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PRELIMINARIES FOR THE THEORY OF PREHOMOGENEOUS VECTOR SPACES

AKIHIKO GYOJA

Department of Fundamental Sciences, Faculty of Integrated Human Studies, Kyoto University, Kyoto 606-01, Japan

Recently the theory of prehomogeneous vector spaces depends on more and more various fields including (1) D-modules and their microlocal analysis, (2) algebraic geometry, especially geometric invariant theory and etale cohomologies. Sometimes to learn these materials needs a lot of effort, especially when the material is assumed to be well known among experts, without references accessible for non-experts. The subsequent notes are included here with the hope that they might be of some use when the situation is such. These notes are written on various occasions; some are for my personal use. Let me explain about each note.

1. On affine open subsets. Here a proof of the following fact is given in a most general setting. *A Zariski open subset U of an affine space $\mathbf{A}^n$ is an affine variety if and only if it can be expressed as $U = \mathbf{A}^n \setminus f^{-1}(0)$ with some polynomial function f.*

2. On Matsushima's theorem. Various proof of this theorem can be found in the references cited at the end of this note. Unfortunately the most transparent one is only alluded in the introduction of Richardson [Ri] as a quote from a letter from A.Borel : "$\cdots$ it is clear that the proof[1] given in my joint paper with Harish-Chandra goes over verbatim in arbitrary characteristic, using étale cohomology. $\cdots$" The purpose of this note is to afford the detail.

3. On Radon transformation. In [DG, 3.3.10], we have indicated how our Radon transformation given in (3.3.9) in loc. cit. relates to the one given in [Br, 9.13]. Here I record the detail of the proof.

4. On non-characteristic pull-back of D-modules. Without doubt, the non-characteristic pull-back for D-modules is well understood. But I could not find a reference except for [SKK, pp.406–418], where only a version for microdifferential equations is given, the zero section of the cotangent bundle being excluded from the consideration. Here I give a D-module version with a detailed proof.

[1]See 3.6 of A.Borel, Harish-Chandra, Arithmetic subgroups of algebraic groups, Ann. of Math. 75 (1962), 485 535.

5. On D-modules associated to a complex power of a function. In (5.4) of "A.Gyoja, Theory of prehomogeneous vector spaces, II, a supplement; to appear in Publ. RIMS, Kyoto University", I gave a lemma without proof because it seems standard. In fact, in Ginsburg [Gi], this lemma is attributed to A.Beilinson and an outline of the proof is given. This note is emerged from my effort to understand it. The proof given in my note is by no means elegant nor transparent.

1. On affine open subsets

(1) *Let A be a normal domain (i.e., a noetherian domain which is integrably closed in its fractional field), $X = \operatorname{spec} A$, $\mathcal{O}_X$ the structure sheaf, and $U \subset X$ an open subset. If $(U, \mathcal{O}_X|U)$ is an affine scheme, then $X \setminus U$ is purely of codimension one.*
(2) *If further A is a regular ring (i.e., a noetherian domain such that every local ring A_p ($p \in \operatorname{spec} A$) is a regular local ring), then for each $x \in X$, there exist $f, g \in A$ such that $X_f \cap U = X_f \cap X_g$ and $x \in X_f$. ($X_f = \{x \in X \mid f(x) \neq 0\}$.)*

Proof. (1) Let Z be an irreducible component of $X \setminus U$ whose codimension is ≥ 2, and z its generic point. Then for any point $x \in X$ of codimension one such that $\overline{\{x\}} \ni z$, $B := \Gamma(U, \mathcal{O}_X) \subset \mathcal{O}_{X,x}$. Since A is normal, $B \subset \mathcal{O}_{X,z}$. Let $m_{X,z}$ be the maximal ideal of $\mathcal{O}_{X,z}$. Since $U = \operatorname{spec} B$ is affine and since $m_{X,z} \cap B$ is a prime ideal of B,

$$m_{X,z} \cap B = m_{X,x} \cap B \text{ for some } x \in U. \tag{3}$$

Assume that there exists $0 \neq h \in \Gamma(X, \mathcal{O}_X)$ such that $h|U = 0$. Then $X_h \cap U = \phi$ and $X_h \neq \phi$, contradicting the irreducibility of X. Hence $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X|U)$ is injective, i.e., $A \subset B$. Thus it follows from (3) that $m_{X,z} \cap A = m_{X,x} \cap A$, contradicting $z \neq x$.

(2) follows from (1). □

2. On Matsushima's theorem

In this note, we shall give a detailed account of the following theorem due to Matsushima.

2.1.1 Theorem. *Let H be a closed subgroup of a reductive group G. Then the following conditions are equivalent:*
(1) *G/H is an affine variety.*
(2) *H is a reductive group.*

Here we understand the quotient G/H in the sense of [Bo, §6]. (In particular, see (6.1), (6.3) and (6.8) in loc. cit.) We do not assume the connectedness for a reductive group. Note that the base field is of an arbitrary characteristic.

2.1.2. Remark. For the history of this theorem, see the introduction of [Ri]. See also [Ha].

2.1.3. Convention. We fix always an algebraically closed field k. Every variety, say X, is assumed to be over k, and identified with the set $X(k)$ of its k-rational points. We write $k[X]$ for $\Gamma(X, \mathcal{O}_X)$.

2.2. Proof of (1)⇒(2).

2.2.1. Let l be a prime number, $\neq \mathrm{char}(k)$. For any variety X over k, put

$$m(X) := \max\{i \mid H^i(X, \mathbf{Q}_l) \neq 0\},$$

where H^i is the l-adic étale cohomology.

2.2.2. Lemma. *For a linear algebraic group G, we have $m(G) \leq \dim G$, and the equality holds if and only if G is reductive.*

Proof. We may assume that G is connected.

(I) First assume G reductive. Then

$$H^\bullet(G, \mathbf{Q}_l) = \Lambda^\bullet(I(\mathrm{Sym}^\bullet(X \otimes \mathbf{Q}_l(-1)[-1])^W)),$$

cf. [De, 8.2], and hence $m(G) = \dim G$. (Here $\Lambda^\bullet(\ )$ denotes the Grassmann algebra, $I(\ )$ denotes the totality of "indecomposable elements", $\mathrm{Sym}^\bullet(\ )$ denotes the symmetric algebra, X denotes the character lattice of *the* maximal torus of G, and $(\ )^W$ denotes the invariant part under the Weyl group action.)

(II) Consider the general case. Let U be the unipotent radical of G, and $\pi : G \to G/U$ be the projection. Then the spectral sequence

$$E_2^{ij} = H^i(G/U, R^j\pi_*\mathbf{Q}_l) \Rightarrow H^{i+j}(G, \mathbf{Q}_l)$$

degenerates. Hence we get $H^i(G, \mathbf{Q}_l) = H^i(G/U, \mathbf{Q}_l)$ and

$$m(G) = m(G/U) = \dim G/U.$$

Therefore

$$m(G) = \dim G \Leftrightarrow \dim U = 0 \Leftrightarrow G \text{ is reductive.} \quad \square$$

2.2.3. Proof of (1)⇒(2). Let $\pi : G \to G/H$ be the projection. Consider the spectral sequence

$$E_2^{ij} = H^i(G/H, R^j\pi_*\mathbf{Q}_l) \Rightarrow H^{i+j}(G, \mathbf{Q}_l). \tag{1}$$

Assume that G/H is affine but H is not reductive. Since G/H is affine, we have

$$E_2^{ij} = 0 \text{ unless } 0 \le i \le \dim G/H. \tag{2}$$

Since π is smooth, we have $(R^j\pi_*\mathbf{Q}_l)_x \simeq H^j(H, \mathbf{Q}_l)$ for all $x \in G/H$. Hence (2.2.2) yields that

$$E_2^{ij} = 0 \text{ unless } 0 \le j < \dim H. \tag{3}$$

Combining (1)–(3), we get

$$H^i(G, \mathbf{Q}_l) = 0 \text{ unless } 0 \le i < \dim G.$$

Hence G is not reductive by (2.2.2). $\square$

2.3. Proof of (2)⇒(1).

We start with the following lemma.

2.3.1. Lemma. *Let $f : X \to Y$ be a surjective morphism of varieties over an algebraically closed field k with finite fibres. Assume that Y is affine and $f^* : k[Y] \to k[X]$ is an isomorphism. Then f is an isomorphism.*

Proof. By Zariski's Main Theorem [EGA, IV, (8.12.6)], there exists a commutative diagram of morphisms of varieties

$$\begin{tikzcd} X \arrow[rr, "\text{inclusion}"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\overline{f}"] \\ & Y & \end{tikzcd}$$

where $X \subset \overline{X}$ is an open dense subset, and where $\overline{f}$ is a finite morphism. Since Y is an affine variety and since $\overline{f}$ is an affine morphism, $\overline{X}$ is also an affine variety. From the above diagram, and from our assumption, we get the following commutative diagram.

$$\begin{tikzcd} k[X] & & k[\overline{X}] \arrow[ll, "\text{injection}"'] \\ & k[Y] \arrow[ul, "\simeq"] \arrow[ur, "\overline{f}^*"'] & \end{tikzcd}$$

Hence $\overline{f}^*$ is an isomorphism, $\overline{f}$ is an isomorphism ($\overline{X}$ and Y being affine), $X = \overline{X}$ (f being surjective), and finally we get the result. □

2.3.2. Proof of (2) ⇒(1). Assume that G is any linear algebraic group and H is reductive. (The reductivity of G is not necessary for this half of the prof.) Let H act on G by the right multiplication. Put $Y := \operatorname{Spec} k[G]^H$. (Cf. (2.1.3).) Let $\alpha : G \to Y$ be the morphism corresponding to the inclusion $k[G]^H \to k[G]$. By the universal property of $\pi : G \to G/H$ [Bo, 6.3], we get the morphism $G/H \xrightarrow{\beta} Y$. By the definition in [Bo, 6.3], the points of G/H are in one-to-one correspondence with $\{gH \mid g \in G\}$. On the other hand, by [Ri, 1.3], the points of Y are also in one-to-one correspondence with the same set. Hence β is bijective. By the definition in [Bo, 6.1, (2)] (with $K = k$, $U = V = G$, and $W = G/H$ in the notation in loc. cit.), we can identify

$$k[G/H] = \{\varphi \in k[G] \mid \varphi \text{ is constant on each } gH \ (g \in G)\} = k[G]^H = k[Y].$$

Now apply (2.3.1) to $\beta : G/H \to Y$. Then we see that β is an isomorphism. Since Y is an affine variety, G/H is also so. □

References

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[Ri] R.W.Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), 38 41.

3. On Radon Transformation

This note is a detail of [DG, (3.3.10)]. We cite [DG, (a.b.c)] simply as (a.b.c). In this note, we describe, when $\chi = 1$, how our 'Radon transformation $\mathcal{R}(\ )$' given in (3.3.9) relates to the one given in [Br, 9.13].

Let $V = \mathbf{A}^n_{\mathbf{F}_q}$, $V^\vee$ be the dual space of V, and $\langle\ \rangle : V^\vee \times V \to \mathbf{A}^1_{\mathbf{F}_q}$ the natural pairing. Let $V^\times := V \setminus \{0\}$, $V^{\vee\times} := V^\vee \setminus \{0\}$, $P := V^\times/\mathbf{G}_m$, $P^\vee := V^{\vee\times}/\mathbf{G}_m$,

$$\begin{array}{ccccc}
V^{\vee\times} & \xleftarrow{pr^\vee} & V^{\vee\times} \times V^\times & \xrightarrow{pr} & V^\times \\
{\scriptstyle g^\vee}\downarrow & & {\scriptstyle g^\vee\times 1}\downarrow & & \| \\
P^\vee & \xleftarrow{\overline{pr}^\vee} & P^\vee \times V^\times & \xrightarrow{\overline{pr}} & V^\times \\
\| & & {\scriptstyle 1\times g}\downarrow & & {\scriptstyle g}\downarrow \\
P^\vee & \xleftarrow{\widetilde{pr}^\vee} & P^\vee \times P & \xrightarrow{\widetilde{pr}} & P
\end{array}$$

be the natural morphisms, $Z := \{(g^\vee(v^\vee), g(v)) \in P^\vee \times P \mid \langle v^\vee, v\rangle = 0\}$, $R := P^\times \times P \setminus Z$, and $\widetilde{pr}_Z$ etc. the restriction of $\widetilde{pr}$ etc. to Z etc. In this paragraph, we always assume that

(1) $K \in D^b_c(V, \overline{\mathbf{Q}_l})$ is the zero extension of $K|_{V^\times}$, and $K|_{V^\times} = g^*\widetilde{K}$ for some $\widetilde{K} \in D^b_c(P, \overline{\mathbf{Q}_l})$.

In [Br, 9.12], the Radon transform $\Phi(\widetilde{K}) \in D^b_c(P^\vee, \overline{\mathbf{Q}_l})$ is defined to be

(2) $$\Phi(\widetilde{K}) := R(\widetilde{pr}^\vee_Z)_!(\widetilde{pr}_Z)^*\widetilde{K}[n-2].$$

Then under the assumption (1), *we have a distinguished triangle*

(3) $$g^{\vee *}R\widetilde{pr}^\vee_!\widetilde{pr}^*\widetilde{K}[n-1] \to \mathcal{R}(K) \to g^{\vee *}\Phi(\widetilde{K})(-1) \xrightarrow{+1}.$$

Proof. Let $\varphi : V^\times(\mathbf{F}_q) \to \overline{\mathbf{Q}_l}$ be a function such that $\varphi(tv) = \varphi(v)$ for all $t \in \mathbf{F}_q^\times$ and $v \in V^\times(\mathbf{F}_q)$, i.e., such that $\varphi(v) = \widetilde{\varphi}(g(v))$ for some $\widetilde{\varphi} : P(\mathbf{F}_q) \to \overline{\mathbf{Q}_l}$. Consider the following calculation:

$$\begin{aligned}
&\sum_{v\in V^\times(\mathbf{F}_q), t\in\mathbf{F}_q} \varphi(v)\psi(t(\langle v^\vee, v\rangle - 1)) \\
=&\sum^*_{v\in V^\times(\mathbf{F}_q), t\in\mathbf{F}_q, s\in\mathbf{F}_q^\times} \widetilde{\varphi}(g(v))\psi(t(\langle v^\vee, v\rangle - s)) \\
=&\,q \sum^*_{\substack{v\in V^\times(\mathbf{F}_q)\\ \langle v^\vee, v\rangle\neq 0}} \widetilde{\varphi}(g(v)).
\end{aligned}$$

Here $\sum^*$ means the sum over the equivalence classes of (v, t, s) or v with respect to $(v, t, s) \sim (\lambda v, \lambda^{-1}t, \lambda s)$ or $v \sim \lambda v$ ($\lambda \in \mathbf{F}_q^\times$). Following this calculation, we get

(4) $$R\pi^\vee_!(\pi^*K \otimes a^*L_\psi)|_{V^{\vee\times}} = g^{\vee *}R(\widetilde{pr}^\vee_R)_!(\widetilde{pr}_R)^*\widetilde{K}(-1)[-2].$$

in place of the first term of (3.3.7, (1)). As for the second term of (3.3.7, (1)), we have

(5)
$$
\begin{aligned}
Rpr^{\vee}_{!}pr^{*}K|_{V^{\vee\times}} &= g^{\vee *}R\widetilde{pr}^{\vee}_{!}(1\times g)^{*}\widetilde{pr}^{*}\widetilde{K}\\
&=g^{\vee *}R(\widetilde{pr}^{\vee})_{!}R(1\times g)_{!}(1\times g)^{*}\widetilde{pr}^{*}\widetilde{K}\\
&=g^{\vee *}R(\widetilde{pr}^{\vee})_{!}(R\Gamma_c(\mathbf{G}_m,\overline{\mathbf{Q}_l})\otimes\widetilde{pr}^{*}\widetilde{K}).
\end{aligned}
$$

Now we obtain the following commutative diagram in $D^b_c(V^{\vee\times},\overline{\mathbf{Q}_l})$, whose rows and columns consisting of three terms are distinguished.

(6)
$$
\begin{tikzcd}
 & g^{\vee *}R(\widetilde{pr}^{\vee})_{!}\widetilde{pr}^{*}\widetilde{K}[-1] \arrow[r, phantom, "="] \arrow[d] & g^{\vee *}R(\widetilde{pr}^{\vee})_{!}\widetilde{pr}^{*}\widetilde{K}[-1] \arrow[d] & \\
g^{\vee *}R(\widetilde{pr}^{\vee}_{R})_{!}\widetilde{pr}^{*}_{R}\widetilde{K}(-1)[-2] \arrow[r] \arrow[d, equal] & Rpr_{!}pr^{*}K \arrow[r] \arrow[d] & \mathcal{R}(K)[-n] \arrow[r, "+1"] \arrow[d] & {} \\
g^{\vee *}R(\widetilde{pr}^{\vee}_{R})_{!}\widetilde{pr}^{*}_{R}\widetilde{K}(-1)[-2] \arrow[r] & g^{\vee *}R(\widetilde{pr}^{\vee})_{!}\widetilde{pr}^{*}\widetilde{K}(-1)[-2] \arrow[r] \arrow[d, "+1"] & g^{\vee *}\Phi(\widetilde{K})(-1)[-n] \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & {} & {} &
\end{tikzcd}
$$

Indeed, we get the first horizontal triangle from (3.3.7), (4) and (3.3.9). The second horizontal triangle is the obvious one. From the distinguished triangle

$$
\overline{\mathbf{Q}_l}[-1]\to R\Gamma_c(\mathbf{G}_m,\overline{\mathbf{Q}_l})\to\overline{\mathbf{Q}_l}(-1)[-2]\xrightarrow{+1}
$$

and (5), we get the first vertical triangle. Then by [BBD, 1.1.11], we get the second vertical triangle. □

References

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4. Non-characteristic pull-back of D-modules

4.1. Differential operators.

4.1.1. Definition. Following [SKK], we define the sheaf of (algebraic) differential operators on a smooth variety X by

$$\mathcal{D}_X = \{R\Gamma_{[X]}(\mathcal{O}_{X\times X})[\dim X] \otimes_{\mathbf{Z}_X \boxtimes \mathcal{O}_X} (\mathbf{Z}_X \boxtimes \omega_X)\} |_X,$$

where X is identified with the diagonal of $X \times X$, and where ω_X is the invertible sheaf of differential forms of highest degree.

