

Equivariant Holomorphic Embeddings of Symmetric Domains and Applications

Ichiro Satake (Chuo University)
(中央大学・理工学部 佐武一郎)

This is a slightly expanded version of my talks given at the Workshop on Automorphic Forms and Zeta Functions held at the RIMS in January, 1997. The purpose of these talks was to give a survey of my old results (with some new aspects) on equivariant holomorphic embeddings of a symmetric domain into another symmetric domain. In the first three sections, I give basic definitions and properties of hermitian symmetric pairs and (strongly) equivariant holomorphic maps (also called "modular embeddings"). Then, in the remaining sections, I explain the solutions to our main problems (P1) and (P2) (see §3). The problem (P1) was raised by Kuga (1963) in connection with the construction of certain fiber spaces whose fibers are abelian varieties. The problem (P2) gives an algebraic interpretation of the theory of boundary components of a symmetric domain and the Siegel domain realizations of it, initiated by Piatetski-Shapiro (1961) and completed (analytically) by Wolf and Koranyi (1965).

1. Hermitian symmetric pairs

A pair (G, \mathcal{D}) formed of a real Lie group G and a complex manifold \mathcal{D} is called a *hermitian symmetric pair* (*h.s.p.* for short) if G is the identity connected component (in the usual topology) of the group of real points in a semisimple algebraic group defined over \mathbf{R} (for simplicity, such a group G is called "a connected semisimple \mathbf{R} -group") and if G is acting transitively and holomorphically on \mathcal{D} in such a way that, for any $o \in \mathcal{D}$, the stabilizer $K = G_o$ is a maximal compact subgroup of G . Then, \mathcal{D} can naturally be identified with the coset space G/K . The largest compact normal subgroup G_0 of G acts trivially on \mathcal{D} and one has $G/G_0 \cong (\text{Aut } \mathcal{D})^\circ$, $^\circ$ denoting the identity connected component.

It is well known that a complex manifold \mathcal{D} appears in a h.s.p. if and only if \mathcal{D} is holomorphically equivalent to a bounded symmetric domain. For such a domain \mathcal{D} , the pair $((\text{Aut } \mathcal{D})^\circ, \mathcal{D})$ is a h.s.p. On the other hand, a semisimple \mathbf{R} -group G appears in a h.s.p. if and only if $\text{Ad}(G)(= G/(\text{center}))$ carries "Hodge structures".

In order to define the Hodge structure (in the sense of Deligne), let $S = \text{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)$. Then $S(\mathbf{C})$ is identified with $\mathbf{C}^* \times \mathbf{C}^*$. One denotes by χ_i ($i = 1, 2$) the characters of S defined by the projections to the first and second factors; then one has $\chi_2 = \bar{\chi}_1$. Let $S^{(1)}$ be the kernel of $\chi_1\chi_2$; then $S^{(1)}$ is an \mathbf{R} -form of \mathbf{G}_m , for which one has $S^{(1)}(\mathbf{R}) \cong \mathbf{C}^{(1)}$ and $S^{(1)}(\mathbf{C}) \cong \mathbf{C}^*$ (by χ_1). By a *Hodge structure* of a semisimple (or, more generally, reductive)

R-group G we mean an **R**-homomorphism $\mu : \mathbf{C}^{(1)} \rightarrow G$ such that the Lie algebra $\mathfrak{g}_\mathbf{C}$ of $G(\mathbf{C})$ is a direct sum of three eigen spaces

$$\mathfrak{g}_\mathbf{C}(\mu; \nu) = \{x \in \mathfrak{g}_\mathbf{C} \mid \text{Ad}(\mu(\xi))x = \chi_1(\xi)^\nu x \ (\forall \xi \in S^{(1)}(\mathbf{C}))\} \quad (\nu = -2, 0, 2)$$

and that, for a maximal compact subgroup K , one has

$$(1) \quad \mathfrak{k}_\mathbf{C} = \mathfrak{g}_\mathbf{C}(\mu; 0), \quad \mathfrak{p}_\mathbf{C} = \mathfrak{g}_\mathbf{C}(\mu; -2) + \mathfrak{g}_\mathbf{C}(\mu; 2),$$

where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g} = \text{Lie } G$ with $\mathfrak{k} = \text{Lie } K$. For a Hodge structure μ of G (belonging to K), there corresponds uniquely an element H_o in \mathfrak{g} by the relation

