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Asymptotic Completeness for Hamiltonians with Time-dependent Electric Fields

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1 Introduction

We consider the following equation,

$$i\partial_t u(t,x) = H(t)u(t,x) \qquad \mathbf{H} = \mathbf{L}^2(\mathbf{R}^\nu), \tag{1}$$

$$H(t) = -\frac{1}{2}\Delta - E(t)\cdot x + V(x) \quad (\nu \geq 1)$$

with $E(t) = E + e(t), E$ being a nonzero constant vector in $\mathbf{R}^\nu$.

We assume $V(x)$ is real valued and short range (i.e. $V(x) = O(|x|^{-1/2-\epsilon})$ $|x| \to \infty$). As is well-known, with some suitable conditions on $V(x)$ and $E(t)$, $H(t)$ generates a unique unitary propagator $\{U(t,s)\}_{-\infty<t,s<\infty}$. We denote the unitary propagator generated by $H_0(t) = H(t) - V(x)$ as $\{U_0(t,s)\}$.

Studies for Schrödinger operators with electric fields have been done mainly for D.C. and A.C. Stark effects. Asymptotic completeness for A.C. Stark Hamiltonian, which is represented by $E(t)\cdot x = (\cos t)x_1$, was first proved by Howland and Yajima in [How] and [Ya]. In these papers they consider operators $K = -i\frac{d}{dt} + H(t)$ and $K_0 = K - V$ on $L^2(\mathbf{T}\times\mathbf{R}^\nu)$ and prove the asymptotic completeness by reducing it to that for K and K_0. These results were extended to the 3-body case by Nakamura [Na]. The asymptotic completeness of modified wave operator for long-range potential was proved by Kitada-Yajima [K-Y]. Recently asymptotic completeness for $E(t) = E + (\cos t)\mu$ by Møller [Mø] (μ is small enough compared with the main field E)

As for the case $E(t) = E$, the asymptotic completeness for long-range many-particle systems was proved by Adachi and Tamura in [AT1] [AT2]. In these papers they show the propagation estimates for the propagator by using the commutator technique of E.Mourre [Mo].

The aim of this paper is to accomodate the propagation estimates for the constant electric fields to the Schrödinger operator of the form (1) allowing $e(t)$ to be nonperiodic but small as $t \to \infty$. And with these results, we prove the existence and asymptotic completeness of wave operators.

We assume that $V(x) \in C^\infty(\mathbf{R}^\nu)$ and there exists $\delta_0 > 1/2$ such that

$$|\partial_x^\alpha V(x)| \le C_\alpha < x >^{-\delta_0 - |\alpha|} \quad {}^\forall\alpha \tag{2}$$

where $< \cdot > = (1 + | \cdot |^2)^{1/2}$.
In this paper, either of the following two assumptions are imposed on $V(x)$ and $e(t)$. The former requires that $V(x)$ is relatively small for $|E|$. And the latter requires $|e(t)| \to 0$ as $t \to \infty$.

Assumption 1 *We assume*

$$|E| > \sup_{x \in \mathbf{R}^\nu} \frac{E}{|E|} \cdot \nabla_x V(x). \tag{3}$$

There exist $c(t) \in C^2(\mathbf{R})$ and $\eta_0 > 0$ satisfying

$$|\dot{c}(t)| = O(t^{-\eta_0}) \quad t \to \infty, \tag{4}$$

$$\ddot{c}(t) = -e(t). \tag{5}$$

With this Assumption we write

$$b(t) = -\dot{c}(t), \tag{6}$$

$$a(t) = \frac{1}{2} \int_0^t |\dot{c}(\theta)|^2 d\theta. \tag{7}$$

Assumption 2 *$e(t)$ is a continuous integrable function on $\mathbf{R}_+$. Let $b(t)$ be defined by*

$$b(t) = -\int_t^\infty e(s) ds. \tag{8}$$

Then $b(t)$ satisfies

$$E \cdot b(t) \equiv 0 \quad t >> 1, \tag{9}$$

and there exists $u_0 > 5/2$ such that $|b(t)| = O(t^{-u_0})$

Under this Assumption we put

$$c(t) = \int_t^\infty b(s) ds, \quad a(t) = -\frac{1}{2} \int_t^\infty |b(s)|^2 ds. \tag{10}$$

