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対称空間の円について

東條晃次 (KOJI TOJO) 千葉大学大学院研究生

(間下克哉氏 (Katsuya Mashimo) との共同研究)

Introduction. Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of (M, g). An arc-length parametrized curve $c(t)$ in (M, g) is called a *circle* if there exist a unit vector field $Y(t)$ along $c(t)$ and a positive constant k such that

$$\nabla_{\dot{c}(t)}\dot{c}(t) = kY(t), \quad \nabla_{\dot{c}(t)}Y(t) = -k\dot{c}(t).$$

The constant k is called the *curvature* of the circle $c(t)$.

In [AMU] Adachi, Maeda and Udagawa studied the global behaviour of circles in complex projective space $P^n(\mathbf{C})$ and proved that a circle in $P^n(\mathbf{C})$ is characterized by the curvature k and the sectional curvature of the 2-plane spanned by $\dot{c}(0)$ and $Y(0)$). Adachi [A] studied the similar problem for quaternion projective space (and its noncompact dual). It is also known that every circle in the complex projective space is obtained as an orbit of some one parameter subgroup of the full isometry group (see Maeda and Ohnita [MO]).

In this paper we consider similar problems as above for rank one symmetric spaces. Moreover we characterize homogeneous spaces whose circles are homogeneous.

1. Orbits of one parameter subgroups.

Let $(M, <,>) = G/K$ be a homogeneous Riemannian manifold such that K is a compact subgroup of a Lie group G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant decomposition of $\mathfrak{g}$ and identify $\mathfrak{p}$ with the tangent space $T_o(g/K)$ at the origin $o = \{K\}$. Let Λ be the connection function of $(G/K, \langle , \rangle)$ (cf. Nomizu [N]). Then for $x, y \in \mathfrak{p}$,

$$\Lambda(x)(y) = \frac{1}{2}[x, y]_{\mathfrak{p}} + U(x, y),$$

where $X_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of the vector $X \in \mathfrak{g}$ and

$$\langle U(x, y), z\rangle = \frac{1}{2}\{\langle [z, x]_{\mathfrak{p}}, y\rangle + \langle [z, y]_{\mathfrak{p}}, x\rangle\}, \quad z \in \mathfrak{p}.$$

Then we have the following.

Lemma 1. *Let X be in $\mathfrak{g}$ and $c(t) = \exp tX \cdot o$. Then for any $Y \in \mathfrak{p}$,*

$$\nabla_{\dot{c}(t)} d\exp tX \cdot (Y) = d\exp tX \cdot \{[X_{\mathfrak{k}}, Y] + \Lambda(X_{\mathfrak{p}})(Y)\}.$$

From Lemma 1 we can easily show the following.

Theorem 1. *Let $(G/K, \langle,\rangle)$ be a homogeneous Riemannian manifold and $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ be as above. Let $\{X, Y\}$ be a pair of mutually orthogonal unit vectors in $\mathfrak{p}$ and H be an element of $\mathfrak{k}$. Then $c(t) = \exp t(H + X) \cdot o$ is a circle in $(G/K, \langle,\rangle)$ of curvature k (> 0) with the initial condition*

$$\dot{c}(0) = X, \ \nabla_{\dot{c}(0)} \dot{c}(t) = kY$$

if and only if the following holds:

$$[H, X] + \Lambda(X)(X) = kY,$$
$$[H, Y] + \Lambda(X)(Y) = -kX.$$

In particular, for the case when (G, K) is a Riemannian symmetric pair, then $c(t)$ is a circle with the same initial condition as above if and only if the following holds:

$$[H, X] = kY, \ \ [H, Y] = -kX.$$

2. Results.

Theorem 2. *Let (M, g) be a homogeneous Riemannian manifold. Then all the circles in (M, g) are orbits of one parameter subgroups in the isometry group of (M, g) if and only if (M, g) is a two point homogeneous space.*

At first we show the 'only if' part of the theorem.

Suppose that all the circles in $(G/K, \langle,\rangle)$ are orbits of one parameter subgroups in G and K is compact. Let $\{X, Y\}$ be an arbitrary pair of mutually orthogonal unit vectors in $\mathfrak{p}$. Take two circles c_1 and c_2 with initial datas $\{X, Y\}$ and $\{X, -Y\}$ respectively for some $k > 0$ (cf. Theorem 1). Then by Theorem 1, there exist elements H_1 and H_2 in $\mathfrak{k}$ such that

$$[H_1, X] + \Lambda(X)(X) = kY, \ \ [H_2, X] + \Lambda(X)(X) = -kY.$$

Therefore we get

$$[H, X] = 2kY \quad (H = H_1 - H_2).$$

Then by the implicit function theorem we can see that $\mathrm{Ad}(K) \cdot X$ is open in the uint hypersphere S of $\mathfrak{p}$. Moreover $\mathrm{Ad}(K)$ is closed in S because K is compact. Therefore $\mathrm{Ad}(K) \cdot X$ coincides with S, so G/K is two point homogeneous.

