数理解析研究所講究録
993 巻 1997 年 200-210



# 長波-短波相互作用方程式の振動孤立波解

阪大・基礎工　吉永隆夫 (Takao YOSHINAGA)

## §1. Introduction

In a previous paper,[1] we have shown that there exist a variety of solitary waves in the following resonant interaction equation between long and short waves:

$$\mathrm{i}\frac{\partial S}{\partial t}+\frac{\partial^2 S}{\partial x^2}=SL, \quad \frac{\partial L}{\partial t}+\alpha L\frac{\partial L}{\partial x}+\beta\frac{\partial^3 L}{\partial x^3}=\frac{\partial |S|^2}{\partial x}, \tag{1.1}$$

where $L$ and $S$ denote, respectively, the long wave and the complex amplitude of the short wave. The interaction can occur when the phase velocity of the long wave is nearly equal to the group velocity of the short wave. The parameters $\alpha$ and $\beta$ depend upon the individual properties of the waves and media concerned. For example, $\alpha, \beta \le 0$ corresponds to the capillary-gravity waves,[2,3] $\alpha \ge 0, \beta \le 0$ to the ion acoustic and electron plasma waves,[4,5] and so on.[6] In ref. 1, it is numerically shown for negative $\beta$ that eq. (1.1) has oscillatory solitary wave (solitary wave with oscillating tails that decay as $|x| \to \infty$) solutions in both long and short wave modes. The solutions have two types of wave profiles in each wave mode, that is, envelope shock and envelope soliton types in the short waves, while elevation and depression soliton types in the long waves.

Oscillatory solitary waves of a single mode were first examined numerically by Kawahara[7] in the generalized K-dV equation with a 5th order derivative term. Although this equation is known to describe long capillary-gravity waves on shallow water, recent numerical studies by Longuet-Higgins[8] and Vanden-Broeck and Dias[9] showed the existence of oscillatory solitary waves in more general case of capillary-gravity waves on deep water. Akylas[10] and Longuet-Higgins[11] showed that such waves are described by a steady envelope soliton solution of the Nonlinear Schrödinger (NLS) equation, in which the condition that the phase velocity of the crest is close to the group velocity of the oscillating tails is satisfied. However, this condition is not generally satisfied for the waves with small wave numbers on deep water, which means that 'long' oscillatory solitary waves do not exist on deep water. On the other hand, Dias and Iooss[12] analytically examined oscillatory wave profiles of the capillary-gravity solitary waves by using the procedure of the normal form analysis which was developed on the basis of the bifurcation theory by Iooss and his co-workers.[13,14] Furthermore, Grimshaw *et al.*[15] showed that the oscillatory solitary wave in the generalized K-dV equation is described by the steady envelope soliton solution of the higher order NLS equation, which coincides with the result obtained through the normal form analysis.

As is seen in the above, the 'long' oscillatory solitary waves do not propagate in the steady state on deep water as far as the single wave mode propagation is concerned. However, if the wave interaction occurs between long gravity and short capillary waves,[2,3] the 'long' oscillatory solitary waves can exist by virtue of the interaction with the short capillary waves even on deep water. In this paper, to analytically examine the solutions of such oscillatory solitary waves due to the above interaction, the normal form analysis is applied to the equation for the steady state which is reduced from eq. (1.1). In the next section, the dispersion relation of eq. (1.1) is examined to physically interpret the steady propagation of solitary waves. In § 3, the normal form analysis is carried out in our system for the steady state and analytical solutions are compared with numerical ones. And finally, in § 4, integrability of the interaction system is briefly discussed in the parameter region in which the solitary wave solutions exist.

## §2. Dispersion relation

Before proceeding to the analysis, it will be instructive to examine linear dispersion relations of eq. (1.1) for physical interpretation to the appearance of oscillatory solitary waves. Equation (1.1) has the following plane wave solution with constant amplitude $C$:

$$S = C\exp[\mathrm{i}(kx - \omega t)], \quad L = 0, \tag{2.1}$$

if the dispersion relation

$$\omega - k^2 = 0, \tag{2.2}$$

is satisfied between $k$ and $\omega$. Furthermore, superposing an infinitesimal sinusoidal disturbance proportional to $\exp[\mathrm{i}(Kx - \Omega t)]$ on the plane wave solution (2.1), another linear dispersion relation is obtained between $K$ and $\Omega$

$$\Omega^3 + (\beta K^3 - 4kK)\Omega^2 + [-K^4(1 + 4k\beta) + 4k^2K^2]\Omega + (-\beta K^7 + 4k^2\beta K^5 + 2C^2K^3) = 0. \tag{2.3}$$

When we assume real $K$ and complex $\Omega$ for $k = 0$ (so that, $\omega = 0$ from (2.2)) in eq. (2.3), it is found[16] that the plane wave is unstable for long wave modulations with small $|K|$. In addition to this, in a certain range of negative $\beta$, waves become unstable in an isolated region of $|K|$ with larger wave numbers.

