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EXPONENTIAL LIFTING AND HECKE CORRESPONDENCE

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ABSTRACT. We give a review of some recent results related with Siegel modular forms with respect to the paramodular groups: Exponential Lifting (the infinite product construction), Symmetrisation, multiplicative Hecke correspondence. We give many new examples of modular forms and new constructions of some classical Siegel modular forms (f.e. of the Igusa modular forms of weight 35).

§1. OPERATORS OF SYMETRIZATION

In what follows we shall consider Siegel modular forms with respect to the paramodular group Γ_t which is, by definition, an integral symplectic group of the skew-symmetric form with elementary divisors $(1,t)$. It can be realized as the following subgroup of $Sp_4(\mathbb{Q})$

$$\Gamma_t := \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & t^{-1}* \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \in Sp_4(\mathbb{Q}) \mid \text{ all } * \text{ are integral} \right\}.$$

The quotient

$$\mathcal{A}_t = \Gamma_t \setminus \mathbb{H}_2$$

is isomorphic to the coarse moduli space of Abelian surfaces with a polarization of type $(1,t)$ (see f.e. [Ig2]).

By $\mathfrak{M}_k(\Gamma_t,\chi)$ (resp. $\mathfrak{N}_k(\Gamma_t,\chi)$) we denote the space of all modular (resp. cusp) forms of weight k with respect to the group Γ_t with a character of finite order $\chi : \Gamma_t \to \mathbb{C}^*$.

In [G2] we studied Hecke operators which transform modular forms with respect to Γ_t into modular forms with respect to Γ_{tp}

$$\mathrm{Sym}_{t,p} : \mathfrak{M}_k(\Gamma_t) \to \mathfrak{M}_k(\Gamma_{tp}), \qquad \mathrm{Sym}_{t,p} : F \mapsto \sum_{M\in(\Gamma_t\cap\Gamma_{tp})\setminus\Gamma_{tp}} F|_k M.$$

We call this operator *the operator of p-symmetrisation.* It can be represented as an action of an element from the Hecke ring $H(\Gamma_{\infty,t})$ of the maximal parabolic subgroup $\Gamma_{\infty,t} \subset \Gamma_t$ which consists of elements fixing an isotropic line.

To see this, we take a system of representatives

$$(\Gamma_t \cap \Gamma_{tp}) \setminus \Gamma_{tp} = \{J_{tp}, \ \ \nabla(\frac{b}{tp}), \ b \bmod p\}$$

where

$$J_t = \begin{pmatrix} o & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \\ -1 & 0 & 1 & 0 \\ 0 & -t & 0 & 0 \end{pmatrix}, \qquad \nabla(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is valid $J_t J_{tp} = \mathrm{diag}(1, p, 1, p^{-1})$, thus for any $F \in \mathfrak{M}_k(\Gamma_t)$ we have

$$\mathrm{Sym}_{t,p}(F) = F|_k\big(\Lambda_p + \sum_{b \bmod p} \nabla(\frac{b}{tp})\big).$$

The last operator is defined by the following element in the Hecke ring of the maximal parabolic subgroup $\Gamma_{\infty,t}$ of Γ_t.

$$\mathrm{Sym}_p = \Lambda_p + \nabla_{t,p} \in H(\Gamma_{\infty,t}), \quad \text{where} \quad \nabla_{t,p} = \sum_{b \bmod p} \Gamma_{\infty,t}\nabla(\frac{b}{tp})$$

and $\Lambda_p = \Gamma_{\infty,t}\mathrm{diag}(p, 1, p^{-1}, 1)$. Using this description of the p-symmetrisation as a Hecke operator in the parabolic extension of the Hecke ring (i.e., in the Hecke ring of a maximal parabolic subgroup, see [G2], [G6]–[G7], [G10] for the theory of such rings) we proved

Theorem of Injectivity. *(see [G2]) The operator of p-symmetrisation is injective if* $(t, p) = 1$.

This theorem has the following application in algebraic geometry.

Corollary 2. *Let $\widetilde{\mathcal{A}_t}$ be a smooth compactification of the moduli space of $(1, t)$-polarized Abelian surfaces, then for arbitrary divisor d of t the operator of symmetrisation defines an embedding of the spaces of the canonical differential forms*

$$Sym : H^{3,0}(\widetilde{\mathcal{A}_d}) \to H^{3,0}(\widetilde{\mathcal{A}_t}).$$

In particular the inequality between the Hodge numbers of Siegel threefolds is valid

$$h^{3,0}(\widetilde{\mathcal{A}_t}) \geq h^{3,0}(\widetilde{\mathcal{A}_d}).$$

We recall that an estimation from bellow for the geometric genus $h^{3,0}(\mathcal{A}_t)$ of the moduli space $\mathcal{A}_t$ was obtained in [G1] and [G4]:

$$h^{3,0}(\widetilde{\mathcal{A}_t}) \geq \dim J^{cusp}_{3,t}$$

where $J^{cusp}_{3,t}$ is the space of Jacobi cusp forms of weight 3 and index t. The elements from $H^{3,0}(\widetilde{\mathcal{A}_t})$ are constructed using an arithmetic lifting, which is a generalization of the Maass lifting. We recall now the corresponding construction.

Let

$$\phi(\tau, z) \in J^{cusp}_{k,t}$$

be a Jacobi cusp form of weight k and index t. By definition $\widetilde{\phi}(Z) = \phi(\tau, z)\exp(2\pi i t\omega)$ where $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$. Let

$$T_-(m) = \sum_{\substack{ad=m \\ b \bmod d}} \Gamma_\infty \begin{pmatrix} a & 0 & b & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H(\Gamma_\infty), \tag{1.2}$$

where $\Gamma_\infty \subset Sp_4(\mathbb{Z})$ is the maximal parabolic subgroup of parabolic rank one, be an element of the Hecke ring of the parabolic subgroup Γ_∞. Then we have the following result

Arithmetic Lifting Theorem. *(see* [G1], [G4]–[G5]*). Let ϕ be as above. Then*

$$\mathrm{Lift}(\phi)(Z) = \sum_{m=1}^{\infty} m^{2-k}\left(\widetilde{\phi}\,|_k\, T_-(m)\right)(Z) \in \mathfrak{N}_k(\Gamma_t) \tag{1.3}$$

is a cusp form of weight k with respect to the paramodular group Γ_t.

The Rankin-Selberg convolution related with these modular forms is equal to the Spin (Andrianov) L-function of Siegel modular forms. In this way one can get a very short and clear construction of analytic continuation (together with the functional equation) of Spin-L-function (see [G3]).

The arithmetic lifting commutes with operator of p-symmetrisation

$$\mathrm{Sym}_{t,p}(\mathrm{Lift}(\phi)) = p^{3-k}\,\mathrm{Lift}(\phi|_k T_-(p)). \tag{1.4}$$

(see Satz 2.10 and Satz 3.1 in [G2]).

Let us define the multiplicative analogue of the p-symmetrisation.

Definition of Multiplicative Symmetrisation. Let $F \in \mathfrak{M}_k(\Gamma_t, \chi)$ be a modular form of weight $k \in \mathbb{Z}/2$ with respect to the paramodular group Γ_t with a character (or a multiplier system) $\chi : \Gamma_t \to \mathbb{C}^*$. Then for a prime p we define *the operator of multiplicative symmetrisation*

$$\mathrm{Ms}_p : F \mapsto p^{-k}\,\overline{\chi}(J_t) \prod_{M_i \in \Gamma_t \cap \Gamma_{tp} \backslash \Gamma_{tp}} F|_k M_i. \tag{1.5}$$

(The additional constant $p^{-k}\overline{\chi}(J_t)$ makes formulae simpler.)