4.1.2. Example. In order to understand this definition, let us take up the case where X is an open subset of $\mathbf{A}^1$. Consider the distinguished triangle

$$R\Gamma_{[X]}(\mathcal{O}_{X\times X}) \to \mathcal{O}_{X\times X} \to R\Gamma_{[X\times X\backslash X]}(\mathcal{O}_{X\times X}) \xrightarrow{+},$$

and then consider (a part of) the associated long exact sequence

$$0 \to \mathcal{O}_{X\times X} \to \mathcal{O}_{X\times X}\left[\frac{1}{x-x'}\right] \to H^1_{[X]}(\mathcal{O}_{X\times X}) \to 0,$$

where x (resp. x') is the coordinate function of the first (resp. second) factor of $X \times X$. Let $\delta(x-x') \in H^1_{[X]}(\mathcal{O}_{X\times X})$ be the image of $1/(x-x')$. Note that any local section of $\mathcal{O}_{X\times X}[\frac{1}{x-x'}]$ can be uniquely expressed in a neighbourhood of a point of the diagonal as

$$\Phi(x,x') + P(x,\partial_x)\frac{1}{x-x'}$$

with $\Phi \in \mathcal{O}_{X\times X}$ and with a regular differential operator $P(X,\partial)$ in the usual sense. (An easy exercise.) Hence we get an isomorphism

$$\left\{\begin{matrix}\text{differential operators in}\\ \text{the usual sense}\end{matrix}\right\} \xrightarrow{\cong} \mathcal{D}_X, \quad P(x,\partial_x) \mapsto P(x,\partial_x)\delta(x-x')dx'$$

If the base field is $\mathbf{C}$, the action of $\mathcal{D}_X$ on $\mathcal{O}_X$ can be described as follows: For any $\varphi \in \mathcal{O}_X$, we have

$$\begin{aligned} P(x,\partial_x)\varphi(x) &= P(x,\partial_x)\left(\frac{1}{2\pi i}\int_{C_x}\frac{\varphi(x')}{x'-x}dx'\right) \\ &= \frac{-1}{2\pi i}\int_{C_x}\left(\Phi(x,x') + P(x,\partial_x)\frac{1}{x-x'}\right)\varphi(x')dx', \end{aligned}$$

where C_x is a small circle turning around x in the positive direction. Note that according to this definition, it is natural to define

$$\int \delta(x)dx := \frac{-1}{2\pi i}\int_{|x|=\varepsilon}\frac{dx}{x} = -1.$$

The product structure of $\mathcal{D}_X$ can be described in a similar way.

4.1.3. The above definition of $\mathcal{D}_X$ gives the following expression for $\mathcal{D}_{Y\to X}$. For a morphism $f: Y \to X$ of smooth varieties, we get

$$\mathcal{D}_{Y\to X} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X = \alpha^{-1}\{R\Gamma_{[Y]}(\mathcal{O}_{Y\times X})[\dim X] \otimes_{\mathbf{Z}_Y \boxtimes \mathcal{O}_X} (\mathbf{Z}_Y \boxtimes \omega_X)\},$$

where the morphism $(f^{-1}\mathcal{O}_X)_y = \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is given by $\varphi(x) \mapsto \varphi(f(y))$, and Y is regarded as a subvariety of $Y \times X$ by $\alpha: y \mapsto (y, f(y))$.

4.2. Non-characteristic restriction.

4.2.0. Notation. For a smooth varieties $Y \subset X$, T^*_YX $(\subset T^*X)$ be the conormal bundle, P^*X and P^*_YX the projective bundles corresponding to T^*X and T^*_YX. For a subset C of T^*X, put $C_x := C \cap T^*_xX$ and $C|_Y = \amalg_{y\in Y} C_y$. For a subset of P^*X, we use the similar notation. Let

$$T^*X|_Y \xrightarrow{\rho} T^*Y, \quad (P^*X - P^*_YX)|_Y \xrightarrow{\tilde\rho} P^*Y, \quad \text{and } T^*X - T^*_XX \xrightarrow{\gamma} P^*X$$

be the natural morphisms. For $0 \neq \xi \in T^*_xX$ (resp. $C' \subset T^*_xX$, resp. $C \subset T^*X$), put $\tilde\xi := \gamma(\xi)$ (resp. $\tilde C' := \gamma(C' - \{0\})$, resp. $\tilde C := \gamma(C - T^*_XX)$).

4.2.1. Lemma. *Let Λ be a conic closed subvariety of T^*X. Consider the following conditions. (The morphisms ρ_Λ, ρ^0_Λ and $\tilde\rho^0_\Lambda$ are those induced by ρ and $\tilde\rho$.)*
(i) $\Lambda \cap T^*_YX \subset T^*_XX|_Y$.
(ii) $\tilde\Lambda \cap P^*_YX = \phi$.
(iii$_f$) *$\rho_\Lambda : \Lambda|_Y \to T^*Y$ is a finite morphism.*
(iv$_f$) *$\rho^0_\Lambda : \Lambda|_Y - T^*_YX \to T^*Y - T^*_YY$ is a finite morphism.*
(v$_f$) *$\tilde\rho^0_\Lambda : \tilde\Lambda|_Y - P^*_YX \to P^*Y$ is a finite morphism.*
Let (#$_p$) be the condition obtained from (#$_f$) by replacing 'finite' with 'proper' (#=iii, iv, or v). Then we have the implications

$$\begin{aligned}&(\mathrm{i}) \Leftrightarrow (\mathrm{ii}) \Leftrightarrow (\mathrm{iii}_f) \Leftrightarrow (\mathrm{iii}_p)\\ &\quad\Rightarrow (\mathrm{iv}_f) \Leftrightarrow (\mathrm{iv}_p) \Leftrightarrow (\mathrm{v}_f) \Leftrightarrow (\mathrm{v}_p).\end{aligned}$$

Moreover, if every irreducible component of $\Lambda|_Y$ contained in T^*_YX is also contained in T^*_XX, then all the above conditions are equivalent.

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (v$_p$) are obvious. Since

$$\begin{array}{ccc} ((T^*X - T^*_YX)\cap\Lambda)|_Y & \longrightarrow & ((P^*X - P^*_YX)\cap\tilde\Lambda)|_Y \\ {\scriptstyle\rho^0_\Lambda}\downarrow & & \downarrow{\scriptstyle\tilde\rho^0_\Lambda} \\ T^*Y - T^*_YY & \xrightarrow{\gamma} & P^*Y \end{array}$$

is a cartesian square, since γ is faithfully flat, and since ρ^0_Λ is an affine morphism, $\tilde\rho^0_\Lambda$ is also an affine morphism. Hence we get the implications (iv$_f$) $\Leftrightarrow$ (iv$_p$) $\Leftrightarrow$ (v$_p$) $\Leftrightarrow$ (v$_f$). Similarly, we get (iii$_f$) $\Leftrightarrow$ (iii$_p$).

Let us deduce (i) assuming (iii$_f$). Since for each $y \in T^*_Y Y$ $(= Y)$, $\rho_\Lambda^{-1}(y) = (T^*_Y X \cap \Lambda)_y$ is a closed conic subvariety of $T^*_y X$, the finiteness of ρ_Λ implies that $(T^*_Y X \cap \Lambda)_y = \{0\}$ for all $y \in Y$, i.e., (i).

Let us deduce (iii$_f$) assuming (i). It suffices to show that for each irreducible affine open subset $\phi \neq Y_0 \subset Y$ and for each irreducible component Z of $\Lambda|_{Y_0}$, the morphism $\rho_\Lambda|_Z : Z \to T^*Y_0$ is finite. In other words, it suffices to show that each $\varphi \in \Gamma(Z, \mathcal{O}_Z)$ satisfies an equation of the form

$$\varphi^n + (\rho^* a_1)\varphi^{n-1} + \cdots + (\rho^* a_n) = 0 \tag{1}$$

with some regular functions a_i on T^*Y_0. Since Z is conic, we may assume from the beginning that φ is homogeneous of degree $d \geq 0$ with respect to the natural $\mathbf{G}_m$-action on the fibres of $Z \to Y_0$. Since we have already proved the implication (i) $\Rightarrow$ (iv$_f$), $\varphi|_{Z - T^*_Y X}$ satisfies an equation of the form (1) with some regular functions a_i on $T^*Y_0 - T^*_{Y_0} Y_0$. Considering the $\mathbf{G}_m$-action, we may assume that a_i is homogeneous of degree di. Since $di \geq 0$, every a_i is regular on T^*Y_0. Hence, if

$$Z \not\subset T^*_Y X, \tag{2}$$

then φ satisfies the same, equation satisfied by $\varphi|_{Z - T^*_Y X}$, and hence $Z \to Y_0$ is finite. If (2) is not satisfied, then $Z \subset T^*_Y X \cap \Lambda$, and hence from (i) follows

$$Z \subset T^*_X X. \tag{3}$$

Hence $Z \to Y_0$ is finite, being the composition of

$$Z \xrightarrow{\text{inclusion}} T^*_X X|_{Y_0} \underset{\rho}{\xrightarrow{\cong}} T^*_{Y_0} Y_0 \xrightarrow{\text{inclusion}} T^* Y_0.$$

Now the last assertion is obvious from the above argument. □

4.2.2. Remark. The above conditions are not necessarily equivalent to each other in general. For example, consider the case where $\Lambda = T^*_{Y'} X$ with some smooth Y' such that $Y \subset Y' \subset X$.

4.2.3. Definition. (1) For a conic closed subvariety Λ of T^*X, we say that a subvariety Y of X is *non-characteristic* for Λ if the equivalent conditions (i), (ii), (iii$_p$), and (iii$_p$) of (4.2.1) are satisfied.
(2) For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we say that Y is *non-characteristic* for $\mathcal{M}$ if Y is so for the characteristic variety $\mathrm{SS}(\mathcal{M})$ of $\mathcal{M}$.

4.2.4. Lemma. *Let X be a smooth variety with global coordinate system $(x_1, \cdots, x_n)$ (i.e., an étale morphism $X \to \mathbf{A}^n$), and Y the subvariety defined by $x_1 = 0$. Let $P = P(x, \partial) = \partial_1^m + P_1(x, \partial')\partial_1^{m-1} + \cdots + P_m(x, \partial')$ be an (algebraic) differential operator of order m, where ∂' means $(\partial_2, \cdots, \partial_n)$. Put $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$. Let $u \in \mathcal{M}$ be the section of $\mathcal{M}$ corresponding to $1 \in \mathcal{D}_X$. Then*
(1) $\mathcal{D}_{Y \to X} \otimes^L_{\mathcal{D}_X} \mathcal{M} = \mathcal{D}_{Y \to X} \otimes_{\mathcal{D}_X} \mathcal{M} =: \mathcal{M}|_Y$, and

(2) $\mathcal{M}|_Y$ *is a free* $\mathcal{D}_Y$*-module with basis* $\{1_{Y\to X} \otimes \partial_1^i u \mid 0 \le i < m\}$

Proof. (1) Since $\mathcal{D}_{Y\to X} = \mathcal{D}_X / x_1 \mathcal{D}_X$, it is enough to prove that $\mathcal{M} \xrightarrow{x_1 \times} \mathcal{M}$ is injective. Assume that $x_1 Q(x,\partial) u = 0$. Note that $Q(x,\partial)$ can be uniquely expressed as

$$Q(x,\partial) = G(x,\partial)P(x,\partial) + \sum_{j=0}^{m-1} S_j(x,\partial')\partial_1^j.$$

(An easy exercise.) Since $x_1 Q \equiv 0 \mod \mathcal{D}_X P$ we have $S_j = 0$ by the uniqueness of the above expression, which implies $Qu = 0$. Thus we get the injectivity.

(2) Since

$$\mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X} \mathcal{M} = \mathcal{M}/x_1\mathcal{M} = \mathcal{D}_X / (x_1 \mathcal{D}_X + \mathcal{D}_X P),$$

this assertion is an easy exercise. □

4.2.5. Lemma. *Let* $\varphi : Y \to X$ *be an inclusion mapping of smooth varieties. Assume that* φ *is non-characteristic for a coherent* $\mathcal{D}_X$*-module* $\mathcal{M}$*. Then*
(1) $\mathcal{D}_{Y\to X} \otimes^L_{\mathcal{D}_X} \mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X} \mathcal{M} =: \varphi^*\mathcal{M}$, *and*
(2) $\varphi^*\mathcal{M}$ *is a coherent* $\mathcal{D}_Y$*-module.*

Proof.

(I) Let $Y \subset Z \subset X$ be smooth varieties. Since $\mathcal{D}_{Z\to X}$ is a locally free left $\mathcal{D}_Z$-module [Bo, VI, 7.3, (7)], we get

$$\mathcal{D}_{Y\to X} = \mathcal{D}_{Y\to Z} \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z\to X} = \mathcal{D}_{Y\to Z} \otimes^L_{\mathcal{D}_Z} \mathcal{D}_{Z\to X}.$$

Hence we can reduce the proof to the case where $\dim X - \dim Y = 1$. Since the problem is local, we may assume that X is an affine variety, X has a global coordinate system $(x_1, \cdots, x_n)$ and Y is defined by $x_1 = 0$.

(II) For any $v \in \Gamma(X, \mathcal{M})$, $\mathrm{SS}(\mathcal{D}_X v) \subset \mathrm{SS}(\mathcal{M}) = \Lambda$. Since

$$T^*_Y X = \{(0, x'; \xi_1, 0, \cdots, 0)\} \text{ and}$$
$$T^*_X X|_Y = \{(0, x'; 0, 0, \cdots, 0)\}, \text{ where } x' = (x_2, \cdots, x_n),$$

there exists $P \in \Gamma(X, \mathcal{D}_X)$ such that

(3) $$Pv = 0, \quad \mathrm{ord}(P) = m, \quad \text{and } \sigma(P)(0, x'; \xi_1, 0, \cdots, 0) = \xi_1^m,$$

where $\sigma(P)(x,\xi)$ denotes the principal symbol of P. Since P is of order m, P can be expressed as

$$P(x,\partial) = a(x)\partial_1^m + P_1(x,\partial')\partial_1^{m-1} + \cdots + P_m(x,\partial')$$

with $\mathrm{ord}(P_j) \le j$. Since

$$\xi_1^m = \sigma(P)(0, x'; \xi_1, 0) = a(0, x')\xi_1^m,$$

we get $a(0,x')=1$. In particular, $a(x)$ is invertible in a neighbourhood of Y. Hence we may assume that

$$P(x,\partial)=\partial_1^m+P_1(x,\partial')\partial_1^{m-1}+\cdots+P_m(x,\partial')$$

replacing X by a small neighbourhood of Y, and P by $a(x)^{-1}P$. Then applying (4.2.4) to $\mathcal{D}_X/\mathcal{D}_XP$, we can see that (1) and (2) are true for $\mathcal{D}_X/\mathcal{D}_XP$.

(III) Let us prove (1). Let $\{(v_1,\cdots,v_l)\}$ be a generator system of the $\Gamma(X,\mathcal{D}_X)$-module $\Gamma(X,\mathcal{M})$. From each v_j, construct P_j which satisfies (3), and put $\mathcal{L}_j:=\mathcal{D}_X/\mathcal{D}_XP_j$. (Here and below, we freely shrink X if necessary.) Let

$$\mathcal{L}:=\mathcal{L}_1\oplus\cdots\oplus\mathcal{L}_l\to\mathcal{M}$$

be the natural surjection, and $\mathcal{N}$ be its kernel. Consider the exact sequence

$$\varphi^*\mathcal{N}\to\varphi^*\mathcal{L}\to\varphi^*\mathcal{M}\to 0.$$

Since the $\mathcal{D}_Y$-module $\varphi^*\mathcal{L}$ is finitely generated, $\varphi^*\mathcal{M}$ is also so and hence coherent. (Here we used the fact that $\Gamma(Y,\mathcal{D}_Y)$ for affine Y is left noetherian. Alternatively, we can also prove in the following way without using this fact: Since $\varphi^*\mathcal{M}$ is finitely generated, since $\mathcal{M}$ is an arbitrary $\mathcal{D}_X$-module satisfying a certain assumption, and sine $\mathcal{N}$ satisfies the same assumption, we can conclude that $\varphi^*\mathcal{N}$ is also a finitely generated $\mathcal{D}_Y$-module. Since $\varphi^*\mathcal{L}$ is already known to be coherent, we can conclude that $\varphi^*\mathcal{M}$ is coherent.)

(IV) Next, let us prove (2). Since $\mathcal{D}_{Y\to X}=\mathcal{D}_X/x_1\mathcal{D}_X$, it suffices to show that $\mathcal{M}\xrightarrow{x_1\times}\mathcal{M}$ is injective. Assume that $v\in\mathcal{M}$ and $x_1v=0$. As we have seen in (II), there is $P\in\mathcal{D}_X$ such that (1) and (2) are satisfied. Since

$$\underbrace{[\cdots[[P,x_1],x_1],\cdots,x_1]}_{m\text{ times}}u=(m!)u=0,$$

we get the desired result. □

4.2.6. Lemma. *Let $Z\subset Y\subset X$ be a triple of smooth algebraic varieties over* **C**. *Then there exists a canonical homomorphism*

$$C_{Y|X}\to C_{Z|X}[\operatorname{codim}_Y Z].$$

Here $C_{Y|X}:=R\Gamma_{[Y]}(\mathcal{O}_X)[\operatorname{codim}_X Y]$. (See [Bo, p.260] *for $\Gamma_{[Y]}$.)*

Proof. We have a morphism

$$\begin{array}{ccc} R\Gamma_Y(\mathbf{C}_X)[2\operatorname{codim}_X Y] & & R\Gamma_Z(\mathbf{C}_X)[2\operatorname{codim}_X Z] \\ \| & & \| \\ \mathbf{C}_Y & \longrightarrow & \mathbf{C}_Z. \end{array}$$

(If $\mathbf{C}_X$ etc. are understood as the orientation sheaves, then the vertical equalities become canonical. So if we choose the orientation of complex manifolds, e.g., so that $\bigwedge_j\frac{1}{2i}(dz_j\wedge d\overline{z_j})>0$, then the above morphism is determined uniquely.) Let

$$R\Gamma_{[Y]}(\mathcal{O}_X[\dim X])[2\operatorname{codim}_X Y]\to R\Gamma_{[Z]}(\mathcal{O}_X[\dim X])[2\operatorname{codim}_X Z]$$

be the corresponding morphism (via DR). Shifting it, we get the desired morphism. □

4.2.7. Lemma. *Let $\varphi : Y \to X$ be the inclusion mapping of smooth varieties. Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module for which Y is non-characteristic. Then we have a canonical isomorphism*

$$ {}^L\varphi^* \mathbf{D}_X(\mathcal{M}) \xleftarrow{\cong} \mathbf{D}_Y({}^L\varphi^* \mathcal{M}) $$

of left $\mathcal{D}_Y$-modules. Here

$$ \mathbf{D}_X(\mathcal{M}) := R\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)[\dim X] \otimes_{\mathcal{O}_X} \omega_X^{\otimes -1}. $$

(The left $\mathcal{D}_X$-modules structure of $\mathcal{D}_X$ is used to define $R\underline{\mathrm{Hom}}$, and the right $\mathcal{D}_X$-module structure of $\mathcal{D}_X$ is used to define the left $\mathcal{D}_X$-module structure of $\mathbf{D}_X(\mathcal{M})$. The sheaf of differential form of highest degree is denoted by ω_X.)