Now, in eq. (2.3), we consider the other case that $k \neq 0$ and both $K$ and $\Omega$ are complex, though $\Omega/K$ is real. Introducing $\lambda = \Omega/K$, eq. (2.3) is replaced by

$$-\beta K^4 + [\beta(\lambda - 2k)^2 - \lambda]K^2 + 2C^2 + \lambda(\lambda - 2k)^2 = 0, \tag{2.4}$$

where we have excluded a trivial solution $K = 0$. Since $\lambda$ is the phase velocity of the modulational wave, while the group velocity of the plane wave is given as $\mathrm{d}\,\omega/\,\mathrm{d}\,k = 2k$ from the dispersion relation (2.2), $\lambda \simeq 2k$ is required for steady propagation of the modulational waves. As a result of this, setting $\lambda = 2k$ in eq. (2.4), it is easily found for $\beta < 0$ that the equation has two pairs of complex conjugate roots corresponding to oscillatory unstable state when $|\lambda| < \lambda_m$, where $\lambda_m = \sqrt{-8\beta C^2}$. On the other hand, the equation has real roots corresponding to stable state when $\lambda \geq \lambda_m$, while purely imaginary roots to exponentially unstable state when $\lambda \leq -\lambda_m$. The above results suggest, in the nonlinear stage, that oscillatory solitary waves emerge from non-oscillatory solitary waves when $\lambda$ ($< 0$) increases through $\lambda = -\lambda_m$, while they emerge from infinitesimal sinusoidal waves when $\lambda$ ($\geq 0$) decreases through $\lambda = \lambda_m$. This is also expected from the numerical results in ref. 1. In the followings, we are concerned with the case $\lambda > 0$, to which the analytical procedure is applicable.

## §3. Normal form analysis

### *3.1 Amplitude equations*

For the steady propagation of waves in eq. (1.1), we introduce the following traveling-wave transformation:

$$S = f(x - \lambda t)\exp[\mathrm{i}\, k(x - \frac{\omega}{k}t)], \quad L = g(x - \lambda t), \tag{3.1}$$

where $f$ and $g$ are assumed to be real functions. Note that both functions $f$ and $g$ correspond to the modulational waves, while the exponential function corresponds to the plane wave in the preceding section. Then, in (3.1), we can set $k = \lambda/2$ from the condition for the steady wave propagation and $\omega/k = k$ from (2.2). Thus, making use of (3.1) into eq. (1.1), the following reduced ordinary differential equations are obtained:

$$f'' = fg, \quad \beta g'' + \frac{\alpha}{2}g^2 - \lambda g = f^2 - C^2, \tag{3.2}$$

where $' \equiv \mathrm{d}/\mathrm{d}\zeta$ and $\zeta \equiv x - \lambda t$. On derivation of the above equations, we have imposed on $f$ and $g$ such boundary conditions that $|f| \to C$ (Const.) and $f', f'', g, g'g'' \to 0$ as $|\zeta| \to \infty$.

Carrying out the normal form analysis in our system (3.2), it is convenient to introduce the vector $\boldsymbol{u} = (\tilde{f}, F, g, G)^T$ in order to rewrite eq. (3.2) in the following form:

$$\boldsymbol{u}' = M(\mu)\boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}), \tag{3.3}$$

where $\tilde{f} = f - C$, $F = \tilde{f}'$ and $G = g'$, while the matrix $M$ and the nonlinear term $\boldsymbol{N}$ are given by

$$M(\mu) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{2C}{\beta} & 0 & \dfrac{\lambda_m + \mu}{\beta} & 0 \end{bmatrix}, \quad \boldsymbol{N}(\boldsymbol{u}) = \begin{bmatrix} 0 \\ \tilde{f}g \\ 0 \\ \dfrac{\tilde{f}^2}{\beta} - \dfrac{\alpha g^2}{2\beta} \end{bmatrix}.$$