It is clear that the result of the multiplicative symmetrisation is a modular form of weight $k(p+1)$

$$\mathrm{Ms}_p(F) \in \mathfrak{M}_{k(p+1)}(\Gamma_{tp}, \chi^{(p)})$$

where $\chi^{(p)}$ is a character of Γ_{tp}. Moreover if the modular form F is zero along a Humbert surface $H_l \subset \mathcal{A}_t$ of discriminant D, then $\mathrm{Ms}_p(F)$ is zero along $\mathrm{Ms}_p^*(H_l)$ which is a sum (with some multiplicities) of Humbert surfaces with discriminant D and $p^2 D$ in $\mathcal{A}_{tp}$ (for the definition of Humber surfaces see the next section). We remark that the operators of symmetrisation and multiplicative symmetrisation were used (in particular case $p = 2$) in the paper of Freitag [F] in 1967.

§2. Exponential Lifting

In this section we describe a construction which gives us modular forms with divisor equals to a union of Humbert surfaces. For our purpose it is more convenient to consider Humbert surfaces as divisors of a double cover of the moduli space of $(1,t)$-polarized Abelian surfaces $\mathcal{A}_t$. To define this covering we consider a double normal extension of the paramodular group Γ_t

$$\Gamma_t^+ = \Gamma_t \cup \Gamma_t V_t, \qquad V_t = \frac{1}{\sqrt{t}}\begin{pmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{pmatrix}.$$

The double quotient

$$\mathcal{A}_t \xrightarrow{2:1} \mathcal{A}_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2$$

of $\mathcal{A}_t$ can be interpreted as a moduli space of lattice-polarized $K3$ surfaces for arbitrary t or as the moduli space of Kummer surfaces of $(1,p)$-polarized Abelian surfaces for a prime $t = p$ (see [GH, Theorem 1.5]).

Any Humbert surface in $\mathcal{A}_t^+$ of discriminant D can be represented in the form

$$H_D^+(b) = \pi_t^+\Big(\bigcup_{g\in\Gamma_t^+} g^*(\{Z\in\mathbb{H}_2 \mid a\tau + bz + t\omega = 0\})\Big)$$

where $a, b \in \mathbb{Z}$, $D = b^2 - 4ta$, $b \bmod 2t$ and $\pi_t^+ : \mathbb{H}_2 \to \mathcal{A}_t^+$ (see [vdG], [GH]). Remark that $H_D^+(b)$ depends only on $\pm b \mod 2t$.

The datum for the exponential lifting is a nearly-holomorphic Jacobi form

$$\phi_{0,t}(\tau,z) = \sum_{n,l\in\mathbb{Z}} f(n,l)q^n r^l \in J_{0,t}^{nh} \tag{2.1}$$

of weight 0 and index t. Nearly holomorphic means that there exist a number m such that $\Delta^m(\tau)\phi(\tau,z)$ is a weak Jacobi form. If we chose the minimal non-negative m with this property then $n \ge -m$ in the Fourier expansion (2.1). We recall the notations

$$q = \exp(2\pi i\tau), \quad r = \exp(2\pi i z), \quad s = \exp(2\pi i\omega), \quad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$$

and $\widetilde{\phi}_{0,t}(Z) = \phi_{0,t}(\tau,z)\exp(2\pi i t\omega)$. Let

$$\phi_{0,t}^{(0)}(z) = \sum_{l\in\mathbb{Z}} f(0,l)\, r^l \tag{2.2}$$

be the q^0-part of $\phi_{0,t}(\tau,z)$. The Fourier coefficient $f(n,l)$ of $\phi_{0,t}$ depends only on the norm $4tn - l^2$ of (n,l) and $l \bmod 2t$. From the definition of nearly holomorphic forms, it follows that the norm of indices of non-zero Fourier coefficients are bounded from bellow. One can prove the following theorem which gives us a construction of Siegel modular forms as infinity products (compare with Borcherds construction in [B]).

Exponential Lifting Theorem. *(see* [GN6], [GN1]*) Assume that the Fourier coefficients of Jacobi form $\phi_{0,t}$ from* (2.1) *are integral. Then the product*

$$\text{Exp-Lift}(\phi_{0,t})(Z) = B_\phi(Z) = q^A r^B s^C \prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}} (1 - q^n r^l s^{tm})^{f(nm,l)}, \tag{2.3}$$

where

$$A = \frac{1}{24}\sum_l f(0,l), \quad B = \frac{1}{2}\sum_{l>0} l f(0,l), \quad C = \frac{1}{4}\sum_l l^2 f(0,l),$$

and $(n,l,m) > 0$ means that if $m > 0$, then l and n are arbitrary integers, if $m = 0$, then $n > 0$ and $l \in \mathbb{Z}$ or $l < 0$ if $n = m = 0$, defines a meromorphic modular form of weight $\frac{f(0,0)}{2}$ with respect to Γ_t^+ with a character (or a multiplier system if the weight is half-integral) induced by $v_\eta^{24A} \times v_H^{2B}$. All divisors of Exp-Lift$(\phi_{0,t})(Z)$ *on $\mathcal{A}_t^+$ are the Humbert modular surfaces $H_D(b)$ of discriminant $D = b^2 - 4ta$ with multiplicities*

$$m_{D,b} = \sum_{n>0} f(n^2a, nb).$$

Moreover

$$B_\phi(V_t(Z)) = (-1)^D B_\phi(Z) \qquad \textit{with} \quad D = \sum_{\substack{n<0\\ l\in\mathbb{Z}}} \sigma_1(-n) f(n,l)$$

where $\sigma_1(n) = \sum_{d|n} d$.

The infinite product expansion of modular form $\Delta_5(Z)$. (see [GN1]–[GN2]). We recall that there exists the unique, up to a constant, weak Jacobi form of weight zero and index one

$$\phi_{0,1}(\tau,z) = \frac{\phi_{12,1}(\tau,z)}{\eta(\tau)^{24}} = \sum_{n\geq 0,l} f_1 n, l q^n r^l$$
$$= (r + 10 + r^{-1}) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) + q^2(\dots)$$

where $\phi_{12,1}$ is the unique Jacobi cusp form of weight 12 and index 1 with integral coprime coefficients. There are several formulae for Fourier coefficients of this Jacobi form. In [EZ] one can find a formula for Fourier coefficients of $\phi_{12,1}$ it in terms of Cohen's numbers (values of special L-functions at integral points)

$$\phi_{12,1}(\tau,z) = 12^{-2}\big(E_4^2(\tau)E_{4,1}(\tau,z) - E_6(\tau)E_{6,1}(\tau,z)\big).$$

For a very convenient formula in terms of Hecke operators see (3.7)–(3.9) below. Let us consider the product of even theta-constants

$$\Delta_5(Z) = 2^{-6} \prod_{\substack{a,b\\ {}^t ab=0\,(mod\,2)}} \Theta_{a,b}(Z) \in \mathfrak{M}_5(Sp_4(\mathbb{Z}),\chi_2)$$

which is a cusp form of weight 5 with non-trivial binary character of $Sp_4(\mathbb{Z})$. The square of $\Delta_5(Z)$ is the first Siegel cusp form of weight 10 (see [Ig1]). Using the exponential lifting for the function $\phi_{0,1}$ we get the following result from [GN1]

$$\Delta_5(Z) = (qrs)^{1/2} \prod_{\substack{n,\, l,\, m \in \mathbb{Z} \\ (n,l,m)>0}} \left(1 - q^n r^l s^m\right)^{f_1(nm,l)} \in \mathfrak{N}_5(\Gamma_1, \chi_2) \tag{2.4}$$

where $f_1(n,l)$ are the Fourier coefficients of $\phi_{0,1}(\tau, z)$.

