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Asymptotic relations between eigenvalues and Fourier coefficients

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In this short note we would like to report on certain asymptotic relations between p-eigenvalues and certain Fourier coefficients of Siegel cusp forms of genus g. In particular, it will turn out that potential strong bounds for the Fourier coefficients will imply potential strong bounds for the eigenvalues. For more details the reader is referred to [2].

We let H_g be the Siegel upper half-space of genus g, with the usual operation of the symplectic group $Sp_g(\mathbf{R})$. We let $\Gamma_g = Sp_g(\mathbf{Z})$ be the Siegel modular group and for a natural number k denote by $S_k(\Gamma_g)$ the space of Siegel cusp forms of weight k and genus g. If $F \in S_k(\Gamma_g)$ we write $a(T)$ for the Fourier coefficients of F, with T a positive definite half-integral matrix of size g. For p a prime, we denote by T_p the usual Hecke operator on $S_k(\Gamma_g)$ defined by

$$T_pF = p^{gk-g(g+1)/2} \sum_{\gamma\in\Gamma_g\backslash\mathcal{O}_{g,p}} F\mid_k \gamma,$$

where $\mathcal{O}_{g,p}$ is the set of integral symplectic similitudes of size $2g$ with scale p and

$$(F\mid_k \gamma)(Z) = \det(CZ+D)^{-k}F((AZ+B)(CZ+D)^{-1}) \quad (Z\in\mathcal{H}_g)$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Let $g \geq 2$ and let F be a Hecke eigenform of all T_p with $T_pF = \lambda_pF$. Both for the Fourier coefficients and the eigenvalues of F there are "generalized Ramanujan-Petersson conjectures" stating that

$$a(T) \ll_{F,\epsilon} (\det T)^{\frac{k}{2}-\frac{g+1}{4}+\epsilon} \qquad (\epsilon>0) \tag{1}$$

(Resnikoff-Saldaña) and

$$\lambda_p \ll_{g,\epsilon} p^{\frac{gk}{2}-\frac{g(g+1)}{4}+\epsilon} \qquad (\epsilon>0) \tag{2}$$

(Kurokawa, Satake, Langlands), respectively.

Both conjectures in general are known to be wrong. If $g=2$, forms F that are Saito-Kurokawa lifts give counterexamples both to (1) and (2). Also (first noticed by Freitag) if $8|g$, there are certain theta series with spherical harmonics of weight $\frac{g}{2}+1$ whose Fourier coefficients do not satisfy (1). Nevertheless, there is some hope that (1) and (2) "generically" should be true.

If $g=2$ and $k\geq 3$, according to Weissauer (2) is true if F is not attached to a CAP-representation.

Here we are interested in the question what bounds for the Fourier coefficients would imply what bounds for the eigenvalues. A first result in this direction is the following

Theorem 1 (Duke-Howe-Li [1]). *One has*

$$\lambda_p a(T) - a(pT) \ll_F (\det T)^{k/2} p^{gk/2-1}.$$

For the *proof* one writes explicitly

$$(T_pF)(Z) = p^{gk-g(g+1)/2} \sum_{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}} (\det D)^{-k} F((AZ+B)D^{-1}),$$

where D runs over all left non-associated (w.r.t. $GL_g(\mathbf{Z})$) right divisors of pE, $A = pD'^{-1}$, and (for fixed D) B runs over all matrices in $\mathbf{Z}^{(g,g)}$ with $B'D = D'B$ modulo the equivalence relation $B_1 \sim B_2$ iff $B_2 = B_1 + SD$ with S symmetric. Putting in the Fourier expansion of F one observes that the term $D = pE$ gives exactly the contribution $a(pT)$. The other terms can be estimated using Hecke's bound $a(T) \ll_F (\det T)^{k/2}$ and the fact that

$$\sum_{\{D,B\}/\sim} 1 = \prod_{j=1}^{g} (1+p^j),$$

hence

$$\sum_{\{D,B\}/\sim, D \neq pE} 1 \ll p^{g(g+1)/2-1}.$$

The assertion now easily follows.

Theorem 1 can be sharpened as follows

Theorem 2 [2]. *Let $\alpha \geq 0$ be fixed and suppose that the bound*

(3) $$a(T) \ll_F (\det T)^{\frac{k}{2}-\alpha}$$

holds for all T. Then one has

$$\lambda_p a(T) - a(pT) \ll_F (\det T)^{\frac{k}{2}-\alpha} p^{\frac{gk}{2}-\kappa_\alpha} \qquad (p > 2,\ p \nmid \det(2T))$$

where

$$\kappa_\alpha := \begin{cases} g\alpha - (2\alpha - 2) & (0 \leq \alpha \leq \frac{3}{2}) \\ g\alpha - (\alpha - \frac{1}{2})^2 & (\alpha \geq \frac{3}{2}). \end{cases}$$

Observe that in the range $0 \leq \alpha \leq \frac{g+1}{4}$ the function $\alpha \mapsto \kappa_\alpha$ is positive and non-decreasing.

For the *proof* one studies the above set of representatives $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ for $\Gamma_g \backslash \mathcal{O}_{g,p}$ in much more detail, making use of older results of Maass. In particular, the presence of certain exponential sums gives rise to cancellations of various terms. Further, one uses results of Siegel on the number of representations of quadratic forms by quadratic forms modulo p.

Theorem 2 very recently has been generalized to Hecke operators of prime power index p^m (p odd) by M. Kuß.

As a corollary to Theorem 2 one obtains immediately

Corollary. *Under the hypothesis (3) one has*

$$\lambda_p \ll_F p^{\frac{gk}{2}-\kappa'_\alpha} \qquad (p \to \infty) \tag{4}$$

where

$$\kappa'_\alpha := \begin{cases} g\alpha & (0 \le \alpha \le 1) \\ \kappa_\alpha & (\alpha \ge 1). \end{cases}$$

It follows from (4) that the bound (1) implies the bound (2) if $g = 3$.

Unfortunately, using Theorem 2 the estimates so far truly proved for the Fourier coefficients imply much weaker estimates for the eigenvalues than obtained previously by other tools, e.g. local representation theory (Duke-Howe-Li) or arithmetic algebraic geometry (Hatada). Nevertheless, we hope that Theorem 2 would give some (more) motivation to study estimates for Fourier coefficients more closely.

References

[1] Duke, W., Howe, R. and Li, J.-S.: Estimating Hecke eigenvalues of Siegel modular forms. Duke Math. J. 67, no.1, 219-240 (1992)

[2] Kohnen, W.: Fourier coefficients and Hecke eigenvalues. To appear in Nagoya Math. J.