Since the parameter $\mu = \lambda - \lambda_m$ denotes a deviation of $\lambda$ from the critical value $\lambda_m$ ($= \sqrt{-8\beta C^2}$, $\beta < 0$), $\mu = 0$ is the bifurcation point below which infinitesimal sinusoidal waves become oscillatory unstable ($\mu < 0$). For $\mu = 0$, $M(0)$ has a pair of eigenvalues $\sigma = \pm \mathrm{i} K_m$ (double and non-semi-simple), where $K_m = \sqrt{\lambda_m/(-2\beta)}$. Since for each eigenvalue two eigenvectors are required in order to complete the eigenspace, one is $\zeta_1$ defined as $(M(0) - \sigma \boldsymbol{I})\zeta_1 = 0$ and the other is the generalized eigenvector $\zeta_2$ as $(M(0) - \sigma \boldsymbol{I})\zeta_2 = \zeta_1$, where $\boldsymbol{I}$ is the unit matrix. In addition to this, it is convenient to introduce the adjoint eigenvectors $\zeta_2^*$ and $\zeta_1^*$ belonging to $\bar{\sigma}$ that denotes complex conjugate of $\sigma$, which are defined as $(M(0)^T + \bar{\sigma}\boldsymbol{I})\zeta_2^* = 0$ and $(M(0)^T + \bar{\sigma}\boldsymbol{I})\zeta_1^* = \zeta_2^*$, respectively. Thus, we find the following normalized eigenvectors for $\sigma = \mathrm{i} K_m$:

$$\zeta_1 = \frac{1}{2}\left[1, \mathrm{i} K_m, -\frac{K_m^2}{C}, -\frac{\mathrm{i} K_m^3}{C}\right]^T, \quad \zeta_2 = \frac{1}{2}\left[\frac{\mathrm{i}}{K_m}, 0, \frac{\mathrm{i} K_m}{C}, -\frac{\mathrm{i} K_m^3}{C}\right]^T,$$
$$\zeta_1^* = \frac{1}{2}\left[1, \frac{2\mathrm{i}}{K_m}, \frac{\beta K_m^2}{2C}, 0\right]^T, \quad \zeta_2^* = \frac{1}{2}\left[\mathrm{i} K_m, -1, \frac{\mathrm{i}\beta K_m^3}{2C}, \frac{\beta K_m^2}{2C}\right]^T. \tag{3.4}$$

We note that these eigenvectors satisfy the orthogonal conditions $< \zeta_i, \zeta_j^* >= \delta_{ij}$ $(i, j = 1, 2)$, while $< \zeta_i, \overline{\zeta_j^*} >=< \overline{\zeta_i}, \zeta_j^* >= 0$, where the inner product $< \zeta_i, \zeta_j >$ is defined as $\zeta_i{}^T \cdot \overline{\zeta_j^*}$.

Assuming weak nonlinearity with respect to $\boldsymbol{u}$ in the vicinity of the bifurcation point $\mu = 0$, we consider the following solution of eq. (3.3):

$$\boldsymbol{u}(\zeta) = A(\zeta)\zeta_1 + B(\zeta)\zeta_2 + \bar{A}(\zeta)\bar{\zeta}_1 + \bar{B}(\zeta)\bar{\zeta}_2 + \boldsymbol{\Phi}(\mu; A, B, \overline{A}, \overline{B}), \tag{3.5}$$

where the nonlinear function $\boldsymbol{\Phi}$ consists of $\mu$ and higher order terms of $A, B, \overline{A}$ and $\overline{B}$. Making use of (3.5) into eq. (3.3) and taking the inner products with $\zeta_1^*$ and $\zeta_2^*$, we obtain the following amplitude equations:

$$A' = \mathrm{i} K_m A + B + D(\mu; A, B, \overline{A}, \overline{B}), \quad B' = \mathrm{i} K_m B + E(\mu; A, B, \overline{A}, \overline{B}), \tag{3.6}$$

where

$$D =< M(0)\boldsymbol{\Phi} - \boldsymbol{\Phi}', \zeta_1^* > + < \boldsymbol{N}(\boldsymbol{u}), \zeta_1^* >, \quad E =< M(0)\boldsymbol{\Phi} - \boldsymbol{\Phi}', \zeta_2^* > . \tag{3.7}$$

According to the procedure of the normal form analysis,[12-15] the nonlinear terms $D$ and $E$ in eq. (3.6) should take the following forms in terms of the functions $P$ and $Q$:

$$D = \mathrm{i} A P(\mu; |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)), \tag{3.8a}$$
$$E = \mathrm{i} B P(\mu; |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)) + A Q(\mu; |A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)), \tag{3.8b}$$