The general explanation of the existence of such infinite product expansion is the fact the modular form Δ_5 defines *a generalized Lorentzian Kac–Moody super-algebra* (see [GN1]).

Siegel theta-constant. Let us consider the "most odd" even Siegel theta-constant

$$\Theta_{1,1}(Z) = \frac{1}{2} \sum_{l_1, l_2 \in \mathbb{Z}} \exp\left(\pi i \left(Z\left[\begin{smallmatrix} l_1 + \frac{1}{2} \\ l_2 + \frac{1}{2}\end{smallmatrix}\right] + l_1 + l_2\right)\right) = \frac{1}{2} \sum_{n,\, m \in \mathbb{Z}} \left(\frac{-4}{n}\right)\left(\frac{-4}{m}\right) q^{n^2/8} r^{nm/4} s^{m^2/8},$$

where $Z[M] = {}^tMZM$. One can prove that the function

$$\Delta_{1/2}\left(\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\right) = \Theta_{1,1}\left(\begin{pmatrix} \tau & 2z \\ 2z & 4\omega \end{pmatrix}\right) \in \mathfrak{M}_{1/2}(\Gamma_4, \chi_8)$$

is a modular form of weight $1/2$ with respect to the paramodular group Γ_4 with a multiplier system of order 8 (see [GN6]). The first non-zero Fourier-Jacobi coefficient of $\Delta_{1/2}$ is the *Jacobi theta-series*

$$\vartheta(\tau, z) = \sum_{n \equiv 1 \bmod 2} (-1)^{\frac{n-1}{2}} \exp\left(\frac{\pi i\, n^2}{4}\tau + \pi i\, nz\right) = \sum_{m \in \mathbb{Z}} \left(\frac{-4}{m}\right) q^{m^2/8}\, r^{m/2},$$

or, equivalently,

$$\vartheta(\tau,\, z) = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1} r)(1 - q^n r^{-1})(1 - q^n). \tag{2.5}$$

We can define a weak Jacobi form

$$\phi_{0,4}(\tau, z) = \frac{\vartheta(\tau, 3z)}{\vartheta(\tau, z)} = \sum_{n \geq 0,\, l \in \mathbb{Z}} f_4(n, l) q^n r^l$$

$$= r^{-1} \prod_{m \geq 1} (1 + q^{m-1} r + q^{2m-2} r^2)(1 + q^m r^{-1} + q^{2m} r^{-2}) \prod_{\substack{n \equiv 1,2 \bmod 3 \\ n \geq 1}} (1 - q^n r^3)(1 - q^n r^{-3})$$

$$= (r + 1 + r^{-1}) - q(r^4 + r^3 - r + 2 - r^{-1} + r^{-3} + r^{-4}) + q^2(\dots) \tag{2.6}$$

where all Fourier coefficients $f_4(n,l)$ of the weak Jacobi form are integral (in fact they are Fourier coefficients of automorphic forms of weight $-1/2$). Thus according to Exponential Lifting Theorem Exp-Lift$(\phi_{0,4})$ is a modular form of weight $1/2$ with respect to the paramodular group Γ_4^+ having irreducible Humbert modular surface H_1 as its divisor. One can show that the quotient $\Delta_{1/2}(Z)/\text{Exp-Lift}(\phi_{0,4})(Z)$ is a holomorphic automorphic function invariant with respect to Γ_4^+, thus it is a constant and we get the following infinite product expansion of $\Delta_{1/2}(Z)$:

$$\frac{1}{2} \sum_{n,m \in \mathbb{Z}} \left(\frac{-4}{n}\right)\left(\frac{-4}{m}\right) q^{n^2/8} r^{nm/2} s^{m^2/2} = q^{1/8} r^{1/2} s^{1/2} \prod_{(n,l,m)>0} (1 - q^n r^l s^{4m})^{f_4(nm,l)}.$$

Modular forms $\Delta_1(Z)$ and $\Delta_2(Z)$. Let us define two weak Jacobi forms of weight 0

$$\phi_{0,2}(\tau,z)=\tfrac{1}{2}\eta(\tau)^{-4}\sum_{m,n\in\mathbb{Z}}(3m-n)\Big(\frac{-4}{m}\Big)\Big(\frac{12}{n}\Big)q^{\frac{3m^2+n^2}{24}}r^{\frac{m+n}{2}}$$

$$=\sum_{n\geq 0,\,l\in\mathbb{Z}}f_2(n,l)q^nr^l=(r+4+r^{-1})+q(r^{\pm 3}-8r^{\pm 2}-r^{\pm 1}+16)+q^2(\dots)\qquad(2.7)$$

($r^{\pm l}$ means that we have two summands with r^l and with r^{-l}) and

$$\phi_{0,3}(\tau,z)=\Big(\frac{\vartheta_{3/2}(\tau,z)}{\eta(\tau)}\Big)^2=\Big(\frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)}\Big)^2=\sum_{n\geq 0,\,l}f_3(n,l)q^nr^l$$

$$=r^{-1}\Big(\prod_{n\geq 1}(1+q^{n-1}r)(1+q^nr^{-1})(1-q^{2n-1}r^2)(1-q^{2n-1}r^{-2})\Big)^2$$

$$=(r+2+r^{-1})+q(-4r^{\pm 3}-4r^{\pm 2}+2r^{\pm 1}+4)+q^2(\dots)\qquad(2.8)$$

of index 2 and 3 respectively. Their Fourier coefficients satisfy the property: if $4tn-l^2<0$ and $f_t(n,l)\neq 0$, then $4tn-l^2=-1$ and $f_t(n,l)=1$.

According to Exponential Lifting Theorem Exp-Lift($\phi_{0,3}$) is a modular form of weight 1 with respect to Γ_3 with a character of order 6 and Exp-Lift($\phi_{0,2}$) is a modular form of weight 2 with respect to Γ_2 with a character of order 4. One can calculate the Fourier expansion of these modular forms. It turns out that (see [GN6])

$$\Delta_1(Z)=\sum_{M\geq 1}\sum_{\substack{m>0,\,l\in\mathbb{Z}\\ n,m\equiv 1\bmod 6\\ 4nm-3l^2=M^2}}\Big(\frac{-4}{l}\Big)\Big(\frac{12}{M}\Big)\sum_{a|(n,l,m)}\Big(\frac{6}{a}\Big)q^{n/6}r^{l/2}s^{m/2}$$

$$=q^{\frac{1}{6}}r^{\frac{1}{2}}s^{\frac{1}{2}}\prod_{\substack{n,l,m\in\mathbb{Z}\\ (n,l,m)>0}}\big(1-q^nr^ls^{3m}\big)^{f_3(nm,l)}\in\mathfrak{M}_1^{cusp}(\Gamma_3^+,\chi_6)$$

where the character $\chi_6:\Gamma_3^+\to\sqrt[6]{1}$ is induced by $v_\eta^4\times v_H$ and

$$\Delta_2(Z)=\sum_{N\geq 1}\sum_{\substack{m>0,\,l\in\mathbb{Z}\\ n,m\equiv 1\bmod 4\\ 2mn-l^2=N^2}}N\Big(\frac{-4}{Nl}\Big)\sum_{a\,|\,(n,l,m)}\Big(\frac{-4}{a}\Big)q^{n/4}r^{l/2}s^{m/2}$$

$$=q^{\frac{1}{4}}r^{\frac{1}{2}}s^{\frac{1}{2}}\prod_{\substack{n,l,m\in\mathbb{Z}\\ (n,l,m)>0}}\big(1-q^nr^ls^{2m}\big)^{f_2(nm,l)}\in\mathfrak{M}_2^{cusp}(\Gamma_2^+,\chi_4)$$

where $\chi_4:\Gamma_2^+\to\{\sqrt[4]{1}\}$. Moreover the divisor of these modular forms is the irreducible Humbert surface H_1

$$\mathrm{Div}_{\mathcal{A}_3^+}(\Delta_1(Z))=H_1,\qquad \mathrm{Div}_{\mathcal{A}_2^+}(\Delta_2(Z))=H_1.$$