Proof. As right $\mathbf{Z}_Y \boxtimes \mathcal{D}_X$-modules, we have

$$
\begin{aligned}
&(\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_{Y\times X}} \omega_{Y\times X}) \otimes_{\mathcal{D}_Y} \varphi^*\mathcal{M}, \quad \text{(cf. (4.2.6) for } \mathcal{B}_{Y|Y\times X})\\
&\qquad (:= (\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_{Y\times X}} \omega_{Y\times X}) \otimes_{\mathcal{D}_Y \boxtimes \mathbf{Z}_X} (\varphi^*\mathcal{M} \boxtimes \mathbf{Z}_X)\ )\\
&= (\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_{Y\times X}} \omega_{Y\times X}) \otimes_{\mathcal{D}_Y \boxtimes \mathcal{D}_X} (\varphi^*\mathcal{M} \boxtimes \mathcal{D}_X)\\
&= (\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_{Y\times X}} \omega_{Y\times X}) \otimes_{\mathcal{D}_Y \boxtimes \mathcal{D}_X} ((\mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X} \mathcal{M}) \boxtimes (\mathcal{D}_{X\to X} \otimes_{\mathcal{D}_X} \mathcal{D}_X))\\
&= (\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_{Y\times X}} \omega_{Y\times X}) \otimes_{\mathcal{D}_{Y\times X}} \mathcal{D}_{Y\times X\to X\times X} \otimes_{\mathcal{D}_{X\times X}} (\mathcal{M}\underline{\boxtimes}\mathcal{D}_X)\\
&= (\mathcal{B}_{Y|X\times X} \otimes_{\mathcal{O}_{X\times X}} \omega_{X\times X}) \otimes_{\mathcal{D}_{X\times X}} (\mathcal{M}\underline{\boxtimes}\mathcal{D}_X), \quad \text{by [Ka, Lemma (4.8)]}\\
&= (\mathcal{B}_{Y|X\times X} \otimes_{\mathcal{O}_{X\times X}} \omega_{X\times X}) \otimes_{\mathcal{D}_X \boxtimes \mathbf{Z}_X} (\mathcal{M} \boxtimes \mathbf{Z}_X)
\end{aligned}
$$

where $\mathcal{M}\underline{\boxtimes}\mathcal{D}_X := \mathcal{D}_{X\times X} \otimes_{\mathcal{D}_X \boxtimes \mathcal{D}_X} (\mathcal{M} \boxtimes \mathcal{D}_X)$. (Here and below, we often write $\mathcal{M}$ etc. for $\varphi^{-1}\mathcal{M}$ etc. in order to simplify notation, but we are exclusively interested in a neighbourhood of Y ($\subset X$).) Hence as left $\mathbf{Z}_Y \boxtimes \mathcal{D}_X$-modules, we have

$$
\begin{aligned}
&(\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_Y \boxtimes \mathbf{Z}_X} (\omega_Y \boxtimes \mathbf{Z}_X)) \otimes_{\mathcal{D}_Y \boxtimes \mathbf{Z}_X} (\varphi^*\mathcal{M} \boxtimes \mathbf{Z}_X)\\
&= (\mathcal{B}_{Y|X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathbf{Z}_X} (\omega_X \boxtimes \mathbf{Z}_X)) \otimes_{\mathcal{D}_X \boxtimes \mathbf{Z}_X} (\mathcal{M} \boxtimes \mathbf{Z}_X).
\end{aligned} \tag{1}
$$

Applying (4.2.6) to $Y \subset X \subset X \times X$, we get

(2) $\mathcal{D}_X = \mathcal{B}_{X|X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathbf{Z}_X} (\omega_X \boxtimes \mathbf{Z}_X) \to \mathcal{B}_{Y|X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathbf{Z}_X} (\omega_X \boxtimes \mathbf{Z}_X)[\mathrm{codim}_X Y]$.

Hence we get

$$
\begin{aligned}
\mathcal{M} &= \mathcal{D}_X \otimes_{\mathcal{D}_X} \mathcal{M} = (\mathcal{B}_{X|X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathbf{Z}_X} (\omega_X \boxtimes \mathbf{Z}_X)) \otimes_{\mathcal{D}_X \boxtimes \mathbf{Z}_X} (\mathcal{M} \boxtimes \mathbf{Z}_X)\\
&\to (\mathcal{B}_{Y|X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathbf{Z}_X} (\omega_X \boxtimes \mathbf{Z}_X)) \otimes_{\mathcal{D}_X \boxtimes \mathbf{Z}_X} (\mathcal{M} \boxtimes \mathbf{Z}_X)[\mathrm{codim}_X Y] \quad \text{by (2)}\\
&= (\mathcal{B}_{Y|Y\times X} \otimes_{\mathcal{O}_Y \boxtimes \mathbf{Z}_X} (\omega_Y \boxtimes \mathbf{Z}_X)) \otimes_{\mathcal{D}_Y \boxtimes \mathbf{Z}_X} (\varphi^*\mathcal{M} \boxtimes \mathbf{Z}_X)[\mathrm{codim}_X Y] \quad \text{by (1)}\\
&= \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \varphi^*\mathcal{M}[\mathrm{codim}_X Y] \quad \text{by (4.1.3).}
\end{aligned}
$$

By using this homomorphism, we obtain

$$
\begin{aligned}
&R\mathrm{Hom}_{\mathcal{D}_Y}(\varphi^*\mathcal{M}, \mathcal{D}_Y)[-\mathrm{codim}_X Y]\\
&\to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \varphi^*\mathcal{M})[\mathrm{codim}_X Y], \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_Y)\\
&\to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_{X\leftarrow Y})\\
&\xleftarrow{\cong} R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\leftarrow Y}.
\end{aligned}
$$

Hence we get the desired morphism. (The last morphism is the natural one. To see that it is an isomorphism, consider a locally $\mathcal{D}_X$-free resolution of $\mathcal{D}_{X\leftarrow Y}$.) □

4.3. Non-characteristic pull-back (general case).

4.3.0. Notation. For a morphism $\varphi : Y \to X$ of smooth varieties, let $\rho : Y \times_X T^*X \to T^*Y$ be the natural morphism. Put $T^*_Y X := \rho^{-1}(T^*_Y Y)$. Define $P^*_Y X$ as in (4.2.0). Let $\tilde{\rho} : (Y \times_X P^*X) - P^*_Y X \to P^*Y$ be the natural morphism.

4.3.1. Lemma. *Let Λ be a conic closed subvariety of T^*X. Consider the following conditions.*

(i) $T^*_Y X \cap (Y \times_X \Lambda) \subset Y \times_X T^*_X X$.

(ii) $P^*_Y X \cap (Y \times_X \tilde{\Lambda}) = \phi$.

(iii$_f$) $\rho_\Lambda : Y \times_X \Lambda \to T^*Y$ *is a finite morphism.*

(iv$_f$) $\rho^0_\Lambda : (Y \times_X \Lambda) - T^*_Y X \to T^*Y - T^*_Y Y$ *is a finite morphism.*

(v$_f$) $\tilde{\rho}^0_\Lambda : (Y \times_X \tilde{\Lambda}) - P^*_Y X \to P^*Y$ *is a finite morphism.*

(vi) $\{(y, \varphi(y)) \mid y \in Y\}$ *is non-characteristic for* $T^*_Y Y \times \Lambda$ $(\subset T^*(X \times Y))$.

Let $(\#_p)$ be the conditions obtained as in (4.2.1). (For the notations, see (4.2.1).) Then we have the implications

$$\begin{aligned}&(\mathrm{i}) \Leftrightarrow (\mathrm{ii}) \Leftrightarrow (\mathrm{iii}_f) \Leftrightarrow (\mathrm{iii}_p) \Leftrightarrow (\mathrm{vi})\\ &\Rightarrow (\mathrm{iv}_f) \Leftrightarrow (\mathrm{iv}_p) \Leftrightarrow (\mathrm{v}_f) \Leftrightarrow (\mathrm{v}_p).\end{aligned}$$

Proof. The equivalence (i) $\Leftrightarrow$ (vi) is obvious. If Y is identified with $\{(y, \varphi(y)) \mid y \in Y\}$ $(\subset Y \times X)$, then we have $Y \times_{Y\times X} (T^*_Y Y \times \Lambda) = Y \times_X \Lambda$. Therefore, the remaining implications immediately follow from (4.2.1). □

4.3.2. Definition. (1) For a conic closed subvariety Λ of T^*X, we say that $\varphi : Y \to X$ is *non-characteristic* for Λ if the equivalent conditions (i), (ii), (iii$_f$), (iii$_p$) and (vi) of (4.3.1) are satisfied. (2) For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we say that $\varphi : Y \to X$ is *non-characteristic* for $\mathcal{M}$ if φ is so for $\mathrm{SS}(\mathcal{M})$.

4.3.3. Theorem. *If a morphism $\varphi : Y \to X$ of smooth varieties is non-characteristic for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, then*

(1) $\mathcal{D}_{Y\to X} \otimes^L_{\mathcal{D}_X} \mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X} \mathcal{M} =: \varphi^*\mathcal{M}$,

(2) $\varphi^*\mathcal{M}$ *is a coherent $\mathcal{D}_Y$-module, and*

(3) *there is a canonical isomorphism* ${}^L\varphi^* \mathbf{D}_X(\mathcal{M}) \xleftarrow{\simeq} \mathbf{D}_Y({}^L\varphi^* \mathcal{M})$.

Proof. Let $Y \xrightarrow{\alpha} Y \times X \xrightarrow{\beta} X$ be the natural morphisms. Then (1), (2) and (3) are obvious for β. They are also true for α by (4.2.5) and (4.2.7). Since

$$\mathcal{D}_{Z\to Y} \otimes^L_{\varphi^{-1}\mathcal{D}_Y} \varphi^{-1}\mathcal{D}_{Y\to X} = \mathcal{D}_{Z\to X}$$

for any morphism $Z \to Y$ of smooth varieties [Ka, Lemma (4.7)], we get the result. □

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5. On D-modules associated to a complex power of a function

5.0. Convention and Notation. We denote by $\mathbf{Z}$ the rational integer ring, and by $\mathbf{C}$ the complex number field.

As for $\mathcal{D}$-modules, we shall work in the algebraic category unless otherwise stated. For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, define the de Rham functor $\mathrm{DR}(-)$ so that $\mathrm{DR}(\mathcal{O}_X) = \mathbf{C}_X$, where $\mathcal{O}_X$ is the structure sheaf. For a morphism $F : X \to Y$ between varieties, and for an $\mathcal{O}_Y$-module $\mathcal{M}$, F^* denotes the usual $\mathcal{O}$-module pull-back; $F^*\mathcal{M} = \mathcal{O}_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}\mathcal{M}$. If it is necessary to clarify that it denotes the *$\mathcal{O}$-module* pull-back, we write ${}^{O}F^*$ for F^*. For a regular holonomic $\mathcal{D}_Y$-module $\mathcal{M}$, we define two kinds of pull-backs ${}^{D}F^*$ and ${}^{D}F^!$ so that

$$\mathrm{DR}_X({}^{D}F^*\,\mathcal{M})[\dim X] = F^*\,\mathrm{DR}_Y(\mathcal{M})[\dim Y], \text{ and}$$
$$\mathrm{DR}_X({}^{D}F^!\,\mathcal{M})[\dim X] = F^!\,\mathrm{DR}_Y(\mathcal{M})[\dim Y].$$

Then ${}^{D}F^! = {}^{O}LF^*[\dim X - \dim Y]$, where ${}^{O}LF^*$ is the left derived functor of ${}^{O}F^*$. For $\mathcal{D}_{X_i}$-module $\mathcal{M}_i$ $(i = 1,2)$, put $\mathcal{M}_1 \boxtimes \mathcal{M}_2 := \mathcal{O}_{X_1\times X_2} \otimes (pr_1^{-1}\mathcal{M}_1 \otimes_{\mathbf{C}} pr_2^{-1}\mathcal{M}_2)$, where $pr_i : X_1 \times X_2 \to X_i$ $(i = 1,2)$ are the projections, and the tensor product is considered over $pr_1^{-1}\mathcal{O}_{X_1} \otimes_{\mathbf{C}} pr_2^{-1}\mathcal{O}_{X_2}$.

We shall refer to [Gy1, (a,b,c)] etc. simply as (a,b,c) etc.

The basic references of this section are [Ka1], [Ka2] for the D-module theory, and [Ka3], [Me] for the Riemann-Hilbert correspondence. We use the latter so constantly that we often use the material without referring to these literatures. The reader is assumed to be familiar with these materials. (Cf. [Ho, Chapter V].)

5.1. Let X be a non-singular irreducible algebraic variety over the complex number field $\mathbf{C}$. (We always assume an algebraic variety to be separable. For the sake of simplicity, we further assume the quasi-compactness.) Let $\mathcal{O} = \mathcal{O}_X$ be the sheaf of regular functions, and $\mathcal{D} = \mathcal{D}_X$ the sheaf of algebraic differential operators. If X is an affine variety, we put $\mathbf{C}[X] := \Gamma(X, \mathcal{O}_X)$ and $D = D_X := \Gamma(X, \mathcal{D}_X)$. (More generally, for a $\mathbf{C}[X]$-module, say M, we denote the corresponding quasi-coherent sheaf on X by the corresponding script letter, and vice versa.) For any $\mathbf{C}$-algebra A, we put $\mathcal{D}_A = \mathcal{D}_{X,A} := \mathcal{D}_X \otimes_{\mathbf{C}} A$, and $D_A = D_{X,A} := D_X \otimes_{\mathbf{C}} A$. In particular, when A is the polynomial ring $\mathbf{C}[s]$, we often write $\mathcal{D}[s] = \mathcal{D}_X[s]$ and $D[s] = D_X[s]$ for $\mathcal{D}_{\mathbf{C}[s]}$ and $D_{\mathbf{C}[s]}$, respectively.

We need the $\mathbf{C}$-algebra $\mathbf{C}[s,t]$ given in (2.3.5), namely, the $\mathbf{C}$-algebra defined by the relation $ts = (s+1)t$. Put $\mathcal{D}[s,t] = \mathcal{D}_X[s,t] := \mathcal{D}_X \otimes_{\mathbf{C}} \mathbf{C}[s,t]$ and $D[s,t] = D_X[s,t] := D_X \otimes_{\mathbf{C}} \mathbf{C}[s,t]$.

In (5.2)–(5.8), we shall work with D-modules assuming X to be an affine variety. Since the category of D_X-modules is equivalent to that of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-modules if X is affine, we can understand the results of (5.2)–(5.8) as those concerning $\mathcal{D}_X$-modules. Thus we can get the same results even if X is not affine as far as the assertion is of local nature.

5.2. Let N be a $D[s,t]$-module, and $b(s,N)$ the (monic) minimal polynomial of $s \in \mathrm{End}_D(N/tN)$. (Possibly $b(s,N)=0$. We define $b(s,\mathcal{N})$ for a $\mathcal{D}[s,t]$-module $\mathcal{N}$ in the same way.) Put

$$A_+(N) := \{\alpha \in \mathbf{C} \mid b(\alpha+j,N) \neq 0 \text{ for } j=0,1,2,\cdots\}, \text{ and}$$
$$A_-(N) := \{\alpha \in \mathbf{C} \mid b(\alpha-j,N) \neq 0 \text{ for } j=1,2,\cdots\}.$$

Put $N(\alpha) := N/(s-\alpha)N$.

Lemma 5.3. *Assume that $t : N \to N$ is injective and $b(\alpha+j,N) \neq 0$ $(0 \leq j \leq l-1)$. Then $t^l : N \to N$ induces an isomorphism $N(\alpha+l) \to N(\alpha)$.*

Proof. We may assume that $l=1$. Put $b(s) := b(s,N)$. Since $b(s)N \subset tN$, we get the natural $D[s]$-homomorphisms $tN \to N \to (tN)[b(s)^{-1}] \to N[b(s)^{-1}]$, and the natural D-homomorphisms

$$\frac{tN}{(s-\alpha)tN} \xrightarrow{A} \frac{N}{(s-\alpha)N} \xrightarrow{B} \frac{(tN)[b(s)^{-1}]}{(s-\alpha)((tN)[b(s)^{-1}])} \xrightarrow{C} \frac{N[b(s)^{-1}]}{(s-\alpha)(N[b(s)^{-1}])}.$$

Since $b(\alpha) \neq 0$, CB and BA are isomorphisms, and consequently, A, B, C are all isomorphisms. Since $t : N \to tN$ is assumed to be an isomorphism,

$$N(\alpha+1) = \frac{N}{(s-\alpha-1)N} \xrightarrow[t]{\simeq} \frac{tN}{(s-\alpha)tN} \xrightarrow[A]{\simeq} \frac{N}{(s-\alpha)N} = N(\alpha).$$

5.4. Assume that X is a connected non-singular affine variety, and $0 \neq f \in \mathbf{C}[X]$. Let N be a $D[s,t]$-module satisfying the following conditions.
(5.4.1) *N is a subholonomic D_X-module.*
(5.4.2) *$N \subset N[f^{-1}]$. (Cf. (2.1.7) for $N[f^{-1}]$.)*
(5.4.3) *If we extend the $D[s,t]$-module structure of N to $N[f^{-1}]$ by $s(f^{-m}u) := f^{-m}(su)$ and $t(f^{-m}u) := f^{-m}(tu)$ $(m \in \mathbf{Z},\ u \in N)$, then $t \in \mathrm{Aut}(N[f^{-1}])$ and $N[t^{-1}]$ $(:= \bigcup_{m\geq 0} t^{-m}N) = N[f^{-1}]$.*
(5.4.4) *$N[f^{-1}]$ is flat over $\mathbf{C}[s]$ (i.e., $\mathbf{C}[s]$-torsion free).*
Since $t : N[f^{-1}] \to N[f^{-1}]$ is assumed to be injective (5.4.3), and $N \subset N[f^{-1}]$ (5.4.2), we can see that N/tN is holonomic by (5.4.1). Similarly, using (5.4.4), we can see that $N(\alpha) = N/(s-\alpha)N$ is holonomic. In particular, $\dim_{\mathbf{C}} \mathrm{End}_D(N/tN) < \infty$, and $b(s,N) \neq 0$.

Lemma 5.5. *Let $\alpha \in \mathbf{C}$. If $b(\alpha,N) \neq 0$, then $(s-\alpha)N + tN = N$ and $(s-\alpha)N \cap tN = (s-\alpha)tN$.*

Proof. Put $b(s) := b(s,N)$, and take $c(s), d(s) \in \mathbf{C}[s]$ so that $c(s)b(s)+d(s)(s-\alpha) = 1$. The action of the left hand side on N/tN is the same as the action of $d(s)(s-\alpha)$. Hence $s-\alpha \in \mathrm{Aut}_D(N/tN)$, from which we get the result.