Since the magnitude of $|\mu|$ is assumed to be of order $|A|^2$ or $|B|^2$ in this analysis, $P$ and $Q$ have the following forms to the leading order:

$$P = p_0\mu + p_1|A|^2 + \frac{\mathrm{i}}{2}p_2(A\bar{B} - \bar{A}B) + \cdots, \tag{3.9a}$$
$$Q = q_0\mu + q_1|A|^2 + \frac{\mathrm{i}}{2}q_2(A\bar{B} - \bar{A}B) + \cdots, \tag{3.9b}$$

where all the coefficients $p_0$ to $q_2$ are assumed to be real. We first calculate the coefficients $p_0$ and $q_0$. With the help of (3.8) and (3.9), we can show that the linearized equations with respect to $A$, $B$, $\bar{A}$ and $\bar{B}$ in eq. (3.6) have the eigenvalues $\pm\mathrm{i}\,K_m[1+p_0\mu/K_m \pm \sqrt{q_0\mu}/(\mathrm{i}\,K_m)]$. On the other hand, the eigenvalues of $M(\mu)$ in the original system (3.3) are given by $\pm\sqrt{\lambda_m/(2\beta)}\sqrt{1+(\mu\pm\sqrt{\mu^2+2\mu\lambda_m})/\lambda_m}$, which are expanded to be $\pm\mathrm{i}\,K_m[1+\mu/(4\lambda_m)\pm \mathrm{i}\sqrt{-2\mu\lambda_m}/(2\lambda_m)+\cdots]$ for small $|\mu|$. Comparison between these two eigenvalues leads to

$$p_0 = -\frac{1}{8\beta K_m}, \quad q_0 = \frac{1}{4\beta}. \tag{3.10}$$

Next, we calculate $\boldsymbol{\Phi}$ to obtain the coefficients $p_1$, $p_2$, $q_1$and $q_2$. Since nonlinear terms including $\mu$ are of higher order nonlinearity than $O(|A|^3, |B|^3)$, when $\boldsymbol{\Phi}$ is assumed up to cubic nonlinearity, it takes the following form with the coefficients $\boldsymbol{a}_0$ to $\boldsymbol{c}_7$:

$$\begin{aligned}\boldsymbol{\Phi} = &(\boldsymbol{a}_0A^2+C.C.)+\boldsymbol{a}_1|A|^2+(\boldsymbol{b}_0B^2+C.C.)+\boldsymbol{b}_1|B|^2+(\boldsymbol{c}_0AB+C.C.)+(\boldsymbol{d}_1\bar{A}B+C.C.)\\ &+(\boldsymbol{a}_2A^3+\boldsymbol{a}_3|A|^2A+\boldsymbol{b}_2B^3+\boldsymbol{b}_3|B|^2B+C.C.)+(\boldsymbol{c}_2A^2B+\boldsymbol{c}_3|A|^2B+\boldsymbol{c}_4A^2\bar{B}+C.C.)\\ &+(\boldsymbol{c}_5B^2A+\boldsymbol{c}_6A|B|^2+\boldsymbol{c}_7\bar{A}B^2+C.C.).\end{aligned} \tag{3.11}$$

In the above expression, the linear terms of $\mu$ have been excluded, since the coefficients $p_0$ and $q_0$ are given in (3.10). Making use of (3.11) into (3.7), while (3.9) into (3.8), we finally find the following coefficients by comparison between the expressions (3.7) and (3.8) (see Appendix):

$$p_1 = \frac{\sqrt{-2\beta}}{864\beta^3K_mC}(7\alpha^2+111\alpha\beta+630\beta^2), \tag{3.12a}$$

$$p_2 = \frac{-1}{216\beta^2C^2}(6\alpha^2+10\alpha\beta^2+95\alpha\beta+42\beta^3 \quad +165\beta^2), \tag{3.12b}$$

$$q_1 = \frac{\sqrt{-2\beta}}{72\beta^3C}(\alpha^2+21\alpha\beta+54\beta^2), \tag{3.12c}$$

$$q_2 = -\frac{\sqrt{-2\beta}}{432\beta^3K_mC}(5\alpha^2+141\alpha\beta+18\beta^2). \tag{3.12d}$$

Thus, the problem is reduced to solving the amplitude equations (3.6) through (3.8) and (3.9) by using (3.10) and (3.12).