$$(2) \quad \mu(e^{it}) = \exp(2tH_o) \quad (t \in \mathbf{R}).$$

Then, H_o is in the center of \mathfrak{k} and one has $(\text{ad } H_o|_{\mathfrak{p}})^2 = -1$; conversely, if one has an element H_o with this property, then one obtains a Hodge structure of $\text{Ad}(G)$ by defining μ by (2) (in $\text{Ad}(G)$). Such an element H_o is called an *H-element* in \mathfrak{g} (belonging to \mathfrak{k}). A semisimple **R**-group G or the corresponding Lie algebra \mathfrak{g} is called of *hermitian type* if $\text{Ad}(G)$ carries a Hodge structure $\mu : \mathbf{C}^{(1)} \rightarrow \text{Ad}(G)$ or, equivalently, if there exists an *H-element* H_o in \mathfrak{g} . It is clear that a compact semisimple **R**-group G carries a unique (trivial) Hodge structure with $H_o = 0$. (Note also that if a reductive **R**-group G carries a Hodge structure, then the center of G is compact.)

Now, it is well known that, for a h.s.p. (G, \mathcal{D}) and $o \in \mathcal{D}$, there exists uniquely a Hodge structure μ of $\text{Ad}(G)$ (belonging to $\text{Ad}(K)$) or, equivalently, an *H-element* H_o (belonging to $\mathfrak{k} = \text{Lie } K$) such that the complex structure and the symmetry of \mathcal{D} at the point o are given by $\mu(e^{\pi i/4}) = \exp(\frac{\pi}{2}H_o)$ and $\mu(i) = \exp(\pi H_o)$, respectively. Conversely, if G is a semisimple **R**-group of hermitian type, the coset space G/K can be realized in a canonical manner as a symmetric bounded domain \mathcal{D} in $\mathfrak{g}_\mathbf{C}(\mu; 2)$ (Harish-Chandra realization), so that the pair (G, \mathcal{D}) becomes a h.s.p.

It should be noted that, in general, a semisimple **R**-group of hermitian type G itself may or may not carry Hodge structures. As we shall see later on, the symplectic group $Sp_{2r}(\mathbf{R})$ (in particular, $SL_2(\mathbf{R})$) has a Hodge structure, whence follows that any G of tube type has one (cf. Th. 8). However, $SU(p, q)$ with $p \neq q$ does not carry Hodge structures, while $U(p, q)$ does.

2. The classification

Let (G, \mathcal{D}) be a h.s.p. In general, G may have a compact factor (which acts trivially on \mathcal{D}). When G has no compact factor (of positive dimension), i.e., when G is isogenous with $(\text{Aut } \mathcal{D})^\circ$, we say that the pair (G, \mathcal{D}) (or the **R**-group G) is *proper*. When one considers G over \mathbf{R} , one may assume G to be proper. However, when one considers **Q**-structures of G , it is important to include the improper case.

A h.s.p. (G, \mathcal{D}) is called (geometrically) *irreducible*, if \mathcal{D} is irreducible, or equivalently, if the non-compact part of G is (almost) simple. Note that, in the irreducible case (with a Hodge structure), $\mu(\mathbf{C}^{(1)})$ coincides with the center of K . Any h.s.p. is isogenous to the direct product of the irreducible ones in an obvious sense. The proper irreducible h.s.p. are classified as follows.

$$\mathcal{D} = (\text{I}_{p,q}) (p \geq q \geq 1), (\text{II}_p) (p \geq 3), (\text{III}_p) (p \geq 1), \\ (\text{IV}_p) (p \geq 3), (\text{V}), (\text{VI}).$$

Correspondingly, one has

$$\mathfrak{g}_{\mathcal{D}} = (\text{A}_{p+q-1}), (\text{D}_p), (\text{C}_p), (\text{BD}_{[p/2]+1}), (\text{E}_6), (\text{E}_7). \\ \dim \mathfrak{g} = (p+q)^2 - 1, 2p^2 - p, 2p^2 + p, \frac{1}{2}(p+1)(p+2), 78, 133. \\ r = \mathbf{R}\text{-rank } \mathfrak{g} = \text{Min}(p, q), \left[\frac{p}{2}\right], p, 2, 2, 3. \\ n = \dim \mathcal{D} = pq, \frac{1}{2}p(p-1), \frac{1}{2}p(p+1), p, 16, 27.$$

EXAMPLE 1. As a typical example of h.s.p. we recall the definition of the Siegel space. Let V be a real vector space of dimension $2r$ endowed with a non-degenerate alternating bilinear form a on $V \times V$ (viewed also as a linear map $a : V \rightarrow V^*$). Then, by definition, one has

$$(3) \quad G = Sp(V, a) = \{g \in GL(V) \mid {}^t g a g = a\}, \\ \mathcal{D} = \mathbf{S}(V, a) = \{I \in GL(V) \mid I^2 = -1, aI > 0\},$$