These modular forms are *discriminant of the moduli space of special $K3$ surfaces* (see [GN3]). One can prove also that Δ_1^6 and Δ_2^4 are the cusp forms of the minimal weights with respect to Γ_3 and Γ_2 respectively. Using information about divisors of these modular forms one can easy prove the *rationality of the moduli space of Kummer surfaces* $\mathcal{A}_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2$ *associated with* $(1,2)$*- and* $(1,3)$*-polarized Abelian surfaces.*

Similar to the arithmetic lifting and the p-symmetrisation (1.4), the exponential lifting commutes with the multiplicative symmetrisation Ms_p (see (1.5)).

Theorem. *Let $\phi \in J_{0,t}^{nh}$ be like in Exponential Lifting Theorem. Then for an arbitrary prime p we have*

$$\mathrm{Ms}_p(\textit{Exp-Lift}(\phi_{0,t})) = \textit{Exp-Lift}(\phi_{0,t}|T_-(p)).$$

Proof. We give a proof of this theorem to illustrate the methods we use and to clarify the exponential lifting construction.

Firstly we use the following special representation of the exponential lifting. Let us decompose the product of the exponential lifting in two factors

$$B_\phi(Z) = q^A r^B s^C \prod_{(n,l,0)>0} (1-q^n r^l)^{f(0,l)} \times \prod_{\substack{n,l,m\in\mathbb{Z}\\ m>0}} (1-q^n r^l s^{tm})^{f(nm,l)} \tag{2.9}$$

and let us calculate the Fourier expansion of the logarithm of the second factor:

$$\log\Bigl(\prod_{\substack{n,l,m\in\mathbb{Z}\\ m>0}} (1-q^n r^l s^{tm})^{f(nm,l)}\Bigr) = -\sum_{n,l\in\mathbb{Z}\,,\, m>0} f(nm,l) \sum_{e\ge 1} \frac{1}{e}\, q^{en} r^{el} s^{emt}$$

$$= -\sum_{\substack{a,b,c\in\mathbb{Z}\\ c>0}} \Bigl(\sum_{d|(a,b,c)} d^{-1} f(\frac{ac}{d^2}, \frac{b}{d})\Bigr)\, q^a r^b s^{tc}.$$

Like in Arithmetic Lifting Theorem the last sum can be written as the action of the formal Dirichlet series (a formal Hecke L-function of $SL_2(\mathbb{Z})$) $\sum_{m\ge 1} m^{-1} T_-(m)$ on the Jacobi form $\phi_{0,t}$ where $T_-(m)$ (see (1.2)) are Hecke elements from the Hecke ring of the parabolic subgroup, Thus we obtain

$$\log\bigl(\prod_{\substack{n,l,m\in\mathbb{Z}\\ m>0}} \dots\bigr) = -\sum_{m\ge 1} m^{-1} \bigl(\widetilde{\phi}_{0,t}\,|\,T_-(m)\bigr)(Z).$$

This expansion shows us that the second factor in (2.9) is invariant with respect to the action of the parabolic subgroup $\Gamma_{\infty,t}$ whenever the product converges.

It is easy to see that the first factor in (2.9) is equal to a product of Jacobi theta-series and Dedekind eta-functions

$$q^A r^B s^C \prod_{(n,l,0)>0} (1-q^n r^l)^{f(0,l)} = \eta(\tau)^{f(0,0)} \prod_{l>0} \Bigl(\frac{\vartheta(\tau, lz) e^{\pi i l^2 \omega}}{\eta(\tau)}\Bigr)^{f(0,l)}.$$

The last identity explains the form of the factor $q^Ar^Bs^C$ in the definition of the function of the theorem. Thus we proved that

$$q^Ar^Bs^C \prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}} (1-q^nr^ls^m)^{f(nm,l)}$$
$$= \eta(\tau)^{f(0,0)} \prod_{l>0} \Big(\frac{\vartheta(\tau,lz)e^{\pi i l^2\omega}}{\eta(\tau)}\Big)^{f(0,l)} \exp\big(-\sum_{m\geq 1} m^{-1}\widetilde{\phi}_{0,t}|T_-(m)(Z)\big) \quad (2.10)$$

whenever the product converges. Thus $B_\phi(Z)$ transforms like a $\Gamma_{\infty,t}$-modular form of weight $\frac{f(0,0)}{2}$ with the multiplier system of the theorem. It is useful to write down the whole product $B_\phi(Z)$ in terms of Hecke operators $T_-(m)$. We can get such expression using the involution V_t

$$\text{Exp-Lift}(\phi_{0,t})(Z) = B_\phi(Z) =$$
$$= q^Ar^Bs^C \exp\big(-\sum_{m\geq 1} m^{-1}\widetilde{\phi}_{0,t}|T_-(m)(Z)\big)\exp\big(-\sum_{m\geq 1} m^{-1}(\widetilde{\phi}^{(0)}_{0,t}+\phi^{(0)}_{0,t})|T_-(m)|V_t(Z)\big). \quad (2.11)$$

The functions $\phi^{(0)}_{0,t}(z) = \sum_l f(0,l)r^l$ and $\widetilde{\phi}^{(0)}_{0,t}(Z) = \sum_l f(0,l)r^ls^t$ are not Jacobi forms, and we fix the standard system of representatives in $T_-(m)$ to define the corresponding formal action. The exponent of the function $\widetilde{\phi}^{(0)}_{0,t}$ in (2.11) defines the subproduct over all $(n,l,0)$ with $n>0$ in (2.9). The exponent with the function $\phi^{(0)}_{0,t}$, which does not depend on τ and ω, defines the finite subproduct over $(0,l,0)$ with $l>0$. The representation (2.11) shows us analogy between the exponential lifting and the arithmetic lifting of holomorphic Jacobi forms.