### *3.2 Solitary wave solutions*

We first assume the following modulational wave solutions of eq. (3.6):

$$A = R(\zeta)\exp[\mathrm{i}(K_m\zeta+\phi)], \quad B = S(\zeta)\exp[\mathrm{i}(K_m\zeta+\psi)]. \tag{3.13}$$

Substituting (3.13) into eq. (3.6) with the help of (3.8), we obtain the following equations:

$$R' = S, \quad S' = RQ(\mu; R^2, 0), \quad \phi' = \psi' = P(\mu; R^2, 0), \tag{3.14}$$

where we have set $\frac{\mathrm{i}}{2}(A\bar{B}-B\bar{A}) = -RS\sin(\phi-\psi)$ to be zero for the solitary wave solutions. Since the representations (3.9) are written as $P = p_0\mu+p_1R^2+\cdots$ and $Q = q_0\mu+q_1R^2+\cdots$ in the use of (3.13), neglecting the higher order terms, eq. (3.14) are simplified to be

$$R'' = q_0\mu R + q_1 R^2, \quad S = R', \quad \phi' = \psi' = p_0\mu + p_1 R^2. \tag{3.15}$$

Consequently, solitary wave solutions of eq. (3.15) are given by

$$\begin{aligned} R &= \pm a\,\mathrm{sech}\,\gamma\zeta, && (3.16a)\\ S &= \mp a\gamma\,\mathrm{sech}\,\gamma\zeta\tanh\gamma\zeta, && (3.16b)\\ \phi = \psi &= p_0\mu\zeta + \frac{p_1 a^2}{\gamma}\tanh\gamma\zeta, && (3.16c)\end{aligned}$$

where $a = \sqrt{-2q_0\mu/q_1}$, $\gamma = \sqrt{q_0\mu}$ and $q_0, q_1 < 0$ for $\beta, \mu < 0$. Thus, making use of (3.16) into (3.5) with the help of (3.13), we have the final forms of the solitary wave solutions

$$\begin{aligned}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} C \\ 0 \end{bmatrix} &\pm \begin{bmatrix} 1 \\ -\dfrac{K_m^2}{C} \end{bmatrix} a\,\mathrm{sech}\,\gamma\zeta\cos(K_m\zeta) \\ &+ \begin{bmatrix} -\dfrac{1}{4C}(5+\dfrac{\alpha}{\beta}) - \dfrac{1}{36C}(\dfrac{\alpha}{\beta}-3)\cos(2K_m\zeta) \\ -\dfrac{\sqrt{-2\beta}}{2C\beta} - \dfrac{\sqrt{-2\beta}}{18\beta C}(\dfrac{2\alpha}{\beta}+3)\cos(2K_m\zeta) \end{bmatrix} a^2\,\mathrm{sech}^2\,\gamma\zeta \\ &\pm \begin{bmatrix} -\dfrac{p_1 a^2}{\gamma} + \dfrac{\gamma}{K_m} \\ \dfrac{K_m^2 p_1 a^2}{\gamma C} + \dfrac{K_m\gamma}{C} \end{bmatrix} a\,\mathrm{sech}\,\gamma\zeta\tanh\gamma\zeta\sin(K_m\zeta). \end{aligned} \tag{3.17}$$

It is noted that the second term of RHS in (3.17) is of $O(|\mu|^{1/2})$, while the third and fourth terms are of $O(|\mu|)$, since $a$ and $\gamma$ are of order $|\mu|^{1/2}$. In the followings, the above analytical solutions are compared with numerical ones which are directly obtained from eq. (3.2) by means of the shooting method used in ref. 1. We first adopt $-$ sign of $\pm$ signs in (3.17). In this case, the numerical solutions are found for $\alpha \leq 0$, which is corresponding to the capillary-gravity waves. For example, for $\alpha = -2$, $\beta = -0.5$ and $C = 1$ ($\lambda_m = 2$ and $K_m = \sqrt{2}$), Figs. 1 show the comparison between analytical (broken lines) and numerical (solid lines) wave profiles. In these figures, the short wave envelope $f$ is of dark soliton type, while the long wave $g$ of elevation soliton type. When we take $\lambda = 1.9$ ($\mu = -0.1$) close to the bifurcation point, Fig. 1(a) shows that the analytical results with small amplitude are in good agreement with the numerical ones except for the small discrepancy in oscillatory parts. However, when $\lambda = 1.6$ ($\mu = -0.4$) corresponding to further deviation from the bifurcation point, as is seen from Fig. 1(b) with larger amplitudes, discrepancy between both results becomes large with respect to the peak amplitudes as well as the oscillatory parts. On the other hand, when $+$ sign is adopted in (3.17), it seems to be difficult to find the corresponding numerical solutions for $\alpha \leq 0$. Instead of this, we can find such numerical solutions for large positive $\alpha$, though this parameter is the case for the ion acoustic and electron plasma waves. For example, when $\alpha = 12$ and $\beta = -0.5$ are taken, Figs. 2 show the comparison between analytical (broken lines) and numerical (solid lines) results where $f$ is of 'bright' soliton type, while $g$ of depression soliton type. In Fig. 2(a) for $\lambda = 1.9$, the analytical results (broken lines) with respect to the peak amplitudes are found to be in fairly good agreement with the numerical ones (solid lines), while some discrepancy between them is found in Fig. 2(b) for $\lambda = 1.6$. Furthermore, in both figures Figs. 2(a) and 2(b), the analytical results with respect to the oscillatory parts do not agree well with the numerical ones.