$S > 0$ meaning that S is symmetric and positive definite. G acts transitively on \mathcal{D} by $g : I \rightarrow gI g^{-1}$. As is well known, one can find a basis $\mathcal{E} = \{e_i (1 \leq i \leq 2r)\}$ of V such that $(a(e_i, e_j)) = \begin{pmatrix} 0 & -B \\ {}^t B & 0 \end{pmatrix}$ with $B \in GL_r(\mathbf{R})$. [\mathcal{E} is called "canonical" if $B = E$. When V is identified with \mathbf{R}^{2r} by a canonical basis, one writes $Sp_{2r}(\mathbf{R})$ for $Sp(V, a)$.] For $I \in \mathcal{D}$, one associates the eigen subspace

$$V_-(I) = \{v \in V_{\mathbf{C}} \mid Iv = -iv\},$$

and introduces the complex coordinates $Z = (z_{ij}) \in M_r(\mathbf{C})$ of I by setting

$$(4) \quad V_-(I) = \{(e_1, \dots, e_{2r}) \begin{pmatrix} Z \\ B^{-1} \end{pmatrix}\}_{\mathbf{C}}.$$

Then, from the condition on I , one has

$$a(w, w) = 0, \quad \sqrt{-1}a(w, \bar{w}) > 0$$

for all $w \in V_-(I)$, $w \neq 0$, whence follows that ${}^tZ = Z$, $\text{Im } Z > 0$, i.e., Z belongs to the "Siegel space" \mathcal{H}_r of degree r (which is of tube type). By this correspondence $I \rightarrow Z$, the space \mathcal{D} is identified with \mathcal{H}_r and the pair (G, \mathcal{D}) thus obtained is a h.s.p. of type (III_r) . Note that the H -element in \mathfrak{g} corresponding to $I \in \mathcal{D}$ is given by $H_o = \frac{1}{2}I$ and the Hodge structure of G is defined by (2).

3. (Strongly) equivariant holomorphic maps

Let (G, \mathcal{D}) and (G', \mathcal{D}') be two h.s.p. A pair (ρ, φ) formed of an \mathbf{R} -homomorphism $\rho : G \rightarrow G'$ and a holomorphic map $\varphi : \mathcal{D} \rightarrow \mathcal{D}'$ is called a (strongly) equivariant holomorphic map (e.h.m. for short) if the following two conditions are satisfied.

$$(5) \quad \varphi(gz) = \rho(g)\varphi(z),$$

$$(6) \quad \varphi(s_z z') = s_{\varphi(z)}\varphi(z')$$

for all $g \in G$ and $z, z' \in \mathcal{D}$, where s_z and $s_{\varphi(z)}$ denote the symmetries of \mathcal{D} and \mathcal{D}' at z and $\varphi(z)$, respectively.

Let $o \in \mathcal{D}$, $o' = \varphi(o) \in \mathcal{D}'$ and let H_o and $H_{o'}$ be the corresponding H -elements in \mathfrak{g} and \mathfrak{g}' . Then one has

$$(H_1) \quad d\rho \circ \text{ad } H_o = \text{ad } H_{o'} \circ d\rho.$$

Conversely, it can be seen easily that, if this condition is satisfied for (ρ, o, o') , then, defining φ by $\varphi(g o) = \rho(g) o'$ (which is well defined), one obtains an e.h.m. (ρ, φ) . The triple (ρ, o, o') or $(\rho, H_o, H_{o'})$ satisfying the above condition (H_1) is said to be *admissible*. It should also be noted that the condition (H_1) is implied by a stronger condition

$$(H_2) \quad d\rho(H_o) = H_{o'},$$

which means that ρ preserves the Hodge structures (if G carries one).

It is a basic problem to determine all e.h.m. (ρ, φ) for the given h.s.p. (G, \mathcal{D}) and (G', \mathcal{D}') . In what follows, we will consider this problem in the following two special cases:

(P1) The case where $\mathcal{D}' = (\text{III}_{r'})$, i.e., the case where

$$(G', \mathcal{D}') = (Sp_{2r'}(\mathbf{R}), \mathcal{H}_{r'}).$$

(P2) The case where $\mathcal{D} = (\text{III}_1)$, i.e., the case where

$$(G, \mathcal{D}) = (SL_2(\mathbf{R}), \mathcal{H}_1).$$

The first problem is to determine all symplectic representations of G giving rise to a (strongly) equivariant holomorphic maps (cf. [1], [2], and [7], Ch.IV). The second one is essentially equivalent to the "Wolf-Koranyi theory" concerning the boundary components of symmetric domains (cf. [7], Ch.III and [8]).

