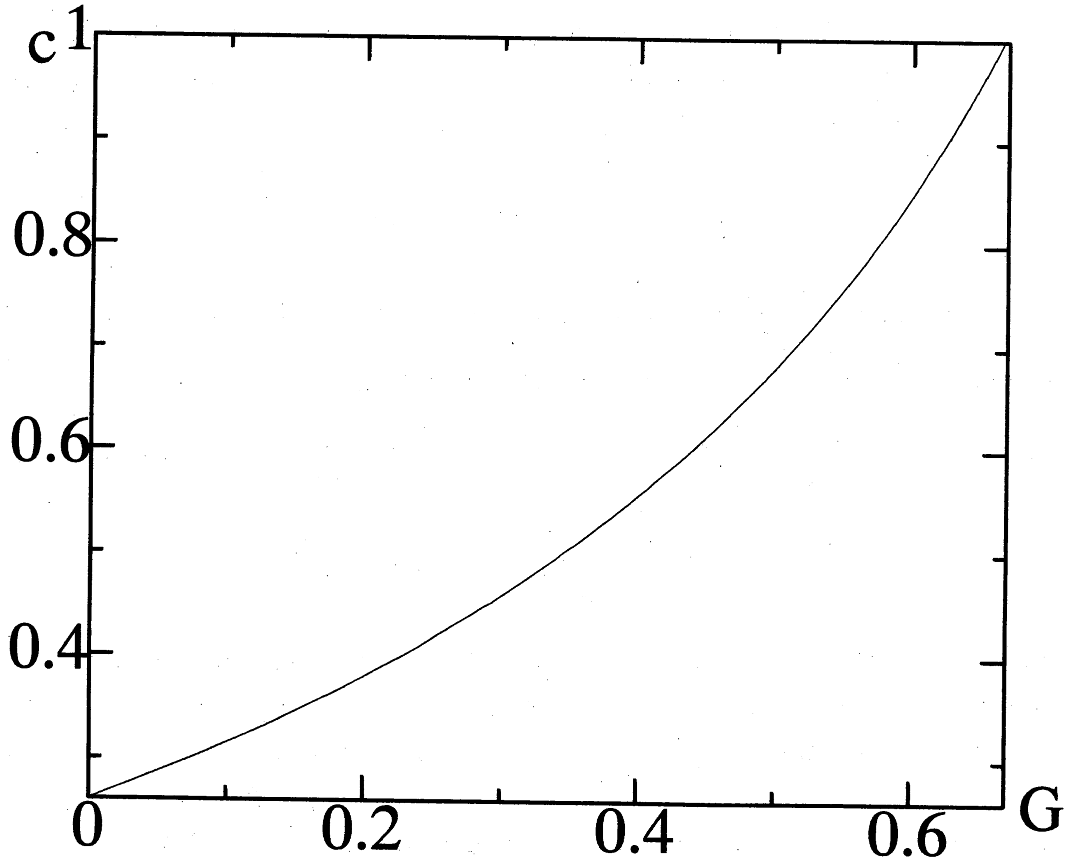

Fig6. The cosine of the angle where the second harmonic resonance occurs

quasipatterns, squares and stripes as G increases from 0 to 1. We note that the selected pattern does not change even for larger β/β_c.

The angles nearby the one which causes the second harmonic resonance are disliked because the Lyapunov functional comes up to 0 for this case. For G between 0 and 2/3, this resonance results in the transitions among patterns, as shown in Fig.6. When G exceeds 2/3, the second harmonic resonance does not occur and the Fig.1 and 4 become to have large flat region except for near $c = 1$. Finally, when G comes up to 1, the curves in Figs.1 $\sim$ 4 become flat. Therefore, stripes may be selected.

The stability of the most favorite pattern cannot be determined in this way. Therefore, the study of its stability is important. The results will be shown in the near future.

§4. Conclusions

The fourth-order quintic amplitude equations for the Faraday waves are derived, which have a Lyapunov functional. We show that the quintic terms in these equations are comparable to the cubic damping and forcing. The coefficients of the cubic damping and cubic forcing between two line modes intersecting at an arbitrary angle, and the coefficients of fourth-order linear damping caused by the forcing are obtained for capillary-gravity waves of infinite depth. Moreover, under the criterion of minimizing the Lyapunov functional, the most favorite pattern is obtained. The most favorite pattern changes from squares to hexagons, 8-fold quasipatterns, 12-fold quasipatterns, squares and stripes as G implying the ratio of the gravitational effect in capillary-gravity waves increases from 0 to 1.

Christiansen *et al.*[3] [4] investigated patterns of capillary Faraday waves in a circular cylinder. They observed transition from disordered stationary state to 8-fold quasipatterns, hexagons and squares in the central region of the container as the forcing amplitude increases. This result is different from ours. However, in contrast to our assumptions, the symmetry of the container may not be neglected in their experiment, because disordered stationary state was expressed as the superposition of a few Bessel's modes.

Secondary instabilities, such as Eckhaus or transverse amplitude instabilities, may occur if the nonlinear dispersion relation is considered. This is beyond the scope of this study.