In ref. 1, numerical solutions of envelope shock type in $f$ are found. However, analytical solutions of this type could not be obtained in the procedure of the normal form analysis, since we consider the weakly nonlinear waves which bifurcate from linear modulational waves on $|f| = C, g = 0$.

## §4. Concluding remarks

In the preceding section, we have shown the analytical solutions when $q_0, q_1 < 0$ for $\beta, \mu < 0$. Since $q_0 < 0$ is always satisfied for $\beta < 0$, the solitary waves can exist for either $\alpha > -18\beta$ or $\alpha < -3\beta$ from the condition of $q_1 < 0$ in (3.12c). Thus, the solutions can always exist for $\alpha \leq 0, \beta < 0$ which corresponds to the capillary-gravity waves. On the other hand, integrability of the resonant system has been examined through the Painlevé test.[1,17] It is shown that eq. (1.1) does not pass the Painlevé PDE test except for $\alpha = \beta = 0$, while the reduced equations (3.2) does pass the Painlevé ODE test only for $\alpha + 6\beta = 0$ when $C \neq 0$. These situations are summarized in Fig. 3, where the hatched region in the $(\alpha, \beta)$ parameter space shows the existence region of the solitary waves, while the results of the PDE and ODE tests are, respectively, shown on the closed circle and on the solid line. Resulting from this, in the hatched region, our system is not integrable, at least, in the sense of Painlevé, which means that the oscillatory solitary waves will not have the soliton properties.

## Appendix:

The leading order in the representations of $D$ and $E$ is found to be $O(|A|^3, |B|^3)$ from (3.8) and (3.9), while it is $O(|A|^2, |B|^2)$ from (3.7) and (3.11). Therefore, setting all the coefficients of quadratic terms of $A$ and $B$ in (3.7) to be vanished, the following coefficients in (3.11) are obtained:

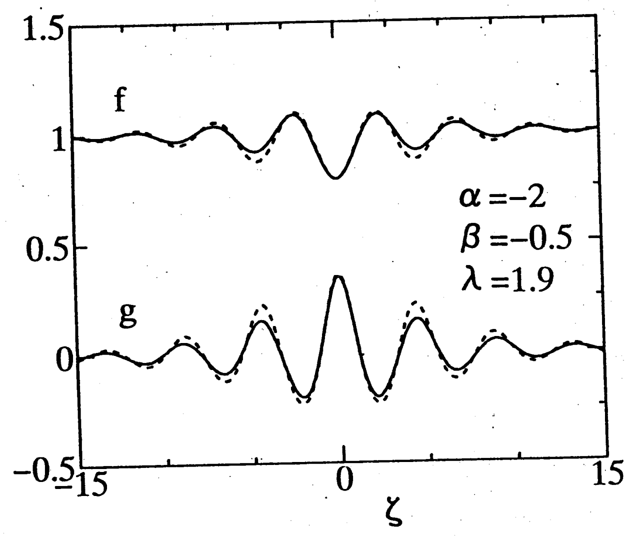
$$\boldsymbol{a}_0 = \begin{bmatrix} -\frac{1}{C}(\frac{\alpha}{72\beta} - \frac{1}{24}) \\ -\frac{\mathrm{i}K_m}{C}(\frac{\alpha}{36\beta} - \frac{1}{12}) \\ -\frac{\sqrt{-2\beta}}{\beta^2 C}(\frac{\alpha}{18} + \frac{\beta}{12}) \\ -\mathrm{i}K_m\frac{\sqrt{-2\beta}}{\beta^2 C}(\frac{\alpha}{9} + \frac{\beta}{6}) \end{bmatrix}, \quad \boldsymbol{a}_1 = \begin{bmatrix} -\frac{1}{C}(\frac{5}{4} + \frac{\alpha}{4\beta}) \\ 0 \\ -\frac{\sqrt{-2\beta}}{2C\beta} \\ 0 \end{bmatrix},$$