On each of these Assumptions 1 or 2, $H(t)$ is essentially self-adjoint on $D(|x|) \cap H^2(\mathbf{R}^\nu)$. And we can construct unique unitary propagator satisfying the following properties (see [Ya2].)
For all $\quad t, t', s \in \mathbf{R}$,

$$U(t,t) = I, \quad U(t,s)U(s,t') = U(t,t'), \tag{11}$$

$$\frac{d}{dt}U(t,s) = -iH(t)U(t,s). \tag{12}$$

We also denote the unitary propagator associated with $H_0(t)$ as $U_0(t,s)$. Our main result is the following.

Theorem 3 *Suppose Assumption 1 or 2 holds. Then the following strong limit exist.*

$$W^+(s) = s - \lim_{t\to+\infty} U_0(t,s)^*U(t,s) \tag{13}$$

$$\tilde{W}^+(s) = s - \lim_{t\to+\infty} U(t,s)^*U_0(t,s) \tag{14}$$

Remark 4 *Theorem 3 holds as $t \to -\infty$, if we replace ∞ in Assumption 1 and 2 by $-\infty$.*

2 Translated Hamiltonians

At first we introduce a Hamiltonian $\hat{H}(t)$, which is obtained by translating $H(t)$. In this section, we give the propagation estimates for the propagator $\hat{U}(t,s)$ associated with $\hat{H}(t)$.

Definition 5

$$\hat{H}(t) = -\frac{1}{2}\Delta - E\cdot x + V(x - c(t)) + E\cdot c(t). \tag{15}$$

We also denote $\hat{H}(t) - V(x - c(t))$ as $\hat{H}_0(t)$.

We can also construct a unique unitary propagator $\hat{U}(t,s)$ and $\hat{U}_0(t,s)$, generated by $\hat{H}(t)$ and $\hat{H}_0(t)$. We remark that $U(t,s)$ and $\hat{U}(t,s)$ ($U_0(t,s)$ and $\hat{U}_0(t,s)$) are related through the following relation.

(Avron-Herbst formula)

$$U(t,s) = \tau(t)\hat{U}(t,s)\tau^*(s), \tag{16}$$

where

$$\tau(t) = \exp(ia(t))\exp(-ib(t)\cdot x)\exp(ic(t)\cdot p) \quad , \quad p = -i\nabla_x. \tag{17}$$

Theorem 6 *We assume Assumption 1. Then there exists $\sigma > 0$ such that for all $0 < u \le 2$ and $h \in C_0^\infty(\mathbf{R})$*

$$\|F(\frac{|x|}{t^2} \le \sigma)\hat{U}(t,s)h(\hat{H}(s)) < x >^{-u/2} \|_{B(\mathbf{H})} = O(t^{-L}) \quad (t\to\infty), \tag{18}$$

with $L = \min\{u, 3/2, 1+\eta_0\}$.

Theorem 7 *We assume Assumption 2. Then there exists* $\sigma > 0$ *such that for all* $0 < u \leq \min\{u_0/2, 3/2\}$ *and* $f \in C_0^\infty(\mathbf{R})$

$$\|F(\frac{|x|}{t^2} \leq \sigma)f(\hat{H}(t))\hat{U}(t,s)h(\hat{H}(s)) < x >^{-u/2} \| = O(t^{-L}) \quad (t \to \infty) \tag{19}$$

where $L = \min\{u_0, 3/2\}$.

Remark 8 *Theorem 3 is obtained if we show the existence of the strong limits of* $\hat{U}_0(t,s)^*\hat{U}(t,s)$ *and* $\hat{U}(t,s)^*\hat{U}_0(t,s)$. *We can prove them by using Cook's method and Theorem 6 (Theorem 7).*

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