Appendix

$$\begin{aligned} T_{lj}^{(1)} &= -\frac{(1+2\,c^2)\,G}{2} + 2\,(1+c)\,C_{lj} + c^2 - 3\,c - \frac{1}{2} \\ &\quad - C_{lj}\sqrt{2+2\,c} + (1-c)\,K_{lj} + \frac{(1-c)\,M_{lj}}{2}, \end{aligned} \tag{A.1}$$

$$T_{lj}^{(2)} = -\frac{(1+2\,c^2)\,G}{4} + \frac{c^2}{2} + \frac{5}{4} + \frac{(1+c)\,M_{-lj}}{4} + \frac{(1-c)\,M_{lj}}{4}, \tag{A.2}$$

$$\begin{aligned} P_{lj}^{(1)} &= \frac{S_{lj}}{16} + 2\,(1+c)\,Q_{lj} + 2\,(1-c)\,Q_{-lj} - \frac{(1-c)\,K_{lj}}{2} - \frac{(1+c)\,K_{-lj}}{2} \\ &\quad - \frac{(1+c)\,C_{lj}}{2} - \frac{C_{-lj}\,(1-c)}{2} - \frac{C_{lj}\sqrt{2+2\,c}}{2} \end{aligned}$$

$$
\begin{aligned}
& -\frac{C_{-lj}\sqrt{2-2c}}{2} + 2N_{lj}\sqrt{2+2c} + 2N_{-lj}\sqrt{2-2c} + 3, && \text{(A.3)}\\
P_{lj}^{(2)} &= \frac{1}{2}T_{lj}^{(1)} - \frac{(1+2c^2)G}{8} - 4(1-c)Q_{-lj} + 2R_{-lj}(1-c) + 4(1+c)V_{-lj} \\
& +\frac{(1+c)M_{-lj}}{2} + \frac{C_{-lj}(1-c)}{2} + \frac{(9-3c)K_{-lj}}{4} - 2V_{-lj}\sqrt{2+2c} \\
& +\frac{C_{-lj}\sqrt{2-2c}}{4} + 4N_{-lj}\sqrt{2-2c} + \frac{c^2}{4} - 2c + \frac{9}{8}, && \text{(A.4)}\\
P_{lj}^{(3)} &= \frac{1}{2}T_{lj}^{(2)} + 2(1-c)V_{lj} + 2(1+c)V_{-lj} - \frac{(1+2c^2)G}{16} - V_{lj}\sqrt{2-2c} \\
& -V_{-lj}\sqrt{2+2c} + \frac{(1+c)M_{-lj}}{4} + \frac{(1-c)M_{lj}}{4} \\
& +R_{lj}(1+c) + R_{-lj}(1-c) + \frac{c^2}{8} - \frac{7}{16}, && \text{(A.5)}\\
h_{lj}^{(1)} &= \frac{G\left(3+6c^2-(1-2c)M_{lj}{}^2 - 4(1+2c)K_{lj}{}^2\right)}{2} \\
& +2C_{lj}(C_{lj}-1)\sqrt{2+2c} - 4(1+c)C_{lj} - 4K_{lj} \\
& +4(1+c)K_{lj}{}^2 + (1-c)M_{lj}{}^2 - 3c^2 + 6c + \frac{5}{2}, && \text{(A.6)}\\
h_{lj}^{(2)} &= \frac{G\{3(1+2c^2) - (1-2c)M_{lj}{}^2 - (1+2c)M_{-lj}{}^2\}}{4} - \frac{7}{4} \\
& -\frac{3}{2}c^2 + \frac{(1-c)M_{lj}{}^2}{2} + \frac{(1+c)M_{-lj}{}^2}{2}, && \text{(A.7)}\\
d_{lj}^{(1)} &= 16(1+c)\sqrt{2+2c}C_{lj}{}^2 - 8C_{lj}(1+c)(3+c) \\
& -8C_{lj}(1+2c)\sqrt{2+2c} - 2(1-c)(1+c)M_{lj} \\
& -4\left(3+c^2\right)K_{lj} - 8(c^2-6c-3) + 4\left(1+2c^2\right)G, && \text{(A.8)}\\
d_{lj}^{(2)} &= 2\left(1+2c^2\right)G - (1+c)(1-c)(M_{lj}+M_{-lj}) - 10, && \text{(A.9)}
\end{aligned}
$$

where

$$
\begin{aligned}
C_{lj} &= \frac{2c^2+5c-1-G(1+c)(1+2c)}{(2+2c)^{3/2} - 4 - G(1+2c)\sqrt{2+2c}}, && \text{(A.10)}\\
K_{lj} &= \frac{(3-c)\sqrt{2+2c} - 4(1+c)}{2(2+2c)^{3/2} - 8 - 2G(1+2c)\sqrt{2+2c}}, && \text{(A.11)}\\
M_{lj} &= \frac{1-c}{2(1-c)-(1-2c)G}, && \text{(A.12)}\\
S_{lj} &= \left(1+2c^2\right)G - 2K_{lj}(1+3c) - 2K_{-lj}(1-3c) + 6C_{lj}\sqrt{2+2c} \\
& +6C_{-lj}\sqrt{2-2c} - 4(1+c)C_{lj} - 4C_{-lj}(1-c) - 9 - 2c^2, && \text{(A.13)}\\
N_{lj} &= -\frac{\sqrt{2+2c}}{8}, && \text{(A.14)}\\
V_{lj} &= \frac{-8M_{lj} - 6c^2 + 13c + 9(1-c) + 3(1-2c)(1-c)G}{32 - 8(2-2c)^{3/2} + 8(1-2c)\sqrt{2-2c}G}, && \text{(A.15)}\\
Q_{lj} &= \frac{-c+2K_{lj}}{8+8c-4(1+2c)G}, && \text{(A.16)}\\
R_{lj} &= \frac{(8M_{lj} - 15 - c)\sqrt{2-2c} + 12(1-c)}{64 - 16(2-2c)^{3/2} + 16(1-2c)\sqrt{2-2c}G}. && \text{(A.17)}
\end{aligned}
$$

Here $c = c_{ij} = \hat{\boldsymbol{k}}_j \cdot \hat{\boldsymbol{k}}_l,\ G = kg/\omega^2$.

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