Lemma 5.6. *(Cf. (2.8.5).) Assume (5.4.1)–(5.4.4). Let $[N(\alpha)]$ be the element in the Grothendieck group of holonomic D-modules. Then $[N(\alpha)]$ depends only on (α mod $\mathbf{Z}$).*

Lemma 5.7. *Assume* (5.4.2)–(5.4.4). *If* $\alpha \in A_{-}(N)$, *then* $N(\alpha) = N(\alpha)[f^{-1}]$.

Proof. Let $m \in \mathbf{Z}_{\geq 0}$. From the diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "s-\alpha+m"] & N \arrow[r] & N(\alpha-m) \arrow[r] & 0 \\ & t^{-m}N \arrow[u, "t^m", "\simeq"'] \arrow[r, "s-\alpha"] & t^{-m}N \arrow[u, "t^m", "\simeq"'] & & \end{tikzcd}$$

together with (5.3), we get the exact sequence

$$0 \to t^{-m}N \xrightarrow{s-\alpha} t^{-m}N \to N(\alpha) \to 0.$$

Taking $\varinjlim_{m}$, we get

$$0 \to N[t^{-1}] \xrightarrow{s-\alpha} N[t^{-1}] \to N(\alpha) \to 0.$$

Since $N[t^{-1}] = N[f^{-1}]$, and since $0 \to N[f^{-1}] \xrightarrow{s-\alpha} N[f^{-1}] \to N(\alpha)[f^{-1}] \to 0$ is exact, we get the desired result.

Remark 5.7.1. The above lemma generalizes [KK, Lemma 2.3].

Lemma 5.8. *Let* N, N' *and* N'' *be* $D_X[s,t]$*-modules satisfying* (5.4.1)–(5.4.4), *and let* $N' \xrightarrow{B} N \xrightarrow{C} N''$ *be* $D_X[s,t]$*-homomorphisms which induce an exact sequence*

$$0 \to N'[f^{-1}] \xrightarrow{B} N[f^{-1}] \xrightarrow{C} N''[f^{-1}] \to 0. \tag{5.8.1}$$

Then the sequence

$$0 \to N'(\alpha) \xrightarrow{B_\alpha} N(\alpha) \xrightarrow{C_\alpha} N''(\alpha) \to 0 \tag{5.8.2}$$

induced by B *and* C *is exact if* $\alpha \in A_\epsilon(N) \cap A_\epsilon(N') \cap A_\epsilon(N'')$ $(\epsilon = +$ *or* $-)$.

Proof. By (5.3), we may and do assume that $\mathrm{Re}(\alpha) \gg 0$ or $\ll 0$.

First, we prove the surjectivity of C_α. Let $N'' = \sum_i Du_i''$ (finite sum). Take $u_i \in N$ and $m \in \mathbf{Z}_{\geq 0}$ so that $C(t^{-m}u_i) = u_i''$ for all i. Take $\alpha \in \mathbf{C}$ so that

$$b(\alpha + j - 1, N'') \neq 0 \quad (1 \leq j \leq m). \tag{5.8.3}$$

Since $t^m u_i'' \in C(N)$, $C(N) + (s-\alpha)N'' \supset t^m N'' + (s-\alpha)N''$. Since our present purpose is to prove that $C(N) + (s-\alpha)N'' \supset N''$, it suffices to show that

$$t^j N'' + (s-\alpha)N'' \supset t^{j-1}N'' + (s-\alpha)N'' \tag{5.8.4}$$

for $1 \leq j \leq m$. The left hand side of (5.8.4) contains $t^{j-1}(tN'' + (s-\alpha-j+1)N'')$, which is equal to $t^{j-1}N''$ by (5.5) and (5.8.3). Thus we get (5.8.4).

Next, we prove that $\ker C_\alpha \subset \operatorname{image} B_\alpha$, or equivalently that $C^{-1}((s-\alpha)N'') \cap N \subset N' + (s-\alpha)N$. (Here and below, we regard $N' \subset N$.) Let $C^{-1}((s-\alpha)N'') \cap N = \sum_i Dv_i$ (finite sum). (Since D is left noetherian, every D-submodule of N is finitely generated.) Let u''_i, u_i and m be as in the first step. Then $C(v_i) = (s-\alpha)\sum_j P_{ij}u''_j$ for some $P_{ij} \in D$. Since $C(t^{-m}\sum_j P_{ij}u_j) = \sum_j P_{ij}u''_j$,

$$v_i - (s-\alpha)t^{-m}\sum_j P_{ij}u_j \in N'[f^{-1}] \cap t^{-m}N = N'[t^{-1}] \cap t^{-m}N, \tag{5.8.5}$$

by the exactness of (5.8.1), and by (5.4.3). Since $N'[t^{-1}] \cap t^{-m}N$ ($\subset t^{-m}N$) is a finitely generated D-module, we can take $l \geq m$ so that

$$t^l(N'[t^{-1}] \cap t^{-m}N) \subset N'. \tag{5.8.6}$$

By (5.8.5) and (5.8.6), we get

$$t^j v_i \in N' + (s-\alpha+j)N \tag{5.8.7$_j$}$$

for $j = l$. Take $\alpha \in \mathbf{C}$ so that

$$\begin{aligned} &b(\alpha-j, N') \neq 0 \quad (1 \leq j \leq l), \text{ and} \\ &b(\alpha-j, N) \neq 0 \quad (1 \leq j \leq l). \end{aligned} \tag{5.8.8}$$

By (5.8.8) and (5.5), we get

$$\begin{aligned} &(s-\alpha+j)N' + tN' = N' \quad (1 \leq j \leq l), \text{ and} \\ &(s-\alpha+j)N \cap tN = (s-\alpha+j)tN \quad (1 \leq j \leq l). \end{aligned} \tag{5.8.9}$$

By (5.8.7$_l$) and (5.8.9), we get

$$\begin{aligned} &t(t^{l-1}v_i) \in tN' + (s-\alpha+l)N, \text{ and hence} \\ &t(t^{l-1}v_i) \in tN' + (s-\alpha+l)tN, \end{aligned}$$

from which follows (5.8.7$_{l-1}$). Repeating this procedure, we finally get (5.8.7$_0$), which is the desired result.

Last, we prove the injectivity of B_α. If $\mathrm{Re}(\alpha) \ll 0$, then (5.8.2) follows from (5.8.1) and (5.7). (In fact,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N'[f^{-1}] \arrow[r] \arrow[d, "s-\alpha"] & N[f^{-1}] \arrow[r] \arrow[d, "s-\alpha"] & N''[f^{-1}] \arrow[r] \arrow[d, "s-\alpha"] & 0 \\
0 \arrow[r] & N'[f^{-1}] \arrow[r] \arrow[d] & N[f^{-1}] \arrow[r] \arrow[d] & N''[f^{-1}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N'(\alpha)[f^{-1}] \arrow[r] \arrow[d] & N(\alpha)[f^{-1}] \arrow[r] \arrow[d] & N''(\alpha)[f^{-1}] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the first two horizontal sequences and three vertical sequences being exact, the third horizontal sequence is also exact, which is nothing but (5.8.2) by (5.7).) Assume that $\mathrm{Re}(\alpha) \gg 0$. By (5.6),

$$\begin{aligned}
[\ker B_\alpha] &= [N'(\alpha)] - [N(\alpha)] + [N''(\alpha)] \\
&= [N'(\alpha - k)] - [N(\alpha - k)] + [N''(\alpha - k)]
\end{aligned}$$

for any $k \in \mathbf{Z}$. Taking k so that $\mathrm{Re}(\alpha - k) \ll 0$, we can see that $[\ker B_\alpha] = 0$, i.e., B_α is injective.

5.9. D-Modules $\mathcal{D}_X[s](f^s\underline{u}|V)$ and $\mathcal{D}_X(f^\alpha\underline{u}|V)$. Let X be a connected non-singular variety over $\mathbf{C}$ and $0 \neq f \in \Gamma(X, \mathcal{O}_X)$. (We do not assume X to be affine.) Let $X_0 := X \setminus f^{-1}(0)$, V be a Zariski open subset of X_0, $\mathcal{M}$ a coherent $\mathcal{D}_V$-module, and $\underline{u} = (u_1, \cdots, u_p)$ a p-tuple of elements of $\Gamma(V, \mathcal{M})$. Consider the left $\mathcal{D}_X[s]$-submodule $\mathcal{I}$ of $\mathcal{D}_X[s]^p$ consisting of $(P_1(s), \cdots, P_p(s)) \in \mathcal{D}_X[s]^p$ such that $\sum_{i=1}^p (f^{m-s}P_i(s)f^s)u_i = 0$ holds in $\mathbf{C}[s] \otimes_{\mathbf{C}} \mathcal{M}$ whenever $m \in \mathbf{Z}$ is sufficiently large. Put $\mathcal{N} := \mathcal{D}_X[s]^p/\mathcal{I}$. Denote by $(f^s\underline{u})_i|V$ the element $((0, \cdots 0, 1, 0, \cdots, 0) \bmod \mathcal{I})$, where 1 appears at the i-th place. Then $\mathcal{N} = \sum_{i=1}^p \mathcal{D}_X[s]((f^s\underline{u})_i|V)$. Put $f^s\underline{u}|V := ((f^s\underline{u})_1|V, \cdots, (f^s\underline{u})_p|V)$. We write $\mathcal{N} = \mathcal{D}_X[s](f^s\underline{u}|V)$. For a complex number α, put $\mathcal{N}(\alpha) := \mathcal{N}/(s-\alpha)\mathcal{N}$, and $f^\alpha\underline{u}|V = ((f^\alpha\underline{u})_1|V, \cdots, (f^\alpha\underline{u})_p|V) := (f^s\underline{u}|V \bmod (s-\alpha)\mathcal{N})$. Then $\mathcal{N}(\alpha) = \mathcal{D}_X(f^\alpha\underline{u}|V) = \sum_{i=1}^p \mathcal{D}_X((f^\alpha\underline{u})_i|V)$. It is easy to see that

$$(\mathcal{D}_X[s](f^s\underline{u}|V))\,|U = \mathcal{D}_U[s](f^s\underline{u}|V \cap U) \tag{5.9.1}$$

for any Zariski open set U of X, and

$$\mathcal{D}_X[s](f^s(g\underline{u})|V) = \mathcal{D}_X[s](g(f^s\underline{u}|V)) \tag{5.9.2}$$

for any $g \in \mathbf{C}[X]$. We shall denote (5.9.2) by $\mathcal{D}_X[s](f^s g\underline{u}|V) = \mathcal{D}_X[s](gf^s\underline{u}|V)$. If X is an affine variety, we define $D_X[s](f^s\underline{u}|V)$ and $D_X(f^\alpha\underline{u}|V)$ in the same way as above. Then

$$\Gamma(X, \mathcal{D}_X[s](f^s\underline{u}|V)) = D_X[s](f^s\underline{u}|V), \text{ and} \tag{5.9.3}$$

$$\Gamma(X, \mathcal{D}_X(f^\alpha\underline{u}|V)) = D_X(f^\alpha\underline{u}|V). \tag{5.9.4}$$

If $V = X_0$, we sometimes write $f^s\underline{u}$ and $f^\alpha\underline{u}$ for $f^s\underline{u}|X_0$ and $f^\alpha\underline{u}|X_0$. It is easy to see that

(5.9.5) *f is not a zero divisor of $\mathcal{D}_X[s](f^s\underline{u}|V)$*

and

(5.9.6) *$\mathcal{D}_X[s](f^s\underline{u}|V)$ is $\mathbf{C}[s]$-flat (i.e., $\mathbf{C}[s]$-torsion free).*

Lemma 5.10. *Let $\mathcal{D}_{X_0}\underline{v}$ and $\mathcal{D}_{X_0}\underline{w}$ be two coherent $\mathcal{D}_{X_0}$-modules with finite global generator systems $\underline{v}$ and $\underline{w}$, respectively. Assume that $b(s, \mathcal{D}_X[s](f^s\underline{v})) \not\equiv 0$ and $b(s, \mathcal{D}_X[s](f^s\underline{w}))$ $\not\equiv 0$. (1) Then a $\mathcal{D}_{X_0}$-homomorphism $\varphi : \mathcal{D}_{X_0}\underline{v} \to \mathcal{D}_{X_0}\underline{w}$ induces canonically a $\mathcal{D}_X$-homomorphism $\varphi_\alpha : \mathcal{D}_X(f^\alpha\underline{v}) \to \mathcal{D}_X(f^\alpha\underline{w})$ if $|\mathrm{Re}(\alpha)| \gg 0$. (2) In particular, $\mathcal{D}_X(f^\alpha\underline{v}) \simeq \mathcal{D}_X(f^\alpha\underline{w})$ if $\mathcal{D}_{X_0}\underline{v} \simeq \mathcal{D}_{X_0}\underline{w}$ and $|\mathrm{Re}(\alpha)| \gg 0$.*

Proof. It is enough to construct a homomorphsim φ_α locally but canonically. Hence we may assume from the beginning that X is an affine variety. Let $\underline{v} = (v_1, \cdots)$, $\underline{w} = (w_1, \cdots)$, $\varphi(v_i) = f^{-k}\sum_j P_{ij}w_j$ with $P_{ij} \in D_X$ and with $k \in \mathbf{Z}_{\geq 0}$, $l \geq k + \max_j(\mathrm{ord}\, P_{ij})$, and $f^{s-k}P_{ij} = Q_{ij}(s)f^{s-l}$ (multiplication of operators) with $Q_{ij}(s) \in D_X[s]$. Define $\varphi_s : D_X[s](f^s\underline{v}) \to D_X[s](f^{-l} \cdot f^s\underline{w})$ by

$$\varphi_s\left(\sum_i R_i(s)(f^s\underline{v})_i\right) := \sum_{i,j} R_i(s)Q_{ij}(s)\left(f^{-l} \cdot (f^s\underline{w})_j\right).$$

Then it is easy to see that φ_s is well defined and that φ_s is independent of the choice of P_{ij}, k and l. Consider the D_X-homomorphism $\varphi_\alpha : D_X(f^\alpha\underline{v}) \to D_X(f^{\alpha-l}\underline{w})$ induced by φ_s. If $|\mathrm{Re}(\alpha)| \gg 0$, then φ_α can be regarded as a D_X-homomorphism $\varphi_\alpha : D_X(f^\alpha\underline{v}) \to D_X(f^\alpha\underline{w})$ by (5.3). Thus we obtain the desired homomorphsim.

Remark 5.11. We can not expect that $\mathcal{D}_X(f^\alpha\underline{u}|V)$ is determined by $\mathcal{D}_V\underline{u}$, even if we impose a "reasonably strong" assumptions on $\underline{u}$. Instead, denoting by $j_V : V \to X_0$ the inclusion mapping, and regarding $\underline{u}$ as global sections in $\Gamma(X_0, (j_V)_*(\mathcal{D}_V\underline{u}))$, we can see from (5.10) that $\mathcal{D}_X(f^\alpha\underline{u}|V)$ is determined by $\mathcal{D}_{X_0}\underline{u}$.

For example, let $X := \mathbf{C}$, $f(x) := x$, $V := \mathbf{C} \setminus \{0, 1\}$, $u_k := (x-1)^k$ $(k \in \mathbf{Z})$, and let α be an arbitrary integer. Then $\mathcal{D}_V u_k = \mathcal{O}_V$, but

$$\mathrm{DR}(\mathcal{D}_{\mathbf{C}^\times}(f^\alpha u_k|V)) = \begin{cases} R(j_V)_*\mathbf{C}, & \text{if } k < 0 \\ \mathbf{C}, & \text{if } k \geq 0. \end{cases}$$

Lemma 5.12. *(Keep notation of (5.9).) Assume that $\mathcal{D}_V\underline{u}$ is holonomic and the inclusion mapping $j_V : V \to X_0$ is an affine morphism. Then there exists a non-zero polynomial $b(s) \in \mathbf{C}[s]$ such that $b(s)\mathcal{D}_X[s](f^s\underline{u}|V) \subset \mathcal{D}_X[s](f^{s+1}\underline{u}|V)$.*

Proof. Since $\Gamma(X_0, (j_V)_*\mathcal{M}) = \Gamma(V, \mathcal{M}) \ni u_i$, we can define $\mathcal{D}_X[s](f^s\underline{u}) = \mathcal{D}_X[s](f^s\underline{u}|X_0$ considering u_i's as sections of $(j_V)_*\mathcal{M}$ on X_0. It is easy to see that

$$\mathcal{D}_X[s](f^s\underline{u}) = \mathcal{D}_X[s](f^s\underline{u}|V). \tag{5.12.1}$$

Since $(j_V)_*\mathcal{M}$ is holonomic, the remainder of the proof goes in the same way as [Ka2, Theorem 2.7].

Lemma 5.13. *Let $X_0 \supset V_1 \supset V_2 \neq \phi$ be Zariski open subsets such that the inclusion mappings $V_i \to X_0$ are affine, $\mathcal{M}$ a coherent $\mathcal{D}_{V_1}$-module, $u_i \in \Gamma(V_1, \mathcal{M})$, and $\underline{u} = (u_1, u_2, \cdots)$ (finite set). Then the following two conditions are equivalent.*
(1) *For any affine open subset U of V_1, $D_U\underline{u} \to D_{V_2\cap U}\underline{u}$ is injective.*
(2) $\mathcal{D}_X[s](f^s\underline{u}|V_1) \xrightarrow{\simeq} \mathcal{D}_X[s](f^s\underline{u}|V_2)$.

Proof. For the sake of simplicity, we assume that $\underline{u}$ consists of only one section $u \in \Gamma(V_1, \mathcal{M})$. Let $\mathcal{I}_i \subset \mathcal{D}_X[s]$ (resp. $\mathcal{J} \subset \mathcal{D}_{V_1}$) be the annihilator of $(f^s u|V_i)$ (resp. u). Then the following conditions are equivalent to (2).

(3) For any affine open subset U of X,

$$D_U[s](f^s u|V_1 \cap U) \xrightarrow{\simeq} D_U[s](f^s u|V_2 \cap U).$$

(4) For any affine open set U of X, for any $P(s) \in D_U[s]$, and for any $m \gg 0$, $(f^{m-s}P(s)f^s)u$ vanishes on $V_1 \cap U$ if it vanishes on $V_2 \cap U$.

(5) For any affine open subset U of X, and for any $P \in D_U$, Pu vanishes on $V_1 \cap U$ if it vanishes on $V_2 \cap U$.

(6) For any affine open subset U of V_1, and for any $P \in D_U$, $P \in \Gamma(U, \mathcal{J})$ if $P|V_2 \cap U \in \Gamma(V_2 \cap U, \mathcal{J})$.

Let U be as in (6). Since U and $V_2 \cap U$ are affine, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(U,\mathcal{J}) & \longrightarrow & \Gamma(U,\mathcal{D}) & \longrightarrow & \Gamma(U,\mathcal{D}u) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \Gamma(V_2\cap U,\mathcal{J}) & \longrightarrow & \Gamma(V_2\cap U,\mathcal{D}) & \longrightarrow & \Gamma(V_2\cap U,\mathcal{D}u) & \longrightarrow & 0
\end{array}$$

with exact rows. From this diagram, we can see that (6) is equivalent to the injectivity of ρ, which is equivalent to (1).