Now we prove the 'if' part. It is known that a space of constant sectional curvature and a complex projective space (and a complex hyperbolic space) have the property stated in the theorem. Thus we may assume that (M, g) is either a quaternion projective space $P^n(\mathbf{Q})$ or a Cayley projective plane $P^2(\mathfrak{C})$. Let G be the isometry group of (M, g) and K the isotropy subgroup of G at a point of M.

Lemma 2. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition and $\{X, Y\}$ a pair of mutually orthogonal unit vectors in $\mathfrak{p}$. Then there exists a 4-dimensional Lie triple system V of $\mathfrak{p}$ corresponding to $P^2(\mathbf{C})$ such that X, $Y \in V$.*

Proof of the lemma.

Case $P^n(\mathbf{Q})$.

Put

$$G = Sp(n+1) = \{g \in M_{n+1}(\mathbf{Q}) : gg^* = I\},$$
$$K = Sp(1) \times Sp(n) = \{(q, g) : q \in Sp(1),\ g \in Sp(n)\},$$

We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. We denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Under the canonical identification

$$\mathfrak{p} \to \mathbf{Q}^n; \begin{bmatrix} 0 & -\overline{q}_1 & \cdots & -\overline{q}_n \\ q_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & \cdots & 0 \end{bmatrix} \mapsto [q_1, \cdots, q_n]$$

the adjoint representation K on $\mathfrak{m}$ is as follows;

$$Ad((\alpha, g))[\cdots, q_j, \cdots] = [\cdots, \sum_{i=1}^{n} g_{ji} q_i \bar{\alpha}, \cdots],$$
$$\alpha \in Sp(1), g = [g_{ij}] \in Sp(n), q_i \in \mathbf{Q}.$$

Since $Sp(n)$ acts transitively on the unit sphere in $\mathbf{Q}^n$, we may assume that $X = [1, 0, \cdots, 0]$. Similarly by the action of

$$Sp(n-1) = \{g \in Sp(n) : gX = X\}$$

on $\mathbf{Q}^n$ we may assume $Y = [u, b, 0, \cdots, 0]$ where $u + \overline{u} = 0$ and $b \in \mathbf{R}$. For an element $q \in Sp(1)$ we put

$$k = \begin{bmatrix} q & & & \\ & q & & \\ & & \ddots & \\ & & & q \end{bmatrix} \in K.$$

It is easily seen that $Ad(k)(X) = X$ and $Ad(k)(Y) = [qu\overline{q}, b, 0, \cdots, 0]$. The group $Sp(1)$ acts transitively on the sphere in $\mathbf{R}i + \mathbf{R}j + \mathbf{R}k$ by $x \to qx\overline{q}$, we may assume that

$$X = [1, 0, \cdots, 0],\ Y = [ai, b, 0, \cdots, 0], \quad a, b \in \mathbf{R}, a^2 + b^2 = 1.$$

Put

$$V = \{[z_1, z_2, 0, \cdots, 0]; z_i \in \mathbb{C}\}.$$

Then it is easy to see that V is a Lie triple system corresponding to $P^2(\mathbb{C})$.

Remark. Set

$$H = k \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & ia & -b & \cdots & 0 \\ 0 & b & -ia & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $\{H, X, Y\}$ satisfy the equations in Theorem 1, so $\exp t(H + X) \cdot o$ is a circle of curvature k with initial condition

$$c'(0) = X, \qquad \nabla_{c'} c'(0) = kY.$$

Case $P^2(\mathfrak{C})$.

In this case $G = F_4$ and $K = Spin(9)$. Let X be a unit vector of $\mathfrak{p}$. Then K acts transitively on the unit hypersphere S^{15} of $\mathfrak{P}$ and the isotropy subgroup of K at X is isomorphic to $Spin(7)$. Then we can see that under the isotropy representation of $Spin(7)$ the tangent space $T_X S^{15}$ is decopmosed into two irreducible components;

$$T_X S^{15} = W_1 \oplus W_2, \quad (\dim W_1 = 8, \quad \dim W_2 = 7).$$

Let Y_i be unit vectors of W_i $(i = 1, 2)$. Then $Spin(7)$ acts transitively on the unit hypersphere S_1 of W_1 and the isotropy subgroup at Y_1 is isomorphic to G_2 $(S_1 = Spin(7)/G_2)$. Moreover G_2 acts transitively on the unit hypersphere S_2 of W_2 $(S_2 = G_2/SU(3))$. In this case there exists a Lie triple system V corresponding to $P^2(\mathbb{C})$ such that V contains X and Y_i, so the lemma is proved.

From Lemma 2 and Theorem 1, any circle in (M, g) is expressed as an orbit of one parameter subgroup of G.

Also, Lemma 2 and Theorem 1 implies the following (cf. [AMU]).

Corollary. *Let (M, g) be one of the complex projective space, the quaternion projective space or the Cayley projective plane. Then there exists a totally geodesically embedded complex projective plane $P^2(\mathbb{C})$ such that for any circle c of (M, g) we can take an isometry ϕ of (M, g) so that $\phi \circ c$ is contained in $P^2(\mathbb{C})$. (Similar result holds for the noncompact dual of M.)*

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