The formula for the multiplicative symmetrisation (see (1.5))

$$\text{Ms}_p(F)(Z) = F(\begin{pmatrix}\tau & pz\\ pz & p^2\omega\end{pmatrix}) \prod_{b \bmod p} F(\begin{pmatrix}\tau & z\\ z & \omega+\frac{b}{tp}\end{pmatrix}) \in M_{k(p+1)}(\Gamma_{tp},\chi^{(p)})$$

shows that Ms_p can be written as the action of Sym_p on the function under the exponent in (2.11). Let us consider the product of the formal Dirichlet series $\sum_{m=1}^\infty T_-(m)m^{-1}$ over the Hecke ring $H(\Gamma_{\infty,t})$ with $\text{Sym}_p = \Lambda_p + \nabla_{t,p}$. According to the definition of the normalizing factor of Hecke operators in the case of weight zero, we can consider the Hecke ring $H(\Gamma_{\infty,t})$ modulo its central element $\Delta(p) = \Gamma_{\infty,t}(pE_4)$. I.e. for any $X\in H(\Gamma_{\infty,t})$ the Hecke operators X and $\Delta(p)X$ are identical. We recall that $\Delta(p)\Lambda_p = T_-(p,p)$ where $T_-(p,p)$ is the embedding of $T(p,p) = SL_2(\mathbb{Z})(pE_2)SL_2(\mathbb{Z}) \in H(SL_2(\mathbb{Z}))$ into the Hecke ring $H(\Gamma_{\infty,t})$. Using the definition one can check that

$$T_-(m)\nabla_{t,p} = \begin{cases} pT_-(m) & \text{if } m\equiv 0 \bmod p\\ \nabla_{t,p}T_-(m) & \text{if } m\not\equiv 0 \bmod p.\end{cases}$$

Thus

$$\Big(\sum_{m=1}^{\infty} T_-(m)\,m^{-1}\Big)(T_-(p,p)+\nabla_{t,p})$$

$$= \sum_{m\ge 1}\Big(T_-(mp)+pT_-(\frac{m}{p})T_-(p,p)\Big)\,m^{-1} + \sum_{(m,p)=1} \nabla_{t,p}T_-(m)\,m^{-1}$$

$$= T_-(p)\sum_{m\ge 1} T_-(m)\,m^{-1} + \nabla_{t,p}\cdot \sum_{(m,p)=1} T_-(m)\,m^{-1}, \quad (2.12)$$

since $T(m)T(p)=T(mp)+pT(p,p)T(\frac{m}{p})$ in $H(SL_2(\mathbb{Z}))$.

Let us consider the representation (2.11) for the exponential lifting of $\phi_{0,t}$. According to (2.12) we have the following identity for the main factor in (2.11)

$$\exp\Big(-\sum_{m\ge 1}(\widetilde{\phi}_{0,t}|T_-(m))|\mathrm{Sym}_p(Z)\Big) = \exp\Big(-\sum_{m\ge 1}(\widetilde{\phi}_{0,t}|T_-(p))|T_-(m)(Z)\Big),$$

since $\nabla_{t,p}$ defines zero operator on the space of Jacobi functions of weight 0 and index t:

$$(\widetilde{\phi}_{0,t}|\nabla_{t,p})(Z) = \widetilde{\phi}_{0,t}(Z)\cdot \sum_{b \bmod p} \exp\big(\frac{2\pi i b}{p}\big) = 0.$$

Let us consider the second exponent in (2.11). We have the formal identity

$$V_t\Big(\Lambda_p + \sum_{b \bmod p} \nabla(\frac{b}{tp})\Big) = T_-(p)V_{tp}\,(\sqrt{p}E_4)^{-1}$$

where we consider $T_-(p)$ as the formal sum of the left cosets fixed in (1.2) and V_t is the involution defined in the beginning of this section. The second exponent in (2.11) is not $\Gamma_{\infty,t}$-invariant, but it is invariant with respect to the minimal parabolic subgroup Γ_{00} which is the intersection of $\Gamma_{\infty,t}$ with the subgroup of the upper-triangular matrices in Γ_t. This parabolic subgroup is the semidirect product of the subgroup of all upper-triangular matrices in $SL_2(\mathbb{Z})$ with the Heisenberg group. Thus we still can consider $T_-(m)$ as an element of the Hecke ring $H(\Gamma_{00})$ of this minimal parabolic subgroup if we take $T_-(m)$ in the standard form (1.2). (See [G9], where Hecke rings of parabolic subgroups of this type were considered for GL_n over a local field.) Thus for the second exponent in (2.11) we get

$$\exp\Big(-\sum_{m\ge 1} m^{-1}(\widetilde{\phi}^0_{0,t}+\phi^0_{0,t})|T_-(m)|V_t|\mathrm{Sym}_p\,(Z)\Big)$$

$$= \exp\Big(-\sum_{m\ge 1} m^{-1}(\widetilde{\phi}^0_{0,t}+\phi^0_{0,t})|T_-(p)|T_-(m)|V_{tp}\,(Z)\Big).$$

This finishes the proof.

□

Many examples of using the multiplicative symmetrisation see in [GN6].

§3. Multiplicative Hecke operators

Let $F \in \mathfrak{M}_k(\Gamma_t, \chi)$ be a modular form of integral weight and

$$X = \Gamma_t M \Gamma_t = \sum_i \Gamma_t g_i \in \mathcal{H}_*(\Gamma_t) = \bigotimes_{(p,t)=1} \mathcal{H}_p(\Gamma_t) \cong \bigotimes_{(p,t)=1} \mathcal{H}_p(\Gamma_1)$$

be a Hecke element with a good reduction modulo all primes dividing t. Then we can define *a multiplicative Hecke operator*

$$[F]_X := \prod_i F|_k g_i. \tag{3.1}$$

This is again a Γ_t-modular form. We call it *the Hecke product of F defined by X.*

It is well known that the arithmetic lifting (1.3) commutes with the action of Hecke operators (see f.e. [G2], [G8]). In Theorem A.7 of [GN4, Appendix A] we proved that the exponential lifting commutes with multiplicative Hecke operators. More exactly we have

Functoriality of Exponential Lifting. *For arbitrary $\phi \in J_{0,t}^{nh}$ and $X \in \mathcal{H}_*(\Gamma_t)$ the identity*

$$[\text{Exp-Lift}(\phi_{0,t})]_X = c \cdot \text{Exp-Lift}\big(\phi|\mathcal{J}_0^{(t)}(X)\big) \tag{3.2}$$

is valid, where $\mathcal{J}_0^{(t)}$ is a natural projection of the Hecke ring $\mathcal{H}_*(\Gamma_t)$ into the Hecke-Jacobi ring of the parabolic subgroup of Γ_t (see [G2], [G6], [G8]) and c is a constant.

Remark. The operator $\mathcal{J}_0^{(t)}$ is the same one which appears in the commutative relation between the arithmetic lifting and Hecke operators (see [G8]).

To prove this theorem one should use only some considerations related with parabolic extension of local Hecke rings, similar to the proof of the commutativity of the exponential lifting with multiplicative symmetrisation.

We consider below some examples related with the Hecke operator of index p $T(p) = \Gamma_t \mathrm{diag}(1,1,p,p)\Gamma_t$ in the case of good and bad reduction. If $T(p) \in H(\Gamma_t)$ ($(p,t) = 1$) we have

$$\begin{aligned} T(p) = \Gamma_t \begin{pmatrix} p&0&0&0\\ 0&p&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} &+ \sum_{a_1,a_2,a_3 \bmod p} \Gamma_t \begin{pmatrix} 1&0&a_1&a_2\\ 0&1&a_2&a_3/t\\ 0&0&p&0\\ 0&0&0&p \end{pmatrix} \\ &+ \sum_{a \bmod p} \Gamma_t \begin{pmatrix} 1&0&a&0\\ 0&p&0&0\\ 0&0&p&0\\ 0&0&0&1 \end{pmatrix} + \sum_{b_1,b_2 \bmod p} \Gamma_t \begin{pmatrix} p&0&0&0\\ -b_1&1&0&b_2/t\\ 0&0&1&b_1\\ 0&0&0&p \end{pmatrix} \end{aligned} \tag{3.3}$$

and

$$\mathcal{J}_0(T(p)) = T_0(p) + p^2 + p \tag{3.4}$$

where $T_0(p)$ is the Hecke-Jacobi operator

$$T_0(p) = \sum_{\substack{b \bmod p\\ c \bmod p^2}} \Gamma_\infty \begin{pmatrix} 1&0&c&b\\ 0&p&pb&0\\ 0&0&p^2&0\\ 0&0&0&p \end{pmatrix} + \sum_{\substack{a,b \bmod p\\ a\neq 0}} \Gamma_\infty \begin{pmatrix} p&0&a&ab\\ 0&p&ab&ab^2\\ 0&0&p&0\\ 0&0&0&p \end{pmatrix} + \sum_{\lambda \bmod p} \Gamma_\infty \begin{pmatrix} p^2&0&0&0\\ p\lambda&p&0&0\\ 0&0&1&-\lambda\\ 0&0&0&p \end{pmatrix}. \tag{3.5}$$

The element $T_0(p)$ defines an operator on the space of Jacobi forms which does not change the index

$$T_0(p) : J_{k,t} \to J_{k,t}.$$

It coincides (up to a constant) with the Hecke operator T_p defined in [EZ, §4].