Lemma 5.14. *Let notation and assumption be as in* (5.13). *Assume further that $\mathcal{M} = \mathcal{D}_{V_1}\underline{u} = \sum_i \mathcal{D}_{V_1}u_i$ and $\mathcal{M}$ is a simple $\mathcal{D}_{V_1}$-module, i.e., a non-zero coherent $\mathcal{D}_{V_1}$-module without proper coherent $\mathcal{D}_{V_1}$-submodules.*

(1) *For any open subset W of V_1, $\mathcal{M}|W$ is a simple $\mathcal{D}_W$-module, or* $= 0$.

(2) *The support of $\mathcal{M}$ is an irreducible variety.*

(3) *If* $\mathcal{M}|W \neq 0$, *then* $\Gamma(V_1, \mathcal{M}) \to \Gamma(W, \mathcal{M})$ *is injective.*

(4) *Exactly one of the following holds.*

(a) $\mathcal{M}|V_2 = 0$.

(b) $\mathcal{D}_X[s](f^s\underline{u}|V_1) \xrightarrow{\simeq} \mathcal{D}_X[s](f^s\underline{u}|V_2)$.

Proof. (1) Assume that $\mathcal{M}|W$ has a proper coherent $\mathcal{D}_W$-submodule $\mathcal{N}_W$. By [Ho, (I, 3.3.2, (ii))], there exists a coherent $\mathcal{D}_{V_1}$-submodule $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}|W = \mathcal{N}_W$. This is absurd.

(2) Assume that $Y := \operatorname{supp} \mathcal{M}$ is not irreducible. Take an affine open subset W of V_1 such that $Z := \operatorname{supp}(\mathcal{M}|W)$ is not irreducible. Put $A := \mathbf{C}[W]$. By [Bo, Chapter 4, §1, Theorem 1], $A\underline{u}$ has an A-submodule, say M', whose support is irreducible. Since $\operatorname{supp}(D_W M') = \operatorname{supp}(M') \subset \operatorname{supp}(A\underline{u}) = \operatorname{supp}(D_W\underline{u}) = Z$, $\mathcal{D}_W M'$ is a proper coherent $\mathcal{D}_W$-submodule of $\mathcal{M}|W$. This contradicts (1).

(3) Assume that $0 \neq u \in \Gamma(V_1, \mathcal{M})$ and $u|W = 0$. Since $\mathcal{M}$ is simple, $\mathcal{M} = \mathcal{D}_{V_1} u$. Hence $\mathcal{M}|W = 0$.

(4) It suffices to prove (b) assuming that (a) does not hold. (Obviously (a) and (b) can not hold in the same time.) By (5.13), it suffices to prove the injectivity of $\varphi : D_U\underline{u} \to D_{V_2 \cap U}\underline{u}$ for any affine open subset U of V_1. Assume contrary that φ is not injective for some $U \subset V_1$. Then $D_U\underline{u} \neq 0$ and $Y \cap U \neq \phi$ ($Y = \operatorname{supp} M$). By (1), $D_U\underline{u}$ is simple. Since $\ker \varphi \neq 0$, $\varphi = 0$. Hence $\underline{u} = 0$ in $D_{V_2 \cap U}\underline{u}$, $D_{V_2 \cap U}\underline{u} = 0$, and $Y \cap V_2 \cap U = \phi$. Since we are assuming (a) does not hold, $Y \cap V_2 \neq \phi$. However these relations contradict the irreducibility of Y (see (2)).

Lemma 5.15. *(Keep notation of* (5.9) *with* $V = X_0$.*) Let* t *be the coordinate of* $\mathbf{C}^\times$, *and* $pr : X_0 \times \mathbf{C}^\times \to X_0$ *the projection. Then by* $s \leftrightarrow -\partial_t t$ *and* $f^s\underline{u} \leftrightarrow \delta(t - f(x))\underline{u}(x)$, *we obtain isomorphisms* $\mu : pr_* \mathcal{D}_{X_0 \times \mathbf{C}^\times} = \mathcal{D}_{X_0} \otimes_{\mathbf{C}} \mathbf{C}[-\partial_t t, t, t^{-1}] \xrightarrow{\simeq} \mathcal{D}_{X_0}[s, t, t^{-1}]$ *and* $\mu : pr_*(\mathcal{D}_{X_0 \times \mathbf{C}^\times}(\delta(t - f(x))\underline{u}(x))) \xrightarrow{\simeq} \mathcal{D}_{X_0}[s, t, t^{-1}](f^s\underline{u}) = \mathcal{D}_{X_0}[s](f^s\underline{u})$. *(We have written* $\underline{u}(x)$ *for* $\underline{u}$ *in some places in order to indicate that they are sections on* X_0.*)*

Proof. We may assume X_0 to be an affine variety. For the sake of simplicity, we assume that $\underline{u} = (u_1, \cdots)$ consists of only one section u. Define $\mathbf{C}$-algebra homomorphisms $\Phi : D_{X_0} \to D_{X_0}[\partial_t t]$ and $\Psi : D_{X_0} \to D_{X_0}[s]$ by $\Phi(\partial_{x_i}) = \partial_{x_i} + (\partial_t t)(\log f)_{x_i}$, $\Psi(\partial_{x_i}) = \partial_{x_i} - s(\log f)_{x_i}$, and $\Phi(x_i) = \Psi(x_i) = x_i$. Then

$$\Phi(P)(\delta(t - f(x))u(x)) = \delta(t - f(x))(Pu(x)),$$
$$\Psi(P)(f(x)^s u(x)) = f(x)^s(Pu(x)), \text{ and}$$
$$\mu \circ \Phi = \Psi.$$

Hence for $P_{ij} \in D_{X_0}$ $(i, j \in \mathbf{Z}_{\geq 0})$,

$$\sum_{i,j\geq 0}(-\partial_t t)^i(t-f(x))^j\Phi(P_{ij})(\delta(t-f(x))u(x))=0$$
$$\Leftrightarrow \sum_{i,j\geq 0}(-\partial_t t)^i(t-f(x))^j\delta(t-f(x))P_{ij}u=0$$
$$\Leftrightarrow \sum_{i\geq 0}(-\partial_t t)^i\delta(t-f(x))P_{i0}u=0$$
$$\Leftrightarrow P_{i0}u=0 \text{ for all } i\geq 0.$$

(The last '$\Leftrightarrow$' follows from the identity of the form $(-\partial_t t)^i\delta(t-f(x))=\sum_{\nu=0}^{i}c_\nu f^\nu\cdot\delta^{(\nu)}(t-f(x))$, with $c_i=1$ and $c_\nu\in\mathbf{C}$.) Using this relation together with the similar relation for $f(x)^s u(x)$, we get

$$R(\delta(t-f(x))u(x))=0\Leftrightarrow \mu(R)(f(x)^s u(x))=0$$

for $R\in D_{X_0\times\mathbf{C}^\times}$.

Theorem 5.16. *Let V be a Zariski open subset of $X_0=X\setminus f^{-1}(0)$ such that inclusion mapping $j_V:V\to X_0$ is an affine morphism, let $j:X_0\to X$ denote the inclusion mapping, and assume that $\mathcal{D}_V\underline{u}=\sum_{i=1}^{p}\mathcal{D}_V u_i$ is a regular holonomic $\mathcal{D}_V$-module. Then*
(1) *$\mathcal{D}_X(f^\alpha\underline{u}|V)$ is a regular holonomic $\mathcal{D}_X$-module,*
(2) *$\mathrm{DR}(\mathcal{D}_X(f^\alpha\underline{u}|V))=Rj_*\mathrm{DR}(\mathcal{D}_{X_0}(f^\alpha\underline{u}|V))$ if $\alpha\in A_-(\mathcal{D}_X[s](f^s\underline{u}|V))$, and*
(3) *$\mathrm{DR}(\mathcal{D}_X(f^\alpha\underline{u}|V))=j_!\mathrm{DR}(\mathcal{D}_{X_0}(f^\alpha\underline{u}|V))$ if $\alpha\in A_+(\mathcal{D}_X[s](f^s\underline{u}|V))$.*

(Recall that $\mathrm{DR}(-)=\mathrm{DR}_X(-)=R\underline{\mathrm{Hom}}_{\mathcal{D}^{an}}(\mathcal{O}^{an},-)$, where $\mathcal{O}^{an}=\mathcal{O}_X^{an}$ is the sheaf of holomorphic functions, $\mathcal{D}^{an}=\mathcal{D}_X^{an}:=\mathcal{O}^{an}\otimes_{\mathcal{O}}\mathcal{D}$, and $\underline{\mathrm{Hom}}$ denotes the sheaf of local homomorphisms. Cf. (2.8).)

Remark 5.17. In the above theorem, the regularity assumption for $\mathcal{D}_V\underline{u}$ can not be removed even for (2) or (3). For example, let $X:=\mathbf{C}$, $f(x):=x$, $\partial:=d/dx$, $\mathcal{M}:=\mathcal{D}/\mathcal{D}(x^{m+1}\partial-1)$ $(m\in\mathbf{Z}_{>0})$, u the class of $1\in\mathcal{D}$ in $\mathcal{M}$, and $\mathcal{N}:=\mathcal{D}[s](x^s u)$. Then

$$(-x^{m-1}(s+1)+x^m\partial)(x^{s+1}u)=x^s u,$$

$b(s,\mathcal{N})=1$, $A_\pm(\mathcal{N})=\mathbf{C}$, and $\mathcal{D}(x^\alpha u)=\mathcal{D}(x^\alpha u)[x^{-1}]$ for any $\alpha\in\mathbf{C}$ by (5.7). However $\sum_{k\geq 0}(-1)^k H^k(\mathrm{DR}(\mathcal{D}_X(x^\alpha u)))_0=-m$ and $\sum_{k\geq 0}(-1)^k H^k(Rj_*\mathrm{DR}(\mathcal{D}_{X_0}(x^\alpha u)))_0=0$. (Cf. [Ka4, Chapter 6].) In particular,

$$[\mathrm{DR}(\mathcal{D}_X(x^\alpha u))]\neq[Rj_*\mathrm{DR}(\mathcal{D}_{X_0}(x^\alpha u))]$$

even in the Grothendieck group of perverse sheaves.

The first assertion of (5.16) is easy. (Since $(j_V)_*$ preserves the regularity, we may assume $V=X_0$ by (5.12.1). If $p=1$, (1) is already settled in (2.8.6). Then we can prove generally, noting that $\mathcal{D}_X(f^\alpha\underline{u})$ is a quotient of $\bigoplus_{i=1}^{p}\mathcal{D}_X(f^\alpha u_i)$.) We shall prove (2) and (3) by reducing successively the proof to easier cases. The second assertion is proved in (5.23). The proof of the third assertion is long, and will come to an end in (5.38).

5.18. First reduction. By (5.3), we may assume that
(A1) $\mathrm{Re}(\alpha) \ll 0$ *(resp.* $\gg 0$*) in (5.16, (2)) (resp. (5.16, (3))).*

5.19. Second reduction. Since our problem is local by (5.9.1), we may assume that
(A2) X *is an affine variety.*
Then
(A2′) V *is an affine variety.*
By the first note of the same volume, we may also assume that
(A2″) *there exists* $g \in \Gamma(X, \mathcal{O}_X)$ *such that* $V = X_0 \setminus g^{-1}(0)$.

5.20. Third reduction. Put $X^\sharp := X^\flat \times \mathbf{C}$, $f^\sharp(x,y) := f^\flat(x)y$, $V^\sharp := V^\flat \times \mathbf{C}$, $\mathcal{M}^\sharp := \mathcal{M}^\flat \boxtimes \mathcal{O}_{\mathbf{C}^\times}$, $u_i^\sharp := u_i^\flat \boxtimes 1$, $\underline{u}^\sharp := (u_1^\sharp, \cdots, u_p^\sharp)$, and $\underline{u}^\flat := (u_1^\flat, \cdots, u_p^\flat)$. By '$(-) \boxtimes \mathrm{DR}(\mathcal{D}_{\mathbf{C}} y^\alpha)$' and conversely by the restriction to $X^\flat \times \{1\}$ $(\subset X^\sharp)$, (5.16) for $X^\sharp$, $f^\sharp, \cdots$ follows from that for $X^\flat$, $f^\flat \cdots$, and vice versa. Thus we may assume that
(A3) X, f, V, $\mathcal{M}$, $\underline{u}$ *can be obtained as* $X^\sharp$ *etc. from those for* $X^\flat$ *etc.*
Note that (A3) implies that
(A3′) *there exists* $E \in \Gamma(X, \mathcal{D}_X)$ *such that* $Eu_i = 0$ *and* $Ef = f$.

5.21. Fourth reduction. Since $(j_V)_*\mathcal{M}$ is regular holonomic, we may assume that
(A4) $V = X_0$.
(Cf. (5.12.1).)

5.22. Fifth reduction. Assume (A1)–(A4). By [Ka2, Theorem 2.5], there exists an affine open neighbourhood X' of $f^{-1}(0)$ such that $\mathcal{D}_X(f^s\underline{u})|X'$ is subholonomic. Since (5.16) is obvious outside of $f^{-1}(0)$, we may assume that $\mathcal{D}_X(f^s\underline{u})$ is subholonomic, replacing X with X'. By (A3′), $\mathcal{D}_X(f^s\underline{u}) = \mathcal{D}_X[s](f^s\underline{u})$. Hence we may assume, without destroying (A1)–(A4), that
(A5) $\mathcal{N} := \mathcal{D}_X[s](f^s\underline{u}|V)$ *is subholonomic.*
(Indeed, we may assume that this procedure preserves (A1)–(A4) except for (A3). If X etc. are obtained as $X^\sharp$ etc. from $X^\flat$ etc. as in (5.20), then first shrink $X^\flat$, and next apply '$^\flat \to ^\sharp$'. Then we can assume (A5) keeping (A3) as well.) Then if we define the t-action on $\mathcal{N}$ by $t(P(s)(f^s u_i)) = P(s+1)(f \cdot f^s u_i)$ for $P(s) \in \mathcal{D}_X[s]$, $\mathcal{N}$ becomes a $\mathcal{D}_X[s,t]$-module satisfying (5.4.1)–(5.4.4). (Indeed, (A5) is (5.4.1), (5.9.5) yields (5.4.2), (5.4.3) can be directly proved, and (5.9.6) is (5.4.4).)

5.23. Proof of (5.16, (2)). By (5.7), $\mathcal{D}_X(f^\alpha\underline{u}) = \mathcal{N}(\alpha) = \mathcal{N}(\alpha)[f^{-1}]$ if $\mathrm{Re}(\alpha) \ll 0$. Hence we get the result by (5.16, (1)).

5.24. Sixth reduction. Let us prove (5.16, (3)) when the length of $\mathcal{D}_V u$ is l, assuming that $l > 1$ and (5.16, (3)) is already proved when the length is $< l$. If V and $\underline{u}$ are obtained as $V^\sharp$ and $\underline{u}^\sharp$ from some $V^\flat$ and $\underline{u}^\flat$ as in (5.20), then consider a proper coherent $\mathcal{D}_{V^\flat}$-submodule $\mathcal{D}_{V^\flat}\underline{u}'^\flat$ of $\mathcal{D}_{V^\flat}\underline{u}^\flat$, put $\mathcal{D}_{V^\flat}\underline{u}''^\flat := \mathcal{D}_{V^\flat}\underline{u}^\flat / \mathcal{D}_{V^\flat}\underline{u}'^\flat$, apply '$^\flat \to ^\sharp$', and denote the resulting $\mathcal{D}$-modules by $\mathcal{D}_V\underline{u}'$ and $\mathcal{D}_V\underline{u}''$. Then applying the reduction of (5.22) once again, we may assume that all the assumptions

(A1)–(A5) are satisfied by $\mathcal{D}_V\underline{u}$, $\mathcal{D}_V\underline{u}'$ and $\mathcal{D}_V\underline{u}''$. Put $\mathcal{N} := \mathcal{D}_X[s](f^s\underline{u})$, $\mathcal{N}' := \mathcal{D}_X[s](f^s\underline{u}')$ and $\mathcal{N}'' := \mathcal{D}_X[s](f^s\underline{u}'')$. Then $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ satisfy (5.4.1)–(5.4.4), and (5.8.1) becomes exact. Hence (5.8.2) is also exact, and applying DR to it, we get the distinguished triangle

$$\mathrm{DR}(\mathcal{D}_X(f^\alpha\underline{u}')) \to \mathrm{DR}(\mathcal{D}_X(f^\alpha\underline{u})) \to \mathrm{DR}(\mathcal{D}_X(f^\alpha\underline{u}'')) \xrightarrow{+},$$

from which we get (5.16, (3)) for the second term. (Note that (5.16, (3)) is assumed to be already proved for the first and the third terms by the induction hypothesis.)

Hence we may assume, without destroying (A1)–(A5), that

(A6) *$\mathcal{D}_V\underline{u}$ is a simple $\mathcal{D}_V$-module, i.e., a coherent $\mathcal{D}_V$-module without proper coherent $\mathcal{D}_V$-submodules.*

Assume (A4) and (A6). Then, there exists a closed irreducible subvariety Z of X, and an irreducible locally constant sheaf L on a non-singular open dense subvariety U of $Z_0 := Z \cap X_0$ such that

$$\mathrm{DR}_{X_0}(\mathcal{D}_{X_0}\underline{u})[\dim X_0] = (i_{Z_0})_*(j_0)_{!*}L[\dim Z], \tag{5.24.1}$$

where $j_0 : U \to Z_0$ and $i_{Z_0} : Z_0 \to X_0$ are the inclusion mappings.

5.25. Normal crossing case (1). Before studying generally (5.16, (3)), let us consider the case where

$$\begin{aligned}
&X = \mathbf{C}^n,\\
&[1,n] = E \sqcup F \sqcup G \sqcup H,\\
&f(x) = \prod_{i\in E} x_i^{e_i} \quad (e_i \in \mathbf{Z}_{>0}), \qquad X_0 = X \setminus f^{-1}(0) = (\mathbf{C}^\times)^E \times \mathbf{C}^{F\sqcup G\sqcup H},\\
&Z = \{x_i = 0\ (i \in H)\} = \mathbf{C}^{E\sqcup F\sqcup G}\ (:= \mathbf{C}^{E\sqcup F\sqcup G} \times \{0\}^H),\\
&U = (\mathbf{C}^\times)^{E\sqcup F} \times \mathbf{C}^G,\\
&V = X_0 \setminus g^{-1}(0) \text{ with } g(x) := \prod_{i\in E} x_i^{e_i'} \times \prod_{i\in F} x_i^{f_i} \quad (e_i' \in \mathbf{Z}_{\geq 0},\ f_i \in \mathbf{Z}_{>0}),\\
&v(x) := \prod_{i\in E\sqcup F} x_i^{\lambda_i} \cdot \prod_{i\in H} \delta(x_i) \quad (\lambda_i \in \mathbf{C}), \text{ and}\\
&\mathcal{M} := \text{the minimal extension of } \mathcal{D}_V v \text{ to } X_0,
\end{aligned}$$

and $\underline{u} \subset \Gamma(V, \mathcal{D}_V v) = \Gamma(X_0, (j_V)_*(\mathcal{D}_V v))$ generates $\mathcal{M}$ as a $\mathcal{D}_{X_0}$-module.