$$b_0 = \begin{bmatrix} -\frac{\sqrt{-2\beta}}{C^2}(\frac{5\alpha}{72}+\frac{7\beta}{24}) \\ \frac{\mathrm{i}}{CK_m}(\frac{5\alpha}{54\beta}+\frac{5}{9}) \\ \frac{1}{C^2}(\frac{13\alpha}{108\beta}+\frac{13}{18}) \\ \frac{\mathrm{i}\sqrt{-2\beta}}{\beta^2 CK_m}(\frac{\alpha}{6}+\beta) \end{bmatrix}, \quad b_1 = \begin{bmatrix} \frac{\sqrt{-2\beta}}{C^2}(\frac{3\alpha}{8\beta}+\frac{19}{8}) \\ 0 \\ -\frac{1}{C^2}(\frac{\alpha}{2\beta}+3) \\ 0 \end{bmatrix},$$

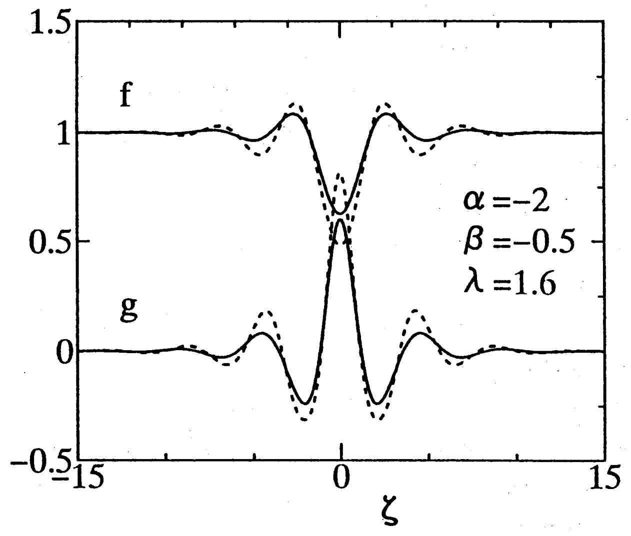
$$c_0 = \begin{bmatrix} -\frac{\mathrm{i}}{CK_m}(\frac{5\alpha}{108\beta}+\frac{1}{36}) \\ \frac{1}{C}(\frac{7\alpha}{108\beta}+\frac{5}{36}) \\ -\mathrm{i}\frac{\sqrt{-2\beta}}{\beta^2 CK_m}(\frac{2\alpha}{27}+\frac{4\beta}{9}) \\ \frac{\sqrt{-2\beta}}{\beta^2 C}(\frac{\alpha}{27}+\frac{13\beta}{18}) \end{bmatrix}, \quad d_1 = \begin{bmatrix} -\frac{\mathrm{i}}{CK_m}(-\frac{\alpha}{4\beta}+\frac{1}{4}) \\ -\frac{1}{C}(\frac{\alpha}{4\beta}+\frac{5}{4}) \\ 0 \\ -\frac{\sqrt{-2\beta}}{2\beta C} \end{bmatrix}.$$

Using the above coefficients, comparison between (3.7) and (3.8) with respect to the cubic nonlinear terms of $A$ and $B$ leads to the coefficients $p_1, p_2, q_1$ and $q_2$.


1) T. Yoshinaga and T. Kakutani: J. Phys. Soc. Jpn. **63** (1994) 445.
2) T. Kawahara, N. Sugimoto and T. Kakutani: J. Phys. Soc. Jpn. **39** (1975) 1379.
3) V. D. Djordjevic and L. G. Redekoop: J. Fluid Mech. **79** (1977) 703.
4) V. E. Zakharov: Sov. Phys. JETP **72** (1972) 908.
5) N. Nishikawa, H. Hojo, K. Mima and H. Ikeji: Phys. Rev. Lett. **33** (1974) 148.
6) see the references cited in ref. 1.
7) T. Kawahara: J. Phys. Soc. Jpn. **33** (1972) 260.
8) M. S. Longuet-Higgins: J. Fluid Mech. **200** (1989) 451.
9) J. -M. Vanden-Broeck and F. Dias: J. Fluid Mech. **240** (1992) 549.
10) T. R. Akylas, Phys. Fluids **A5** (1993) 789.
11) M. S. Longuet-Higgins: J. Fluid Mech. **252** (1993) 703.
12) F. Dias and G. Iooss: Physica **D 65** (1993) 399.
13) G. Iooss and Kirchigässner: C. R. Acad. Sci. Paris **311 I** (1990) 265.
14) G. Iooss and M. Adelmeyer, *Topics in Bifurcation Theory and Applications*, Advanced Series in Nonlinear Dynamics (World Science, Singapore, 1992) Vol. 3.
15) R. Grimshaw, B. Malomed and E. Benilov: Physica **D 77** (1994) 473.
16) T. Yoshinaga, M. Wakamiya and T. Kakutani: Phys. Fluids **A3** (1991) 83.
17) T. Yoshinaga: J. Phys. Soc. Jpn. **64**(1995) 3193.