We want to use (3.2) with $X = T(p)$ for the modular forms $\Delta_5(Z)$, $\Delta_2(Z)$, $\Delta_1(Z)$ and $\Delta_{1/2}(Z)$ from §2. To write down the right hand side of (3.2), we need a formula for the action of $T_0(p)$ on the Jacobi forms of index t for a divisor p of t. Let $\phi_{0,t}(\tau, z)$ be an arbitrary Jacobi form of weight zero and index t with Fourier expansion

$$\widetilde{\phi_{0,t}}(Z) = \sum_{n,l\in\mathbb{Z}} g(n,l)\, q^n r^l s^t = \sum_N g(N) \exp\left(2\pi i\, \mathrm{tr}(NZ)\right)$$

where $N = \left(\begin{smallmatrix} n & l/2 \\ l/2 & t \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ (t is fixed). In accordance with (3.5) we get

$$\widetilde{\phi_{0,t}}|T_0(p)(Z) =$$

$$\sum_{M_i\in\Gamma_\infty\backslash T_0(p)} \widetilde{\phi_{0,t}}|M_i(Z) = p^3 \sum_{\substack{l\equiv 0 \, mod\, p \\ n\equiv 0\, mod\, p^2}} g\left(\left(\begin{smallmatrix} n & l/2 \\ l/2 & t \end{smallmatrix}\right)\right) \exp\left(2\pi i\, \mathrm{tr}(p^{-2} N[\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)] Z)\right)$$

$$+ \sum_{n,l\in\mathbb{Z}} G_p(N) g(N) \exp\left(2\pi i\, \mathrm{tr}(NZ)\right) + \sum_{n,l\in\mathbb{Z}} g(N) \sum_{\lambda\, mod\, p} \exp\left(2\pi i\, \mathrm{tr}(N[\left(\begin{smallmatrix} p & 0 \\ \lambda & 1 \end{smallmatrix}\right)] Z)\right)$$

where $N[M] = {}^tMNM$ and we denote by $G_p(N)$ the Gauss sum

$$G_p(n,l,t) = -p + \sum_{a,b\, mod\, p} \exp\left(2\pi i\, \frac{na + lab + tab^2}{p}\right).$$

Changing N to $N[M^{-1}]$ in the first and third sum, we get the formula for the Fourier coefficient $g_p(n,l)$ of Jacobi form $\phi_{0,t}|T_0(p)(\tau, z)$ of index t

$$g_p(n,l) = p^3 g(p^2 n, pl) + G_p(n,l,t) g(n,l) + \sum_{\lambda\, mod\, p} g\left(\tfrac{n+\lambda l+\lambda^2 t}{p^2},\ \tfrac{l+2\lambda t}{p}\right) \tag{3.6}$$

where we put $g(n,l) = 0$ if $n \notin \mathbb{Z}$ or $l \notin \mathbb{Z}$. In the case of a good reduction, when $(p,t) = 1$, the $G_p(N)$ is given by the formula

$$G_p(n,l,t) = p\left(\frac{-(4nt - l^2)}{p}\right), \qquad (p,t) = 1.$$

If $p|t$, we can represent the formula for G_p in the form useful for exact calculations

$$G_p(n,l,t) = p \begin{cases} 0 & \text{if } l \not\equiv 0 \bmod p \\ p-1 & \text{if } (n,l) \equiv (0,0) \bmod p \\ -1 & \text{if } n \not\equiv 0 \bmod p, \text{ and } l \equiv 0 \bmod p. \end{cases}$$

For the Fourier coefficients depending on λ in (3.6) we have $\det\left(N[\left(\begin{smallmatrix} p & 0 \\ \lambda & 1 \end{smallmatrix}\right)]\right) = p^2 \det(N)$. If $(t,p) = 1$, then $n + \lambda l + \lambda^2 t$ is a full square mod p for $4nt - l^2 \equiv 0 \bmod p^2$. Thus there exists only one λ mod t which gives us a non-trivial term in the third summand in (3.6). Therefore we prove

Lemma. *Let us suppose that the Fourier coefficient $g(n,l)$ of the Jacobi form $\phi_{0,t}$ of weight zero and index t depends only on the norm $N = 4nt - l^2 \in \mathbb{Z}$. We denote $g(N) = g(n,l)$. For any prime p such that $(t,p) = 1$, the Fourier coefficients of $\phi_{0,t}|T_0(p)$ are given by the formula*

$$g_p(N) = p^3 g(p^2 N) + p\Big(\frac{-N}{p}\Big) g(N) + g\Big(\frac{N}{p^2}\Big)$$

where we set $g(\frac{N}{p^2}) = 0$ if $\frac{N}{p^2} \notin \mathbb{Z}$.

The Igusa modular form $\Delta_{35}(Z)$. The Igusa modular form $\Delta_{35}(Z)$ is the first Siegel modular form of odd weight with respect to $\Gamma_1 = Sp_4(\mathbb{Z})$. It has weight 35 (see [Ig1]). We defined this modular form in [GN4] as a Hecke product of $\Delta_5(Z)$.

Let us take the modular form $\Delta_5(Z)$ which has the divisor H_1 in $\mathcal{A}_1$. Using the system of representatives $T(p)$, we then get

$$[\Delta_5(Z)]_{T(2)} = \prod_{a,b,c \bmod 2} \Delta_5(\tfrac{z_1+a}{2}, \tfrac{z_2+b}{2}, \tfrac{z_3+c}{2}) \prod_{a \bmod 2} \Delta_5(\tfrac{z_1+a}{2}, z_2, 2z_3)\, \Delta_5(2z_1, z_2, \tfrac{z_3+a}{2})$$
$$\times\, \Delta_5(2z_1, 2z_2, 2z_3) \prod_{b \bmod 2} \Delta_5(2z_1, -z_1+z_2, \tfrac{z_1-2z_2+z_3+b}{2}).$$