Let $\mathcal{M}_1 := ((j_V)_*((\mathcal{D}_V v)^*))^*$. (Here $j_V \to X_0$ is the inclusion mapping, and $(-)^*$ denotes the dual $\mathcal{D}$-module. Cf. (2.6.3).) Then $\mathcal{M} = \mathcal{M}_1/\Gamma_{X_0\setminus V}(\mathcal{M}_1)$. Since $v \in \Gamma(V, \mathcal{D}_V v) = \Gamma(V, \mathcal{M}) \supset \Gamma(X_0, \mathcal{M})$, we have $g^m v \in \Gamma(X_0, \mathcal{M})$ and $\mathrm{Re}(\lambda_i + f_i m) \geq 0$ $(i \in F)$ if $m \in \mathbf{Z}$ is sufficiently large. Then the $\mathcal{D}_{X_0}$-module $\mathcal{M}$ is generated by $g^m v$, and simple. Hence applying (5.14, (4)) with $V_1 = X_0$ and $V_2 = V$, we get

$$\begin{aligned}(5.25.1)\qquad &\mathcal{D}_X[s](f^s g^m v) = \mathcal{D}_X[s](f^s g^m v|V), \text{ and}\\ &\mathcal{D}_X(f^\alpha g^m v) = \mathcal{D}_X(f^\alpha g^m v|V) \quad (\alpha \in \mathbf{C}).\end{aligned}$$

(Cf. also (5.9.2).) Assume that $\mathrm{Re}(\alpha) \gg 0$. By a direct calculation, we can show that

$$\begin{aligned}&\mathrm{DR}_X(\mathcal{D}_X(f^\alpha g^m v|V))\\ &= \left(\bigotimes_{i\in E} j'_! \mathbf{C}x_i^{-\alpha e_i - \lambda_i}\right) \otimes \left(\bigotimes_{i\in F} j'_{!*}\mathbf{C}x_i^{-\lambda_i}\right) \otimes \mathbf{C}^{\otimes G} \otimes \mathbf{C}_{\{0\}}^{\otimes H}[\operatorname{card} H]\\ &= j_!\, \mathrm{DR}_{X_0}(\mathcal{D}_{X_0}(f^\alpha g^m v|V)),\end{aligned}$$

where $j' : \mathbf{C}^\times \to \mathbf{C}$ and $j : X_0 \to X$ are the inclusion mappings. Hence we get (5.16, (3)) if $\underline{u} = g^m v$. Then by (5.10, (2)), we get the same for an arbitrary global generator system $\underline{u}$ of the $\mathcal{D}_{X_0}$-module $\mathcal{M}$.

Lemma 5.26. *Keep the notation and the assumption of* (5.25). *Let $\pi : T^*X \to X$ be the natural projection, and $\underline{u} = (u_1, \cdots)$ a finite global generator system of the $\mathcal{D}_{X_0}$-module $\mathcal{M}$. Then $\mathcal{D}_X[s](f^s\underline{u})$ is $\mathcal{D}_X$-coherent, and*

$$\mathrm{ch}(\mathcal{D}_X[s](f^s\underline{u})) \cap \pi^{-1}(f^{-1}(0))$$

is holonomic.

Proof. Put $F' := \{i \in F \mid \lambda_i \in \mathbf{Z}\}$ and $F'' := F \setminus F'$. Then

$$\begin{aligned}&\mathrm{ch}(\mathcal{D}_X[s](f^s g^m v))\\ &= W_E \times (T^*_{\mathbf{C}}\mathbf{C})^{F'} \times (T^*_{\mathbf{C}}\mathbf{C} \cup T^*_{\{0\}}\mathbf{C})^{F''} \times (T^*_{\mathbf{C}}\mathbf{C})^G \times (T^*_{\{0\}}\mathbf{C})^H,\end{aligned}$$

where $m \gg 0$, W_E is the Zariski closure of

$$\bigcup_{c\in\mathbf{C}^\times} \{(x_i, \xi_i)_{i\in E} \in T^*\mathbf{C}^E \mid e_i^{-1}x_i\xi_i = c \ (i \in E)\},$$

and, $T^*_{\mathbf{C}}\mathbf{C}$ and $T^*_{\{0\}}\mathbf{C}$ are conormal bundles of $\mathbf{C}$ and $\{0\}$, respectively. Hence we get the result when $\underline{u} = g^m v$.

In the general case, put $N := D_X[s](f^s g^m v)$ $(= D_X(f^s g^m v))$ and $N' := D_X[s](f^s\underline{u})$. Since $N \subset N[f^{-1}] = N'[f^{-1}] \supset N'$, and since both N and N' are finitely generated $D_X[s]$-modules, $t^m N \subset N'$ and $t^m N' \subset N$ for $m \gg 0$. Since $t^m : N \xrightarrow{\simeq} t^m N$ and $t^m : N' \xrightarrow{\simeq} t^m N'$, we get the result.

5.27. Normal crossing case (2). Next, we prove (5.16, (3)), assuming that X, f, Z, U, and V are the same as in (5.25), but as for $\mathcal{D}_V\underline{u}$, simply assuming that its characteristic variety is (the closure of) the conormal bundle of U.

We regard $\underline{u}$ as a subset of $\Gamma(X_0, (j_V)_*(\mathcal{D}_V\underline{u}))$, where $j_V : V \to X_0$ is the inclusion mapping. As is easily seen, the characteristic variety of $(j_V)_*(\mathcal{D}_V\underline{u})$ is a union of conormal bundles of $\mathbf{C}^I$ with subsets $I \subset [1,n]$. Hence the characteristic variety of each composition factor, say $\mathcal{M}'$, of $\mathcal{D}_{X_0}\underline{u}$ is also of the same form. Since $\operatorname{supp}\mathcal{M}' =: Z'$ is an irreducible variety by (5.14, (2)), $Z' = (\mathbf{C}^\times)^E \times \mathbf{C}^{I'} \times \{0\}^{H'}$ with some partition $E \sqcup I' \sqcup H' = [1,n]$. (Here E is the same as in (5.25).) Moreover, considering the characteristic variety, we can see that there is a partition $I' = F' \sqcup G'$ such that the restriction of $\mathrm{DR}(\mathcal{M}')$ to $U' := (\mathbf{C}^\times)^{E\sqcup F'} \times \mathbf{C}^{G'} \times \{0\}^{H'}$ is an irreducible locally constant sheaf (up to shift). Since $\pi_1(U') = \mathbf{Z}^{E\sqcup F'}$, we can see that the restriction of $\mathcal{M}'$ to $V' := (\mathbf{C}^\times)^{E\sqcup F'} \times \mathbf{C}^{G'\sqcup H'}$ is of the form $\mathcal{D}_{V'}v'$ with v' as in (5.25). Hence $\mathcal{M}'$ itself is also of the form as in (5.25).

Now, in order to prove (5.16, (3)), we may assume (A1)–(A5) even in our present special situation, as we can see from the argument so far. Moreover we can reduce the proof to the case where $\mathcal{D}_{X_0}\underline{u}$ is some $\mathcal{M}'$ as above, by the argument of (5.24). But then we have already proved it in (5.25).

Similarly, we can generalize (5.26) to the case considered here. We omit the detail.

In (5.25) and (5.27), the similar results hold in the category of $\mathcal{D}^{an}$-modules.

5.28. Let us return to the general situation, keeping all the assumptions (A1)–(A6). Let $g_1, \cdots, g_r$ be regular functions on X such that $g_1 = \cdots = g_r = 0$ are (minimal) defining equations of Z at its generic point. (See (5.24) for notation.) Let S be the locus where $f \cdot \prod_i g_i =: h$ is not normal crossing. Shrinking the open subset U of Z_0, we may assume, keeping (A1)–(A6), that

(A7) *U is non-singular, $U \cap S = \phi$ and $U = Z_0 \setminus p^{-1}(0)$ with some $p(x) \in \mathbf{C}[X]$.*
Let $F : X' \to X$ be a proper modification of S such that $\{F^*h = 0\}$ is normal crossing [Hi]. Let $f' := F^*f$, $U' := F^{-1}(U)$ ($\simeq U$), and Z' be the Zariski closure of U' in X'. By a further proper modification of $Z' \setminus U'$, we may assume that Z' is non-singular and $Z' \setminus U'$ is a normal crossing divisor of Z'. (Here we claim Z' to be the closure of U', again.)

In the process of the above proper modification, each center of blowing-up has an image in X which is closed in X and does not intersect U. Hence there is an affine open set $\tilde{V} \subset X_0$ such that $F : F^{-1}(\tilde{V}) \to \tilde{V}$ is an isomorphism, and $\tilde{V} \cap Z = U$. Shrinking X, U, and $\tilde{V}$, we may assume (A2″) for $\tilde{V}$, i.e., $\tilde{V} = X_0 \setminus \tilde{g}^{-1}(0)$ with some $\tilde{g} \in \mathbf{C}[X]$. Put $X_0' := X' \setminus f'^{-1}(0)$, $\tilde{V}' := F^{-1}(\tilde{V})$ and $\tilde{g}' := F^*\tilde{g}$. Then $\tilde{V}' = X_0' \setminus \tilde{g}'^{-1}(0)$. By a further proper modification of $\tilde{g}'^{-1}(0)$, we may assume that $\{F^*\tilde{g} \cdot F^*h = 0\}$ is a normal crossing divisor of X'. Note that $\tilde{V}' \cap Z' = U'$. Note also that we may assume that the above procedure always preserves (A3). Indeed, if X etc. are obtained as $X^\sharp$ etc. from some $X^\flat$ etc., then first apply the above procedure to $X^\flat$ etc., and then apply '$\flat \to \sharp$'.

Now, locally with respect to the classical topology, X', f', Z', U', $\tilde{V}'$ and $\tilde{g}'$ can be regarded as those studied in (5.25) and (5.27). (Indeed, since $\{F^*\tilde{g} \cdot F^*h = 0\}$ is normal crossing, and since Z' is non-singular, we may regard locally with respect to the classical topology that $X' = \mathbf{C}^n$, $Z' = \mathbf{C}^I \times \{0\}^H$ with some partition $I \sqcup H = [1,n]$, and that f' and $\tilde{g}'$ are monomials in the coordinate functions $\{x_i\}_{1\le i\le n}$. Let $f' = \prod_{i\in E} x_i^{e_i}$ ($e_i \in \mathbf{Z}_{>0}$). Since $\{f' = 0\} \not\supset Z'$, $E \subset I$. Let

$I = E \sqcup I'$. Since

(5.28.1)
$$\begin{aligned}&(\mathbf{C}^{\times})^{E} \times \mathbf{C}^{I'} \times \{0\}^{H} \setminus \{\tilde{g}' = 0\}\\ &= \mathbf{C}^{E\sqcup I'} \times \{0\}^{H} \setminus (\{f' = 0\} \cup \{\tilde{g}' = 0\}) = \tilde{V}' \cap Z' = U',\end{aligned}$$

we may regard that

(5.28.2)
$$U' = (\mathbf{C}^{\times})^{E\sqcup F} \times \mathbf{C}^{G} \times \{0\}^{H}$$

with some partition $F \sqcup G = I'$. Since $\tilde{g}'$ is a monomial in x_i's, we may assume that $\tilde{g}'(x) = \prod_{i\in E} x_i^{e'_i} \times \prod_{j\in F} x_j^{f_j}$ ($e'_i \in \mathbf{Z}_{\geq 0}$, $f_j \in \mathbf{Z}_{>0}$). Thus all the conditions for X, f, Z, U, V and $\tilde{g}$ in (5.25) are satisfied with X', f', Z', U', $\tilde{V}'$ and $\tilde{g}'$.) Identifying U' with U, the locally constant sheaf L on U (cf. (5.24.1)) can be regarded as a locally constant sheaf on U', which we shall denote by L'. Put $Z'_0 := Z' \cap X'_0$, and let $j'_0 : U' \to Z'_0$ and $i_{Z'_0} : Z'_0 \to X'_0$ be the inclusion mappings. Let $\mathcal{M}'$ be a regular holonomic $\mathcal{D}_{X'_0}$-module such that

$$\mathrm{DR}_{X'_0}(\mathcal{M}')[\dim X'] = (i_{Z'_0})_*(j'_0)_{!*}L'[\dim Z'].$$

If we identify $\tilde{V}' = F^{-1}(\tilde{V})$ and $\tilde{V}$, then $\mathcal{M}'|\tilde{V}' = \mathcal{M}|\tilde{V}$. Take $0 \neq u \in \Gamma(\tilde{V}, \mathcal{M})$ and $0 \neq u' \in \Gamma(\tilde{V}', \mathcal{M}')$ which correspond to each other. Consider u as a section of $(j_{\tilde{V}})_*(\mathcal{M}|\tilde{V})$ on X_0, where $j_{\tilde{V}} : \tilde{V} \to X_0$ is the inclusion mapping. Since $\mathcal{M} \subset (j_{\tilde{V}})_*(\mathcal{M}|\tilde{V})$ and since $(j_{\tilde{V}})_*(\mathcal{M}|\tilde{V})/\mathcal{M}$ is supported by $\tilde{g}^{-1}(0)$, $\tilde{g}^m u \in \mathcal{M}$ for a sufficiently large m. Replacing u with $\tilde{g}^m u$, and by the similar argument for u' and $\tilde{g}'$, we may assume that

(A8) *u (resp. u') can be extended to a section of $\mathcal{M}$ (resp. $\mathcal{M}'$) on X_0 (resp. X'_0).*

Then $\mathcal{M} = \mathcal{D}_{X_0}u$ and $\mathcal{M}' = \mathcal{D}_{X'_0}u'$. Moreover, if $\mathcal{M}$ and $\mathcal{M}'$ are obtained as a result of '$\sharp \to \flat$', then we may and do assume that u and u' are also obtained as a result of the similar procedure (cf. (5.20)).

Lemma 5.29. *Put $\int^0_{F|X'_0} \mathcal{D}_{X'_0}u' =: \mathcal{D}_{X_0}\underline{v}$. If $\mathrm{Re}(\alpha) \gg 0$, then $\int^0_F \mathcal{D}_{X'}(f'^{\alpha}u') = \mathcal{D}_X(f^{\alpha}\underline{v})$.*

(Here $\int^i_F(-)$ denotes the i-th cohomology sheaf of $\int_F(-) := RF_*(\mathcal{D}_{X\leftarrow X'} \otimes^L_{\mathcal{D}_{X'}} (-).)$

The proof is long and come to an end in (5.38). Before embarking in the proof, let us show that we can finish the proof of (5.16, (3)) assuming (5.29).

5.30. Proof of '(5.29) $\Rightarrow$ (5.16, (3))'. First note that, if $\mathrm{Re}(\alpha) \gg 0$, then

(5.30.1)
$$\begin{aligned}&\mathrm{DR}\Big(\int^0_F \mathcal{D}_{X'}(f'^{\alpha}u')\Big)\\ &= {}^pH^0(RF_!\,\mathrm{DR}_{X'}(\mathcal{D}_{X'}(f'^{\alpha}u')))\\ &= {}^pH^0(RF_!\,j'_!\,\mathrm{DR}_{X'_0}(\mathcal{D}_{X'}(f'^{\alpha}u'))), \quad \text{by (5.25) and (5.27)},\\ &= j_!({}^pH^0(RF_!\,\mathrm{DR}_{X'_0}(\mathcal{D}_{X'}(f'^{\alpha}u')))), \quad \text{since } j \text{ is affine},\\ &(= j_!(\mathbf{C}f^{-\alpha} \otimes {}^pH^0(RF_!\,\mathrm{DR}_{X'_0}\mathcal{M}'))).\end{aligned}$$

(Here ${}^pH^0$ denotes the perverse cohomology; ${}^pH^0 \circ \mathrm{DR} = \mathrm{DR} \circ H^i$ for regular holonomic $\mathcal{D}$-modules.) Then (5.16, (3)) holds for the special type of $\mathcal{D}_{X_0}$-module $\mathcal{D}_{X_0}\underline{v}$ by (5.29). Using this, and assuming (A1)–(A6), we prove (5.16, (3)) for $\mathcal{M}$ by the induction on $d := \dim \operatorname{supp} \mathcal{D}_{X_0}u$. From the construction, $\mathcal{D}_{X_0}\underline{v}$ is obtained as $\mathcal{K}^\sharp$ from some $\mathcal{K}^\flat$. Consider a composition series of $\mathcal{K}^\flat$, and then apply '$\flat \to \sharp$' as in (5.20). Then we get a composition series

$$\mathcal{D}_{X_0}\underline{v} = \mathcal{D}_{X_0}\underline{v}^{(0)} \supsetneqq \mathcal{D}_{X_0}\underline{v}^{(1)} \supsetneqq \cdots \supsetneqq \mathcal{D}_{X_0}\underline{v}^{(k)} = 0.$$

Note that each composition factor is obtained as a result of '$\flat \to \sharp$' as in (5.20). Since $\mathcal{D}_{X_0}\underline{v}|\tilde{V} = \mathcal{M}|\tilde{V}$, exactly one of the composition factor is $\mathcal{M}$, and the remaining factors $\mathcal{D}_{X_0}\underline{w}^{(i)} := \mathcal{D}_{X_0}\underline{v}^{(i)}/\mathcal{D}_{X_0}\underline{v}^{(i+1)}$ are supported by subvarieties of dimension $< d$. (Indeed, $\operatorname{supp}(\mathcal{D}_{X_0'}u') = Z_0'$, $\operatorname{supp}(\mathcal{D}_{X_0}\underline{v}) \subset Z$ and $\dim(Z \setminus V) < \dim Z = d$.) Here we take as $\underline{w}^{(i)}$ the image of $\underline{v}^{(i)}$. By the first step, (5.16, (3)) holds for $\mathcal{D}_{X_0}\underline{v}$. By the induction hypothesis, it holds also for $\mathcal{D}_{X_0}\underline{w}^{(i)}$ except for exactly one i. (In order to apply the induction hypothesis to $\mathcal{D}_{X_0}\underline{w}^{(i)}$, we need to verify (A1)–(A6) for this $\mathcal{D}$-module, replacing X with a neighbourhood of $f^{-1}(0)$ if necessary. There would be no difficulty except for (A5). As for (A5), note that, if (A3) is satisfied, we can apply the procedure (5.22) keeping (A3).) Therefore we get (5.16, (3)) for $\mathcal{M} = \mathcal{D}_{X_0}\underline{u}$. (Cf. the argument of (5.24).) □

Now we embark in the proof of (5.29).