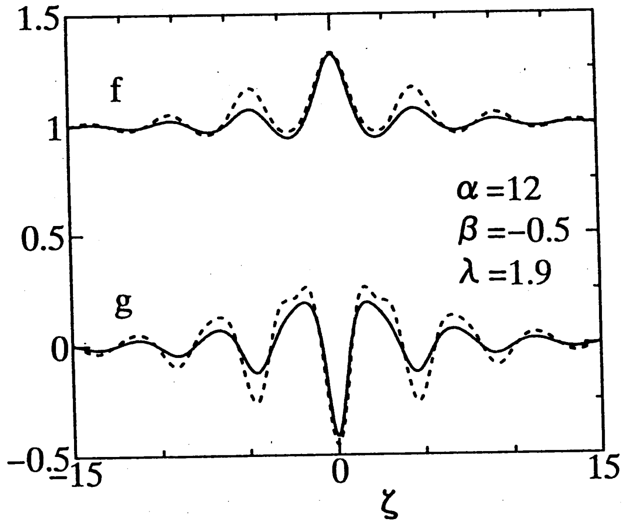



(a)

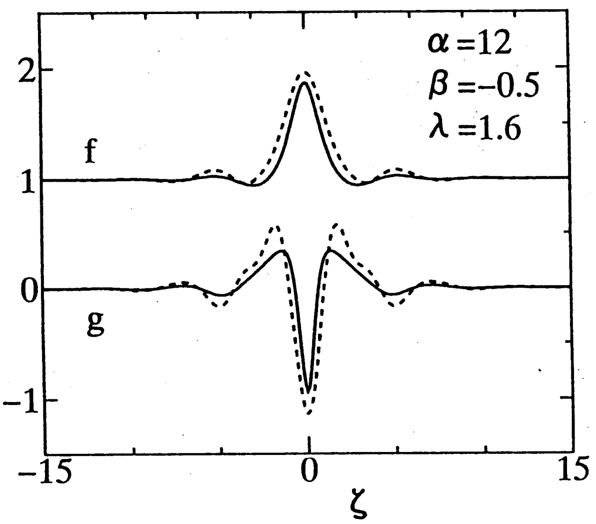



(b)

Fig. 1. Comparison between analytical (broken lines) and numerical (solid lines) profiles of the oscillatory solitary waves for (a) $\lambda = 1.9$ and (b) $\lambda = 1.6$, in which $\alpha = -2$, $\beta = -0.5$ and $C = 1$, and $-$ sign of $\pm$ signs in eq. (3.17) is adopted in the analytical solution.




(a)




(b)

Fig. 2. Comparison between analytical (broken lines) and numerical (solid lines) profiles of the oscillatory solitary waves for (a) $\lambda = 1.9$ and (b) $\lambda = 1.6$, in which $\alpha = 12$, $\beta = -0.5$ and $C = 1$, and + sign of $\pm$ signs in eq. (3.17) is adopted in the analytical solution.

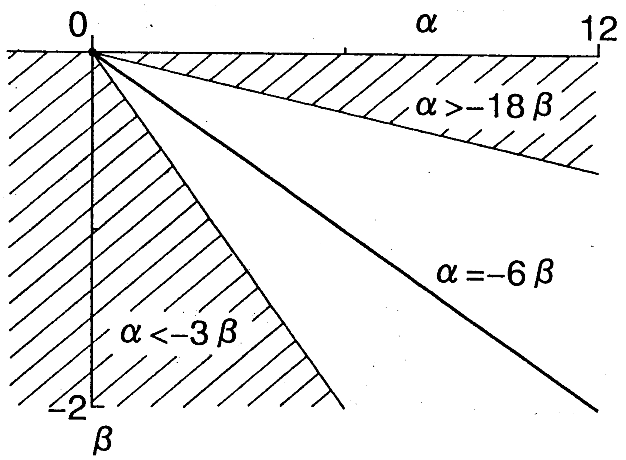



Fig. 3. Parameter region of $\alpha$ and $\beta$, where analytical solitary wave solutions (3.17) can exist in the hatched region, while eq. (1.1) passes the Painlevé PDE test on the closed circle (•) and eq. (3.2) passes the Painlevé ODE test on the solid line.