One can check that $\operatorname{div}_{\mathcal{A}_1}\big([\Delta_5(Z)]_{T(p)}\big) = (p+1)^2 H_1 + H_{p^2}$. Thus

$$\Delta_{35}(Z) = \frac{[\Delta_5(Z)]_{T(2)}}{\Delta_5(Z)^8} = \text{Exp-Lift}\big(\phi_{0,1}|(T_0(2)-2)\big) \in \mathfrak{M}_{35}(\Gamma_1)$$

and

$$\Delta_{35}(Z) = q^2 r s^2\,(q-s) \prod_{\substack{n,\,l,\,m \in \mathbb{Z}\\ (n,l,m)>0}} \big(1 - q^n r^l s^m\big)^{f_1^{(2)}(4nm-l^2)}$$

where $f_1(4n-l^2) = f_1(n,l)$ are the Fourier coefficients of $\phi_{0,1}(\tau,z)$ (see (2.4)) and

$$f_1^{(2)}(N) = 8 f_1(4N) + 2(\Big(\frac{-N}{2}\Big) - 1) f_1(N) + f_1(\frac{N}{4})$$

according to Lemma above. Remark that we cannot construct $\Delta_{35}(Z)$ as an arithmetic lifting of a holomorphic Jacobi form. Nevertheless we get $\Delta_{35}(Z)$ as a finite Hecke product of the lifted form $\Delta_5(Z)$. In particular, from the infinite product formula follows the formula for the second Fourier-Jacobi coefficient $\phi_{35,2}$ of the Igusa modular form

$$\phi_{35,2}(\tau,z) = \eta^{69}(\tau)\vartheta_{11}(\tau,2z) = -q^2 r^{-1} \prod_{n \geq 1} (1-q^{n-1}r^2)(1-q^n r^{-2})(1-q^n)^{70}.$$

(Remark that $\phi_{35,1} = 0$.)

The modular form $D_6(Z)$. Using (3.2)–(3.3) and Lemma above we get modular forms with divisor H_{p^2} in $\mathcal{A}_2^+$ and $\mathcal{A}_3^+$ respectively

$$F_p^{(2)}(Z) = c_2 \frac{[\Delta_2(Z)]_{T(p)}}{\Delta_2(Z)^{(p+1)^2}} = \text{Exp-Lift}\big(\phi_{0,2}|(T_0(p)-p-1)\big) \in \mathfrak{M}_{2p(p^2-1)}(\Gamma_2),$$
$$F_p^{(3)}(Z) = c_3 \frac{[\Delta_1(Z)]_{T(p)}}{\Delta_1(Z)^{(p+1)^2}} = \text{Exp-Lift}\big(\phi_{0,3}|(T_0(p)-p-1)\big) \in \mathfrak{M}_{p(p^2-1)}(\Gamma_3, \chi_3^{(p)})$$

where $p \neq 2$ for Δ_2 and $p \neq 3$ for Δ_1. The character $\chi_2^{(p)}$ is trivial or has order two. (The c_p are constants which can be easily calculated.) In particular we get the modular form

$$D_6(Z) = \frac{2^{22}[\Delta_1(Z)]_{T(2)}}{\Delta_1(Z)^9} = \text{Exp-Lift}\big(\phi_{0,3}|(T_0(2)-3)\big).$$

Thus according to (3.9) where

$$\phi_{0,3}^{(6)}(\tau, z) = \phi_{0,3}|(T_0(2)-3)(\tau, z) = \sum_{n,l} g_3(n,l) q^n r^l = r^2 - r + 12 - r^{-1} + r^{-2} + q(\dots).$$

Example. *Hecke product for $p = t = 2$ and $p = t = 3$.* Using (3.6) we can consider the cases of bad reduction $p = t = 3$ and $p = t = 3$. The Hecke operator $T^+(2) = \Gamma_2^+ \operatorname{diag}(1,1,2,2)\Gamma_2^+$ from the Hecke ring $H(\Gamma_2^+)$ of the maximal normal extension Γ_2^+ contains 18 left cosets: the 15 cosets from (3.3) and

$$\sum_{\substack{a,b \bmod 2\\ (a,b)\neq(0,0)}} \Gamma_2^+ \begin{pmatrix} -a & 2 & b & 0\\ 1 & 0 & 0 & b/2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 2 & a \end{pmatrix}.$$

Therefore the modular form $[\Delta_2]_{T^+(2)}$ of weight 36 has divisor $2H_4 + 9H_1$ and we obtain the identity

$$\begin{aligned}\Delta_{11}(Z)^2 &= \big(\text{Lift}(\eta^{21}(\tau)\vartheta(\tau,2z))\big)^2 = \big(\text{Exp-Lift}(\phi_{0,2}^{(11)}(Z))\big)^2\\ &= c[\Delta_2]_{T^+(2)}(Z)/\Delta_2(Z)^7 = \text{Exp-Lift}(\phi_{0,2}|T_0(2))(Z)\end{aligned}$$

where

$$\phi_{0,2}^{(11)} = \phi_{0,1}|T_-(2) - 2\phi_{0,2} = \phi_{0,1}^2 - 20\phi_{0,2} = \frac{1}{2}\phi_{0,2}|T_0(2). \tag{3.7}$$

because

$$\phi_{0,2}|T_0(2)(\tau,z) = 2r^2 + 44 + 2r^{-2} + q(\dots).$$

The identity (3.7) gives us relations between the "fundamental" Jacobi forms $\phi_{0,1}$ and $\phi_{0,2}$.

Similarly to the case $p = 2$ we obtain for $p = t = 3$ the Jacobi form $\phi_{0,3}|T_0(3)(\tau,z) = 2r^3 + 68 + 2r^{-3} + q(\dots)$. Therefore the Hecke product for $p = 3$ of Γ_3-modular form Δ_1 is related with

$$F_{16}^{(3)}(Z) = \text{Exp-Lift}\,\big(\tfrac{1}{2}\phi_{0,3}|(T_0(3)-2)\big) \in \mathfrak{M}_{16}(\Gamma_3, v_\eta^8 \times \text{id}_H)$$

with divisor H_9. One can construct this modular form using multiplicative 3-symmetrisation of Δ_5. In terms of Jacobi forms it is equivalent to the relation

$$\phi_{0,3}|T_0(3) = 2\phi_{0,1}|T_-(3) - 6\phi_{0,3}.$$

Recall that the Jacobi form $\phi_{0,3}(\tau, z)$ is the square of an infinite product (see (2.8)).

Example. *Γ_4-modular forms with H_{p^2}-divisors.* Our next examples are connected with the Jacobi form $\phi_{0,4}$ (see (2.6)). In the case of good reduction ($p \neq 2$), we can define the same function as above. Since

$$\phi_{0,4}|T_0(p)(\tau, z) = r^p + pr + (p^3+1) + r^{-p} + pr^{-1} + q(\dots) \qquad (p \neq 2)$$

we obtain a modular form

$$F_p^{(4)}(Z) = \text{Exp-Lift}\big(\phi_{0,4}|(T_0(p) - p - 1)\big) \in \mathfrak{M}_{p(p^2-1)/2}(\Gamma_4, v_\eta^{p(p^2-1)} \times \mathrm{id}_H)$$

with divisor H_{p^2}. For instance for $p = 3$ we get a modular form of weight 12 with divisor H_9 in $\mathcal{A}_4^+$. Using (3.6) for $p = 2$, we obtain a very nice identity

$$\phi_{0,4}|T_0(2)(\tau, z) = \phi_{0,1}(\tau, 2z). \tag{3.8}$$

It gives us the second new formula for the generator $\phi_{0,1}(\tau, z)$ (see [2.?]). We can represent it in an equivalent form using the operator Λ_2^* dual to Λ_2^*. One can check (see [G2]) that

$$p^{-1}T_0(p) \cdot \Lambda_2^* = T_+(1, p^2) = \sum_{a,b,c \bmod p^2} \Gamma_\infty \begin{pmatrix} 1 & 0 & a & pb \\ 0 & 1 & b & c \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} + \sum_{\substack{a,b \bmod p \\ c \bmod p^2 \\ a \neq 0}} \Gamma_\infty \begin{pmatrix} p & 0 & a & pb \\ 0 & 1 & b & c \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}$$

$$+ \sum_{\substack{a \bmod p \\ c \bmod p^2}} \Gamma_\infty \begin{pmatrix} p^2 & 0 & 0 & 0 \\ -a & 1 & 0 & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & p \end{pmatrix}.$$

Thus (3.8) is equivalent to

$$8\phi_{0,1}(\tau, z) = \phi_{0,4}|T_+(1,4)(\tau, z). \tag{3.9}$$

We recall that $\phi_{0,4}(\tau, z)$ is given by an infinite product (see (2.6)). One can find a formula for the action of $T_+(1,4)$ on Fourier coefficients of Jacobi forms similar to the formula (3.19) for $T_+(2)$.