Lemma 5.31.

$$\int_{F|X_0'}^0 \mathcal{D}_{X_0'}[s](f'^s u') = \mathcal{D}_{X_0}[s](f^s \underline{v}).$$

Proof. By (5.15), it suffices to show that

$$\begin{aligned}(5.31.1) \qquad & \int_{(F|X_0')\times\mathbf{C}^\times}^0 \mathcal{D}_{X_0'\times\mathbf{C}^\times}(\delta(t - f'(x'))u'(x')) \\ &= \mathcal{D}_{X_0'\times\mathbf{C}^\times}(\delta(t - f(x))\underline{v}(x)).\end{aligned}$$

By the isomorphism $X_0 \times \mathbf{C}^\times \to X_0 \times \mathbf{C}^\times$, $(x,t) \mapsto (x, t - f(x))$ and by the similar isomorphism $X_0' \times \mathbf{C}^\times \to X_0' \times \mathbf{C}^\times$, (5.31.1) is transformed into

$$\begin{aligned}(5.31.2) \qquad & \int_{(F|X_0')\times\mathbf{C}^\times}^0 \mathcal{D}_{X_0'\times\mathbf{C}^\times}(u'(x)\delta(t)) \\ &= \mathcal{D}_{X_0'\times\mathbf{C}^\times}(\underline{v}(x)\delta(t)),\end{aligned}$$

which is obvious.

5.32. By (A3) for X' etc., $\mathcal{D}_{X'}[s](f'^s u') = \mathcal{D}_{X'}(f'^s u')$. Let $X'^* \subset X'$ be a Zariski open neighbourhood of $f'^{-1}(0)$ such that $\mathcal{D}_{X'}[s](f'^s u')|X'^*$ is subholonomic. Since $X^* := X \setminus F(X' \setminus X'^*)$ is a Zariski open neighbourhood of $f^{-1}(0)$, and since (5.29) holds outside of $f^{-1}(0)$ by (5.31), we may assume that $\mathcal{D}_{X'}[s](f'^s u')$

is subholonomic, replacing X and X' with X^* and $F^{-1}(X^*)$ ($\subset X'^*$). Further shrinking X, we may assume that

(A9) $\mathcal{D}_{X'}[s](f'^s u')$ *and* $\mathcal{D}_X[s](f^s \underline{v})$ *are subholonomic.*

5.33. Put

$$N := D_X[s](f^s \underline{v})$$
$$N' := \Gamma(X, \int_F^0 \mathcal{D}_{X'}[s](f'^s u')), \text{ and}$$
$$N'' := \mathrm{image}(\varphi : N' \to N'[f^{-1}]),$$

where φ is the natural morphism. Then these are $D_X[s,t]$-modules, and finitely generated D_X-modules by (A9). By (5.31),

$$N[f^{-1}] = N'[f^{-1}] = N''[f^{-1}], \tag{5.33.1}$$

and this is a $D_X[s,t,t^{-1}]$-module. Obviously

$$N[f^{-1}] = N[t^{-1}]. \tag{5.33.2}$$

Let us show that

$$N''[f^{-1}] = N''[t^{-1}]. \tag{5.33.3}$$

Since $N''[f^{-1}]$ ($= N[f^{-1}]$) is a $D_X[s,t,t^{-1}]$-module, the inclusion '$\supset$' is obvious. Since $N''[f^{-1}] = D_{X_0}[s](f^s \underline{v})$ (cf. (5.29) for $\underline{v}$, and (5.31)), it suffices to show that $f^{-m} \cdot f^s v_i \in N''[t^{-1}]$ ($m > 0$), i.e., $t^l(f^{-m} \cdot f^s v_i) \in N''$ for $l \gg 0$. The latter is obvious. By (5.33.1)–(5.33.3),

$$t^m N \subset N'' \text{ and } t^m N'' \subset N \tag{5.33.4}$$

if $m \in \mathbf{Z}$ is sufficiently large. (Note that $N \subset N[f^{-1}]$ and $N'' \subset N''[f^{-1}]$. Note also that, to obtain (5.33.4), $D_X[s]$-finiteness of N and N'' is enough.) Since $N'' \simeq t^m N'' \subset N$ as D_X-modules, N'' is subholonomic D_X-module. Moreover, we can see that N and N'' satisfy (5.4.1)–(5.4.4), using the above relations. (The $\mathbf{C}[s]$-flatness of $N''[f^{-1}]$ follows from (5.33.1).) By (5.33.4), we get natural morphisms

$$\frac{t^{3m}N''}{(s-\alpha)t^{3m}N''} \xrightarrow{A} \frac{t^{2m}N}{(s-\alpha)t^{2m}N} \xrightarrow{B} \frac{t^m N''}{(s-\alpha)t^m N''} \xrightarrow{C} \frac{N}{(s-\alpha)N}. \tag{5.33.5}$$

Since $N \xrightarrow[t^k]{\simeq} t^k N$ and $N'' \xrightarrow[t^k]{\simeq} t^k N''$ for any $k \in \mathbf{Z}_{\geq 0}$, (5.33.5) can be identified with

$$N''(\alpha+3m) \xrightarrow{A} N(\alpha+2m) \xrightarrow{B} N''(\alpha+m) \xrightarrow{C} N(\alpha). \tag{5.33.6}$$

By (5.3), BA and CB become isomorphims if $\mathrm{Re}(\alpha) \gg 0$. Then B is an isomorphism. Using (5.3) again, we get

$$N(\alpha) \xrightarrow{\simeq} N''(\alpha) \text{ if } \mathrm{Re}(\alpha) \gg 0. \tag{5.33.7}$$

Now, let $\pi : T^*X \to X$ be the natural projection, $W := \mathrm{ch}(N)$, and $W' := \mathrm{ch}(N')$, where $\mathrm{ch}(-)$ denotes the characteristic variety. (Cf. (2.2).)

Lemma 5.34. (1) $W \cap \pi^{-1}(X_0) = W' \cap \pi^{-1}(X_0)$. (2) $W' \setminus \pi^{-1}(X_0)$ *is isotropic.*

Proof. We get (1) from (5.33.1). Let $\tilde{W}' := \mathrm{ch}(\mathcal{D}_{X'}[s](f'^s u'))$, and $T^*X' \xleftarrow{\rho} T^*X \times_X X' \xrightarrow{\varpi} T^*X$ the natural morphisms. (See (5.28) for X'.) By [Ka1, Theorem 4.2],

$$W' \setminus \pi^{-1}(X_0) \subset \varpi\rho^{-1}(\tilde{W}' \setminus \pi'^{-1}(X'_0)),$$

where $\pi' : T^*X' \to X'$ denotes the natural projection. By (5.26) (cf. also (5.27)), and by [Ka1, Proposition 4.9], we get (2).

5.35. By (A9) and (5.34),
(5.35.1) N' *is a subholonomic* D_X*-module, and*
(5.35.2) $K := \ker(\varphi : N' \to N'[f^{-1}])$ *is a holonomic* $D_X[s,t]$*-module.*
Then by [Ka1, Proposition 5.11],

$$t^m K = 0 \text{ for } m \gg 0. \tag{5.35.3}$$

For such m,

$$\varphi : t^m N' \xrightarrow{\simeq} t^m N''. \tag{5.35.4}$$

In fact, if $u \in N'$ and $\varphi(t^m u) = 0$, then $t^m \varphi(u) = 0$. Since $t : N'[f^{-1}] \to N'[f^{-1}]$ is injective, $\varphi(u) = 0$. Hence $u \in K$, $t^m u = 0$, and (5.35.4) is injective. The surjectivity is obvious. From (5.35.4), it follows that

$$\frac{t^m N'}{(s-\alpha)t^m N'} \xrightarrow{\simeq} \frac{t^m N''}{(s-\alpha)t^m N''} \xleftarrow[t^m]{\simeq} \frac{N''}{(s-\alpha-m)N''} \simeq N''(\alpha) \tag{5.35.5}$$

if $\mathrm{Re}(\alpha) \gg 0$. (The last isomorphism follows from (5.3).)

Lemma 5.36. *Let M be a $D[s]$-module which is finitely generated as a D-module. Put $T(\alpha) := \ker(s-\alpha \mid N \to N)$. Then $T(\alpha) = 0$ except for a finite number of α's.*

Proof. Since each $T(\alpha)$ ($\alpha \in \mathbf{C}$) is a D-submodule of M, and since for any mutually distinct $\alpha_0, \alpha_1, \cdots, \alpha_l \in \mathbf{C}$, $T(\alpha_0) \cap \sum_{i=1}^{l} T(\alpha_i) = 0$, we get the result. (Note that D is left noetherian.)

5.37. Put $R := \ker(t^m : N' \to t^m N')$, where m is a sufficiently large integer. Since $\operatorname{supp} R \subset f^{-1}(0)$, R is a holonomic D_X-module by (5.34, (2)). By (5.36), $\ker(s-\alpha-m|R \to R) = 0$ if $m \in \mathbf{Z}_{\geq 0}$ and $\mathrm{Re}(\alpha) \gg 0$. In this case, $s-\alpha-m : R \to R$ is an isomorphism, since R is holonomic. Applying the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "s-\alpha-m"] & N' \arrow[r, "t^m"] \arrow[d, "s-\alpha-m"] & t^m N' \arrow[r] \arrow[d, "s-\alpha"] & 0 \\
0 \arrow[r] & R \arrow[r] & N' \arrow[r, "t^m"] & t^m N' \arrow[r] & 0,
\end{tikzcd}$$

we get

$$\frac{N'}{(s-\alpha-m)N'} \simeq \frac{t^m N'}{(s-\alpha)t^m N'} \tag{5.37.1}$$

if $\mathrm{Re}(\alpha) \gg 0$ and $m \in \mathbf{Z}_{\geq 0}$.

5.38. Put $\tilde{\mathcal{N}} := \mathcal{D}_{X'}[s](f'^s u')$ and $\tilde{\mathcal{N}}(\alpha) = \tilde{\mathcal{N}}/(s-\alpha)\tilde{\mathcal{N}}$. If $\mathrm{Re}(\alpha) \gg 0$ and $m \in \mathbf{Z}_{\geq 0}$, we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N}' \arrow[r, "s-\alpha"] & \mathcal{N}' \arrow[r] & \int_F^0 \tilde{\mathcal{N}}(\alpha) \arrow[r] & \int_F^1 \tilde{\mathcal{N}} \\
0 \arrow[r] & \mathcal{N}' \arrow[r, "s-\alpha-m"] \arrow[u, "t^m"] & \mathcal{N}' \arrow[r] \arrow[u, "t^m"] & \int_F^0 \tilde{\mathcal{N}}(\alpha+m) \arrow[r] \arrow[u, "\simeq"] & \int_F^1 \tilde{\mathcal{N}} \arrow[u, "t^m"]
\end{tikzcd}$$

whose first row is exact. Here $\mathcal{N}'$ is the $\mathcal{D}_X$-module associated to N', i.e., $\mathcal{N}' = \int_F^0 \tilde{\mathcal{N}}$. The injectivity of $s-\alpha$ follows from (5.36). Let $\mathcal{K}$ be the largest holonomic submodule of $\int_F^1 \tilde{\mathcal{N}}$. Since $\mathcal{K}$ has a $\mathcal{D}_X[s,t]$-module structure, $t^m\mathcal{K} = 0$ for $m \gg 0$. Since $\int_F^0 \tilde{\mathcal{N}}(\alpha+m)$ is holonomic, its image in $\int_F^1 \tilde{\mathcal{N}}$ is annihilated by t^m ($m \gg 0$). Hence we get the sequence

$$0 \to \mathcal{N}' \xrightarrow{s-\alpha} \mathcal{N}' \to \int_F^0 \mathcal{D}_{X'}(f'^\alpha u') \to 0 \tag{5.38.1}$$

if $\mathrm{Re}(\alpha) \gg 0$, i.e.,

$$\frac{N'}{(s-\alpha)N'} \simeq \Gamma(X, \int_F^0 \mathcal{D}_{X'}(f'^\alpha u')). \tag{5.38.2}$$

Summing up, we get

$$\begin{aligned}\Gamma(X, \mathcal{D}_X(f^\alpha \underline{v})) &= N(\alpha) \simeq N''(\alpha) \quad \text{by (5.33.7)}\\ &\simeq \frac{t^m N'}{(s-\alpha)t^m N'} \quad \text{by (5.35.5)}\\ &\simeq \frac{N'}{(s-\alpha-m)N'} \quad \text{by (5.37.1)}\\ &\simeq \frac{N'}{(s-\alpha)N'} \quad \text{by (5.3)}\\ &= \Gamma(X, \int_F^0 \mathcal{D}_{X'}(f'^\alpha u')) \quad \text{by (5.38.2)}\end{aligned}$$

if $\mathrm{Re}(\alpha) \gg 0$. Thus

$$\int_F^0 \mathcal{D}_{X'}(f'^\alpha u') = \mathcal{D}_X(f^\alpha \underline{v})$$

and we get (5.29), and hence we have also completed the proof of (5.16).

5.39. Let $\mathrm{Mod}(\mathcal{D}_X)$ denote the category of (left) $\mathcal{D}_X$-modules, $\mathrm{Mod}_h(\mathcal{D}_X)$ (resp. $\mathrm{Mod}_{rh}(\mathcal{D}_X)$) its full subcategory consisting of holonomic (resp. regular holonomic) $\mathcal{D}_X$-modules, $D(\mathcal{D}_X)$ the derived category of $\mathrm{Mod}(\mathcal{D}_X)$, and $D^b_h(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$) the full subcategory of $D(\mathcal{D}_X)$ consisting of bounded complexes with holonomic (resp. regular holonomic) cohomologies. By [Be] (together with [Ka3], [Me]),

$$\text{(5.39.1)} \qquad D^b_{rh}(\mathcal{D}_X) = D^b(\mathrm{Mod}_{rh}(\mathcal{D}_X)).$$

If X is an affine variety, we can define $\mathrm{Mod}(D_X)$, $\mathrm{Mod}_h(D_X)$, $\mathrm{Mod}_{rh}(D_X)$, $D(D_X)$, $D^b_h(D_X)$ and $D^b_{rh}(D_X)$ in the same way as above.

Let $\mathrm{Mod}(\mathbf{C}_X)$ denote the category of $\mathbf{C}_X$-modules, $D(\mathbf{C}_X)$ its derived category, and $D^b_c(\mathbf{C}_X)$ the full subcategory of $D(\mathbf{C}_X)$ consisting of bounded complexes with (algebraically) constructible cohomologies.

5.40. $\mathcal{D}_X$-Modules $(f^\alpha, \mathcal{M})_*$ and $(f^\alpha, \mathcal{M})_!$. Let $0 \neq f \in \mathbf{C}[X]$, $j : X_0 = X \setminus f^{-1}(0) \to X$ be the inclusion mapping, and $\mathcal{M}$ a regular holonomic $\mathcal{D}_{X_0}$-module. If $\mathcal{M}$ is generated by global sections $\underline{u} = (u_1, \cdots, u_p)$ ($u_i \in \Gamma(X_0, \mathcal{M})$), then we can define $\mathcal{D}_X(f^\alpha \underline{u})$ as in (5.9). Let $\underline{v} = (v_1, \cdots, v_q)$ be another global generator system of the $\mathcal{D}_{X_0}$-module $\mathcal{M}$. Then for $m \in \mathbf{Z}$,

$$\mathrm{DR}_X(\mathcal{D}_X(f^{\alpha+m}\underline{u})) = \mathrm{DR}_X(\mathcal{D}_X(f^{\alpha+m}\underline{v})) = \begin{cases} Rj_*(\mathbf{C}f^{-\alpha} \otimes \mathrm{DR}_{X_0}(\mathcal{M})) & \text{if } m \ll 0\\ j_!(\mathbf{C}f^{-\alpha} \otimes \mathrm{DR}_{X_0}(\mathcal{M})) & \text{if } m \gg 0\end{cases}$$

by (5.16). Hence the natural isomorphism $\mathcal{D}_{X_0}(f^{\alpha+m}\underline{u}) \simeq \mathcal{D}_{X_0}(f^{\alpha+m}\underline{v})$ ($\simeq \mathcal{D}_{X_0} f^\alpha \otimes_{\mathcal{O}_{X_0}} \mathcal{M}$) uniquely extends to $\mathcal{D}_X(f^{\alpha+m}\underline{u}) \simeq \mathcal{D}_X(f^{\alpha+m}\underline{v})$ if $m \gg 0$ or $m \ll 0$. By the same reason, $\mathcal{D}_X(f^{\alpha+m}\underline{u})$ is independent of a special choice of $m \in \mathbf{Z}$ as far as $m \gg 0$ or $m \ll 0$.

Generally, let $X = \bigcup_i U_i$ be a finite open covering, $\underline{u}^{(i)} \in \Gamma(U_i, \mathcal{M})^{p_i}$ ($p_i \in \mathbf{Z}_{\geq 0}$) a finite generator system of $\mathcal{M}|U_i$, and consider $\mathcal{D}_{U_i}(f^{\alpha+m}\underline{u}^{(i)})$ ($m \gg 0$ or $m \ll 0$). By what we have seen above, these $\mathcal{D}_{U_i}$-modules patch together. In other words, there uniquely exist regular holonomic $\mathcal{D}_X$-modules $(f^\alpha, \mathcal{M})_* = (f^\alpha, \mathcal{M})_{*,X}$ and $(f^\alpha, \mathcal{M})_! = (f^\alpha, \mathcal{M})_{!,X}$ such that

$$\mathcal{D}_{U_i}(f^{\alpha+m}\underline{u}^{(i)}) = \begin{cases} (f^\alpha, \mathcal{M})_*|U_i & \text{if } m \ll 0 \\ (f^\alpha, \mathcal{M})_!|U_i & \text{if } m \gg 0. \end{cases} \tag{5.40.1}$$

Then

$$\mathrm{DR}_X((f^\alpha, \mathcal{M})_*) = Rj_*(\mathbf{C}f^{-\alpha} \otimes \mathrm{DR}_{X_0}(\mathcal{M})), \text{ and} \tag{5.40.2}$$

$$\mathrm{DR}_X((f^\alpha, \mathcal{M})_!) = j_!(\mathbf{C}f^{-\alpha} \otimes \mathrm{DR}_{X_0}(\mathcal{M})). \tag{5.40.3}$$

By (5.40.2) and (5.40.3), $\mathcal{M} \mapsto (f^\alpha, \mathcal{M})_*$ and $\mathcal{M} \mapsto (f^\alpha, \mathcal{M})_!$ are exact functors $\mathrm{Mod}_{rh}(\mathcal{D}_{X_0}) \to \mathrm{Mod}_{rh}(\mathcal{D}_X)$. By (5.39.1), these functors have natural extensions $D^b_{rh}(\mathcal{D}_{X_0})$
$\to D^b_{rh}(\mathcal{D}_X)$, which we shall denote by the same notation. Then (5.40.2) and (5.40.3) hold for $\mathcal{M} \in D^b_{rh}(\mathcal{D}_{X_0})$ as well.

5.41. Duality. For $\mathcal{L}^\cdot \in D^b_h(\mathcal{D}_X)$, its dual $\mathbf{D}(\mathcal{L}^\cdot) = \mathbf{D}_X(\mathcal{L}^\cdot) = {}^D\mathbf{D}_X(\mathcal{L}^\cdot)$ is defined by

$$\mathbf{D}(\mathcal{L}^\cdot) := R\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{L}^\cdot, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\dim X}[\dim X],$$

where Ω^i_X denotes the sheaf of regular differential forms of degree i, $\underline{\mathrm{Hom}}$ denotes the sheaf of local homomorphisms, and $R\underline{\mathrm{Hom}}$ its derived functor. For $L^\cdot \in D^b_h(D_X)$, $\mathbf{D}L^\cdot$ is defined in a similar way.

We denote the Verdier duality by $\mathbf{D} = \mathbf{D}_X = {}^{\mathbf{C}}\mathbf{D}_X$;

$$\mathbf{D}(-) := R\underline{\mathrm{Hom}}_{\mathbf{C}_X}(-, \mathbf{C}_X[2\dim X]).$$

It is known that

$$\mathbf{D}(\mathrm{DR}(\mathcal{L}^\cdot)[\dim X]) = \mathrm{DR}(\mathbf{D}\mathcal{L}^\cdot)[\dim X]$$

for $\mathcal{L}^\cdot \in D^b_h(\mathcal{D}_X)$ ([Ho, Chapter V, §5, 5.1]). By (5.40.2) and (5.40.3),

$$\mathbf{D}_X(f^\alpha, \mathcal{M}^\cdot)_* = (f^{-\alpha}, \mathbf{D}_{X_0}\mathcal{M}^\cdot)_! \tag{5.41.1}$$

for $\mathcal{M}^\cdot \in D^b_{rh}(\mathcal{D}_{X_0})$.

5.42. Integration of $\mathcal{D}$-modules. Let $F : X \to Y$ be a morphism between non-singular algebraic varieties over $\mathbf{C}$. Let $\int_F$ denote the usual functor of integration of

$\mathcal{D}_X$-modules along fibres [Ka1] (cf. [Ho, Chapter I, 3.4]). Put $\int_{F!} := \mathbf{D}_Y \circ \int_F \circ \mathbf{D}_X$. Let $0 \neq g \in \mathbf{C}[Y]$, and assume that $f := F^*g \not\equiv 0$. Then by (5.40.2) and (5.40.3),

$$\int_F (f^\alpha, \mathcal{M}^\cdot)_* = (g^\alpha, \int_{F|X_0} \mathcal{M}^\cdot)_*, \text{ and} \tag{5.42.1}$$

$$\int_{F!} (f^\alpha, \mathcal{M}^\cdot)_! = (g^\alpha, \int_{F|X_0,!} \mathcal{M}^\cdot)_! \tag{5.42.2}$$

for $\mathcal{M}^\cdot \in D^b_{rh}(\mathcal{D}_{X_0})$. See [Ho, Chapter V, 5.3]. (Note that we do not need to assume F to be proper.)

5.43. Pull-back. Keep the notation of (5.42). Define the functors ${}^DF^!$ and ${}^DF^*$ from $D^b_{rh}(\mathcal{D}_Y)$ to $D^b_{rh}(\mathcal{D}_X)$ so that

$$(\mathrm{DR}_X \circ {}^DF^!)[\dim X] = (F^! \circ \mathrm{DR}_Y)[\dim Y], \text{ and}$$
$$(\mathrm{DR}_X \circ {}^DF^*)[\dim X] = (F^* \circ \mathrm{DR}_Y)[\dim Y].$$

(See [Ho, Chapter I, 3.5.1], where ${}^DF^!$ is denoted by $F^!$. The functor ${}^DF^*$ is defined by $\mathbf{D}_X \circ {}^DF^! \circ \mathbf{D}_Y$ [Ho, Chapter III, 3.1.1].) Then by (5.40.2) and (5.40.3),

$${}^DF^!(g^\alpha, \mathcal{M}^\cdot)_* = (f^\alpha, {}^D(F|X_0)^! \mathcal{M}^\cdot)_*, \text{ and} \tag{5.43.1}$$

$${}^DF^*(g^\alpha, \mathcal{M}^\cdot)_! = (f^\alpha, {}^D(F|X_0)^* \mathcal{M}^\cdot)_! \tag{5.43.2}$$

for $\mathcal{M}^\cdot \in D^b_{rh}(\mathcal{D}_{Y_0})$.

5.44. Tensor product. In the notation of (5.40), it is easy to see that

$$(f^\alpha, \mathcal{D}_{X_0} f^\beta \otimes_{\mathcal{O}_{X_0}} \mathcal{M}^\cdot)_* = (f^{\alpha+\beta}, \mathcal{M}^\cdot)_*, \text{ and} \tag{5.44.1}$$

$$(f^\alpha, \mathcal{D}_{X_0} f^\beta \otimes_{\mathcal{O}_{X_0}} \mathcal{M}^\cdot)_! = (f^{\alpha+\beta}, \mathcal{M}^\cdot)_! \tag{5.44.2}$$

for $\mathcal{M}^\cdot \in D^b_{rh}(\mathcal{D}_{X_0})$. Let X_i $(i = 1, 2)$ be non-singular varieties, $0 \neq f_i \in \mathbf{C}[X_i]$, and $X_{i,0} := X_i \setminus f_i^{-1}(0)$. Then

$$((f_1 \boxtimes f_2)^\alpha, \mathcal{M}^\cdot_1 \boxtimes \mathcal{M}^\cdot_2)_* = (f_1^\alpha, \mathcal{M}^\cdot_1)_* \boxtimes (f_2^\alpha, \mathcal{M}^\cdot_2)_*, \text{ and} \tag{5.44.3}$$

$$((f_1 \boxtimes f_2)^\alpha, \mathcal{M}^\cdot_1 \boxtimes \mathcal{M}^\cdot_2)_! = (f_1^\alpha, \mathcal{M}^\cdot_1)_! \boxtimes (f_2^\alpha, \mathcal{M}^\cdot_2)_!, \tag{5.44.4}$$

for $\mathcal{M}^\cdot_i \in D^b_{rh}(\mathcal{D}_{X_0})$. Now consider the case where $X_1 = X_2 = X$, $f_1 = f_2 = f$, $\mathcal{M}^\cdot_1 = \mathcal{M}^\cdot$, and $\mathcal{M}^\cdot_2 = \mathcal{D}_{X_0} f^{\beta-\alpha} \otimes_{\mathcal{O}_{X_0}} \mathcal{M}^\cdot$. Let $\Delta : X \to X \times X$ be the diagonal morphism, and consider the pull-back of (5.44.3) using (5.43.1) and (5.44.1). Then we get

$$(f^{\alpha+\beta}, \mathcal{M}^\cdot_1 \otimes^L_{\mathcal{O}_{X_0}} \mathcal{M}^\cdot_2)_* = (f^\alpha, \mathcal{M}^\cdot_1)_* \otimes^L_{\mathcal{O}_X} (f^\beta, \mathcal{M}^\cdot_2)_* \tag{5.44.5}$$

for $\mathcal{M}^\cdot_i \in D^b_{rh}(\mathcal{O}_{X_{i,0}})$.

Lemma 5.45. *If $\mathcal{M} \in \mathrm{Mod}_{rh}(\mathcal{D}_{X_0})$ is locally free $\mathcal{O}_{X_0}$-module of rank r, then*

$$\underline{\mathrm{ch}}(f^\alpha, \mathcal{M})_* = \underline{\mathrm{ch}}(f^\alpha, \mathcal{M})_! = r \cdot \underline{\mathrm{ch}}(\mathcal{D}_X f^\alpha).$$

(Here $\underline{\mathrm{ch}}(-)$ denotes the characteristic cycle. See (2.2.4).)
Proof. By (5.16), [Hi] and [La], we may assume that $f^{-1}(0)$ is normal crossing. Since the problem is local (with respect to the classical topology), we may assume that $X = \mathbf{C}^n = \{(x_1, \cdots, x_n)\}$ and f is a monomial. By the usual devissage, we may assume that $\mathcal{M} = \mathcal{D}_{X_0}(x_1^{\beta_1} \cdots x_n^{\beta_n})$ ($\beta_i \in \mathbf{C}$). Then the assertion becomes obvious. □

Lemma 5.46. *If $\mathcal{M} \in \mathrm{Mod}_{rh}(\mathcal{D}_{X_0})$, then $\underline{\mathrm{ch}}(f^\alpha, \mathcal{M})_* = \underline{\mathrm{ch}}(f^\alpha, \mathcal{M})_!$ and it is independent of $\alpha \in \mathbf{C}$.*

Proof. Since the assertion is of local nature, we may assume that $\mathcal{M}$ has a finite global generator system $\underline{u}$. Then it is enough to prove that
(5.46.1) $\underline{\mathrm{ch}}\, \mathcal{D}_X(f^\alpha \underline{u})$ *is independent of* $\alpha \in \mathbf{C}$.
As we can see from (5.40), we can reduce the proof to the case where $\mathcal{M}$ is a simple $\mathcal{D}_{X_0}$-module. By the argument of (5.30), we can reduce the proof to the case considered in (5.25). Thus we get the assertion by a direct calculation.

Characteristic cycle of $\mathcal{D}_X[s](f^s\underline{u})$

Lemma 5.50. *Let $\mathcal{D}_{X_0}\underline{u}$ be a regular holonomic $\mathcal{D}_{X_0}$-module such that $\mathrm{DR}_{X_0}(\mathcal{D}_{X_0}\underline{u})$ is a locally constant sheaf of rank r. Then $\mathcal{D}_X[s](f^s\underline{u})$ is $\mathcal{D}_X$-coherent, and*

$$\underline{\mathrm{ch}}(\mathcal{D}_X[s](f^s\underline{u})) = r \cdot [W_f],$$

where W_f is the Zariski closure in T^*X of

$$\{(x, s d\log f(x)) \in T^*X_0 \mid s \in \mathbf{C}^\times\}.$$

(Recall that $\underline{\mathrm{ch}}(-)$ denotes the characteristic cycle. Cf. (2.2.4).)
Proof. (1) First consider the case where

$$\begin{aligned} X &= \mathbf{C}^n, \\ [1,n] &= E \sqcup F, \\ f(x) &= \prod_{i\in E} x_i^{e_i} \ (e_i \in \mathbf{Z}_{>0}), \\ u(x) &= \prod_{i \in E} x_i^{\lambda_i} \ (\lambda_i \in \mathbf{C}), \end{aligned}$$

and $\underline{u}$ consists of only one section u. Then we can prove by a direct calculation.

(2) Next consider the case where X, $f(x)$ and $u(x)$ are the same as in (1), but $\underline{u} = (u_1, \cdots, u_p)$ ($u_i \in D_{X_0}(\prod_i x_i^{\lambda_i})$, $1 \le i \le p$) is an arbitrary global generator system of $D_{X_0}u$. Put Euler $:= \sum_{i=1}^n x_i \partial_{x_i}$, and decompose u_i as $u_i = \sum_{j=1}^{q_i} v_{ij}$ so that (Euler)$v_{ij} \in \mathbf{C}v_{ij}$. Put $\underline{v} = (v_{ij})_{ij}$. Then we get a natural morphism

$$N := D_X[s](f^s\underline{u}) \xrightarrow{\varphi} N' := D_X[s](f^s\underline{v}) = D_X(f^s\underline{v}).$$

Since $N[f^{-1}] \xrightarrow{\simeq} N'[f^{-1}]$ and since $N \subset N[f^{-1}]$ and $N' \subset N'[f^{-1}]$ by (5.9.5), φ is injective. Since N is a finite D_X-submodule of the finite D_X-module N', N is also D_X-finite. Since $N[t^{-1}] = N[f^{-1}] = N'[f^{-1}] = N'[t^{-1}]$, and since N and N' are $D_X[s]$-finite, $N \simeq t^m N \subset N'$ and $N' \simeq t^m N' \subset N$ for a sufficiently large integer m. Hence the assertion holds in this case.

(3) Consider the case where X and $f(x)$ are the same as in (1), but $D_{X_0}\underline{u}$ is arbitrary. We prove by the induction on the length l $(= r)$ of $D_{X_0}\underline{u}$. If $l = 1$, we have already done in (2). Assume $l > 1$, and let $D_{X_0}\underline{u}'$ be a simple submodule of $D_{X_0}\underline{u}$. Then $u'_i = f^{-k}\sum_j Q_{ij}u_j$ with some $k \in \mathbf{Z}_{\geq 0}$ and $Q_{ij} \in D_X$. Take $m \gg 0$ so that $f^{s+m-k}Q_{ij}f^{-s} =: P_{ij}(s) \in D_X[s]$. Then $f^{s+m}u'_i = \sum_j P_{ij}(f^s u_j) \in D_X[s](f^s\underline{u})$. Hence, replacing $\underline{u}'$ with $f^m\underline{u}'$, we may assume that $D_X[s](f^s\underline{u}') \subset D_X[s](f^s\underline{u})$.

Let $\underline{u}''$ be the image of $\underline{u}$ in $D_{X_0}\underline{u}/D_{X_0}\underline{u}'$. Put $N := D_X[s](f^s\underline{u})$, $N' := D_X[s](f^s\underline{u}')$ and $N'' := D_X[s](f^s\underline{u}'')$. Then the length of $D_{X_0}\underline{u}''$ is $l-1$, and

$$0 \longrightarrow N' \xrightarrow{B} N \xrightarrow{C} N'' \longrightarrow 0, \quad \text{and}$$
$$0 \longrightarrow N'[f^{-1}] \xrightarrow{B'} N[f^{-1}] \xrightarrow{C'} N''[f^{-1}] \longrightarrow 0$$

are exact sequences of $D_X[s,t]$-modules, possibly except for the middle term of the first sequence. Regard $N' \subset N$ by B. Put $R := \ker C$. Then

$$R \simeq t^m R = t^m(N'[f^{-1}] \cap N) = t^m(N'[t^{-1}] \cap N) \subset N' \subset R \quad (m \gg 0). \tag{5.50.1}$$

(Note that $D_X[s]$ is left noetherian, and hence $N'[t^{-1}] \cap N$ is a finite $D_X[s]$-module.) Since $t^m R$ is a D_X-submodule of the finite D_X-module N', R is also D_X-finite. From (5.50.1) and (2), we get $\underline{\mathrm{ch}}\, R = \underline{\mathrm{ch}}\, N' = [W_f]$. On the other hand, N'' is D_X-finite and $\underline{\mathrm{ch}}\, N'' = (r-1)[W_f]$ by the induction hypothesis. Hence we get the result in this case.

(4) Consider the case where X is general and $f^{-1}(0)$ is normal crossing. It is enough to prove the similar assertion in the category of $\mathcal{D}_X^{an}$-modules. Thus it follows from the $\mathcal{D}^{an}$-analogue of (3), which follows from (3).

(5) Consider the general case. We may and do assume X to be affine. Let $F : X' \to X$ be a proper modification of $f^{-1}(0)$ such that the zero locus of $f' := F^*f$ is normal crossing. Put $X'_0 := X' \setminus f'^{-1}(0)$ $(\simeq X_0)$, identify $\mathcal{D}_{X_0}\underline{u}$ with a $\mathcal{D}_{X'_0}$-module, and denote it by $\mathcal{D}_{X'_0}\underline{u}'$. Put

$$N := D_X[s](f^s\underline{u}),$$
$$N' := \Gamma(X, \int_F^0 \mathcal{D}_{X'}[s](f'^s\underline{u}')), \text{ and}$$
$$N'' := \mathrm{image}(N' \to N'[f^{-1}]).$$

Then these are $D_X[s,t]$-modules, and, N' and N'' are finite D_X-modules by (4). Although we know only the $D_X[s]$-finiteness of N at the first stage, we can apply (5.33.4) to the present N and N''. Since we have already obtained the D_X-finiteness of N'', N is also D_X-finite. Then the argument of (5.33) and (5.35) works in the present situation, and we get

$$\begin{aligned}\underline{\mathrm{ch}}(N) &= \underline{\mathrm{ch}}(N'') \quad \text{by (5.33.4)}\\ &= \underline{\mathrm{ch}}(N') - \sum_{\nu}[\Lambda_\nu] \quad \text{by (5.35.2)}\\ &= r\cdot[W_f] + \sum_{\mu}[\Lambda'_\mu] \quad \text{by (4), (5.26) and [Ka1, Proposition 4.9]}\end{aligned}$$

where Λ_ν and Λ'_μ are some holonomic varieties. Thus we get the assertion in the same way as the proof of [Ka1, Corollary 5.12].

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List of Symbols

5.1. $\mathcal{O} = \mathcal{O}_X$, $\mathcal{D} = \mathcal{D}_X$, $\mathcal{D}_A = \mathcal{D}_{X,A} := \mathcal{D}_X \otimes_{\mathbf{C}} A$, $D_A = D_{X,A} := D_X \otimes_{\mathbf{C}} A$, $\mathcal{D}[s] = \mathcal{D}_X[s] := \mathcal{D}_X \otimes_{\mathbf{C}} \mathbf{C}[s]$, $\mathcal{D}[s,t] = \mathcal{D}_X[s,t] := \mathcal{D} \otimes_{\mathbf{C}} \mathbf{C}[s,t]$
5.2. $b(s,N)$, $A_+(N)$, $A_-(N)$, $N(\alpha)$
5.9. $X_0 := X \setminus f^{-1}(0)$, $\mathcal{D}_X[s](f^s\underline{u}) = \sum_i \mathcal{D}_X[s](f^s\underline{u})_i$, $\mathcal{D}_X(f^\alpha\underline{u}) = \sum_i \mathcal{D}_X(f^\alpha\underline{u})_i$, $D_X[s](f^s\underline{u})$, $D_X(f^\alpha\underline{u})$, $f^s\underline{u}|V$, $(f^s\underline{u}|V)_i$, $f^\alpha\underline{u}|V$, $(f^\alpha\underline{u}|V)_i$, $gf^s\underline{u}|V$
5.16. $V \xrightarrow{j_V} X_0 \xrightarrow{j} X$
5.24. Z, L, $U \xrightarrow{j_0} Z_0 \xrightarrow{i_{Z_0}} X_0$
5.39. $\mathrm{Mod}_h(\mathcal{D}_X)$, $\mathrm{Mod}_{rh}(\mathcal{D}_X)$, $D^b_h(\mathcal{D}_X)$, $D^b_{rh}(\mathcal{D}_X)$, $D^b_c(\mathbf{C}_X)$
5.40. $(f^\alpha, \mathcal{M})_*$, $(f^\alpha, \mathcal{M})_!$
5.47. $\mathbf{K}$ (algebraic closure of $\mathbf{C}(s)$)