Anti-symmetric Modular Forms.

The arithmetic lifting provides us with modular forms which are invariant with respect to the main exterior involution V_t of the group Γ_t (see the beginning of §2). Using the exponential lifting, one can construct anti-invariant modular forms, i.e. forms satisfying $F(V_t(Z)) = -F(Z)$. For example for $t = 1$ the Igusa modular form $\Delta_{35}(Z)$ is anti-invariant.

Existence Theorem. *(see [GN6]) For arbitrary $t > 1$ there exists an anti-symmetric modular form of weight 12 with respect to Γ_t with trivial character.*

We construct below such forms for $t = 2, 3, 4$ and show that *for these t anti-symmetric form of weight 12 is unique.*

Let us consider the function $\psi_{0,t}(\tau, z) = \Delta(\tau)^{-1}E_{12,t}(\tau, z)$ where

$$\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + 253q^3 + \dots$$

and $E_{12,t}(\tau, z)$ is a Jacobi–Eisenstein series of weight 12 and index t.

There exists a formula for Fourier coefficients of $E_{k,1}$ in terms of H. Cohen's numbers (see [EZ, §2]). One can find the table of the values of Fourier coefficients of $E_{4,1}(\tau, z)$ and $E_{6,1}(\tau, z)$ in [EZ, §1]. Using the basic Jacobi forms $\phi_{0,2}(\tau, z) = r^{\pm 1} + 4 + \dots$, $\phi_{0,3}(\tau, z) = r^{\pm 1} + 2 + \dots$ and the forms $\phi_{0,2}^{(11)} = r^{\pm 2} + 22 + \dots$ and $\phi_{0,3}^{(6)} = r^{\pm 2} - r^{\pm 1} + 12 + \dots$, which are the data for the exponential liftings in §2. We define

$$\begin{aligned}
\psi_{0,2}(\tau, z) &= \Delta(\tau)^{-1}E_{6,1}(\tau, z)^2 - 2\phi_{0,2}^{(11)}(\tau, z) + 176\phi_{0,2}(\tau, z)\\
&= \sum_{n\geq 0,\, l\in\mathbb{Z}} c_2(n, l)q^n r^l = q^{-1} + 24 + q(\dots),\\
\psi_{0,3}(\tau, z) &= \Delta(\tau)^{-1}E_{4,1}(\tau, z)^3 - 3\phi_{0,3}^{(6)}(\tau, z) - 171\phi_{0,3}(\tau, z)\\
&= \sum_{n\geq 0,\, l\in\mathbb{Z}} c_3(n, l)q^n r^l = q^{-1} + 24 + q(\dots).
\end{aligned}$$

The Jacobi forms $\psi_{0,p}$ ($p = 2, 3$) contain the only type of Fourier coefficients with indices of negative norm. This is q^{-1} of norm $-4p$. Thus we can use both functions to produce the exponential liftings

$$\Psi_{12}^{(2)}(Z) = \text{Exp-Lift}(\psi_{0,2}) = q\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}(1 - q^n r^l s^{2m})^{c_2(nm,l)} \in \mathfrak{M}_{12}(\Gamma_2), \tag{3.10}$$

$$\Psi_{12}^{(3)}(Z) = \text{Exp-Lift}(\psi_{0,3}) = q\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}(1 - q^n r^l s^{3m})^{c_3(nm,l)} \in \mathfrak{M}_{12}(\Gamma_3). \tag{3.11}$$

According to Exponential Lifting Theorem

$$\Psi_{12}^{(p)}(V_p < Z >) = -\Psi_{12}^{(p)}(Z) \quad (p = 2, 3) \quad \text{and} \quad \mathrm{Div}_{\mathcal{A}_p}(\Psi_{12}^{(p)}) = \begin{cases} H_8 & \text{for } p = 2\\ H_{12} & \text{for } p = 3.\end{cases}$$

The Fourier-Jacobi expansion of $\Psi_{12}^{(p)}$ starts with coefficients

$$\Psi_{12}^{(p)}(Z) = \Delta_{12}(\tau) - \Delta_{12}(\tau)\psi_{0,p}(\tau, z)\exp(2\pi i p\omega) + \dots.$$

Therefore the constructed modular forms $\Psi_{12}^{(p)}(Z)$ ($p = 2, 3$) are not cusp forms.

If we do the same for $t=4$, we get a Jacobi form we used to construct $\Delta_{35}(Z)$. Let us take the Jacobi form

$$\phi_{0,1}|(T_0(2)-2)(\tau,2z)=q^{-1}+(r^4+70+r^{-4})+q(\dots).$$

Its exponential lifting is zero along two Humbert surfaces with discriminant 16. To delete the second component, we consider the additional Jacobi–Eisenstein series which has the constant term equals zero (such a series exists if the index contains a perfect square). For $t=4$ this Jacobi–Eisenstein series is the eight power of the Jacobi theta-series $\vartheta(\tau,z)$.

Using $\vartheta(\tau,z)^8$, we define

$$\begin{aligned}\psi_{0,4}(\tau,z)&=\big(\phi_{0,1}|(T_0(2)+26)\big)(\tau,2z)-\Delta(\tau)^{-1}E_4(\tau)\vartheta(\tau,z)^8-8\big(\phi_{0,4}|(T_0(3)+4)\big)(\tau,z)\\&=\sum_{n\ge0,\,l\in\mathbb{Z}}c_4(n,l)q^nr^l=q^{-1}+24+q(\dots).\end{aligned}$$

Similarly to $\psi_{0,2}$ and $\psi_{0,3}$ the Jacobi form $\psi_{0,4}$ contains only the Fourier coefficients of type q^{-1} with index of negative norm. Taking its exponential lifting we obtain the Γ_4-modular form of weight 12

$$\Psi_{12}^{(4)}(Z)=\text{Exp-Lift}(\psi_{0,4})(Z)=q\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}(1-q^nr^ls^{4m})^{c_4(nm,l)}\in\mathfrak{M}_{12}(\Gamma_4). \qquad (3.12)$$

The modular form $\Psi_{12}^{(4)}(Z)$ is anti-invariant and $\mathrm{Div}_{\mathcal{A}_4}(\Psi_{12}^{(4)})=H_8(0)$.

Uniqueness of $\Psi_{12}^{(t)}$ for $t=2,3,4$. Arbitrary anti-invariant modular form is automatically zero along the Humbert surface $H_{4t}(0)=\pi_t\{\tau-t\omega=0\}$. The modular forms constructed above have this surfaces as full divisor. Thus they are unique.

Remark. After this conference a preprint of Ibukiyama and Onodera [OK] has been appeared where an anti-invariant modular form of weight 12 with respect to Γ_2 was constructed in terms of Siegel theta-constants.

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