4. The problem (P1)

Let V' be a real vector space of dimension $2r'$ endowed with a non-degenerate alternating bilinear form a' on $V' \times V'$, and set

$$G' = Sp(V', a'), \quad \mathcal{D}' = \mathcal{S}(V', a').$$

Suppose there is given a h.s.p. (G, \mathcal{D}) with an H -element $H_o \in \mathfrak{g}$. Then our problem is to find all symplectic representation $\rho : G \rightarrow G' = Sp(V', a')$ along with $I' \in \mathcal{D}'$ such that $(\rho, H_o, \frac{1}{2}I')$ is admissible. For simplicity, we call the quadruple (V', ρ, a', I') satisfying this condition a "solution" to the problem (P1). Since all solutions are fully reducible in an obvious sense, it is enough to consider the case where the representation ρ is " \mathbf{R} -primary", i.e., the case where ρ is the direct sum of mutually equivalent \mathbf{R} -irreducible representations. One may further assume that $\rho(\mathfrak{g})$ is not compact.

For simplicity, we consider our problem in the Lie algebra level. So in what follows, ρ denotes a representation of the Lie algebra \mathfrak{g} . Then we obtain the following results.

THEOREM 1. Let D_1 be one of $\mathbf{R}, \mathbf{C}, \mathbf{H}$. Then all \mathbf{R} -primary solution (V', ρ, a', I') for which $\rho(\mathfrak{g})$ is not compact is obtained in the following form.

$$(7) \quad \begin{aligned} V' &= V_1 \otimes_{D_1} V_2, \quad \rho = \rho_1 \otimes 1, \\ a' &= \text{tr}_{D_1/\mathbf{R}}(\bar{h}_1 \otimes h_2), \quad I' = I_1 \otimes 1, \end{aligned}$$

where V_1 (resp. V_2) is a right (resp. left) D_1 -vector space, h_1 (resp. h_2) is a D_1 -skew hermitian (resp. positive definite D_1 -hermitian) form on $V_1 \times V_1$ (resp. $V_2 \times V_2$), I_1 is a D_1 -linear complex structure on V_1 such that $h_1 I_1$ is D_1 -hermitian positive definite, and

$$(8) \quad \rho_1 : \mathfrak{g} \rightarrow \mathfrak{g}^{(1)} = \text{su}(V_1/D_1, h_1)$$

is an absolutely irreducible representation in D_1 satisfying the condition (H_2) with respect to H_o . (Note that $\mathfrak{g}^{(1)}$ is of type (III),(I),(II) according as $D_1 = \mathbf{R}, \mathbf{C}, \mathbf{H}$.)

THEOREM 2. Let \mathfrak{g} be a semisimple Lie algebra of hermitian type with an H -element H_o and let

$$(9) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s,$$

be the direct sum decomposition of \mathfrak{g} with \mathfrak{g}_0 compact and \mathfrak{g}_i ($1 \leq i \leq s$) simple and non-compact. Then any absolutely irreducible representation $\rho_1 : \mathfrak{g} \rightarrow \mathfrak{g}' = \text{su}(V_1/D_1, h_1)$ satisfying (H_2) with respect to H_o can be written in the form $\rho_1 = \rho_{10} \otimes_{D'_1} 1 + 1 \otimes_{D'_1} \rho_{1i}$ for some $i \geq 1$, where $D'_1 = \mathbf{R}, \mathbf{C}$, or \mathbf{H} and ρ_{10} (resp. ρ_{1i}) is an absolutely irreducible representation of \mathfrak{g}_0 (resp. \mathfrak{g}_i) in D'_1 , ρ_{1i} satisfying the condition (H_2) with respect to the \mathfrak{g}_i -component of H_o .

By virtue of these theorems, our problem is reduced to the determination of all absolutely irreducible representations

$$\rho_1 : \mathfrak{g} \rightarrow \mathfrak{g}^{(1)} = \mathfrak{su}(V_1, h_1)$$

satisfying the condition (H₂) with respect to H_o in the case where \mathfrak{g} is simple and non-compact. A list of solutions is given in [1] and [7] (p.188). In the case where \mathfrak{g} is of type (I),(II),(III), one has the "standard" solution(s) given by the identity map (and its conjugate) of $\mathfrak{g} = \mathfrak{su}(V_1, h_1)$. There are "non-standard" solutions for \mathfrak{g} of type (I_{p,1}) and (IV_p), given, respectively, by a skew-symmetric tensor representations and by spin representations. (One has also non-standard solutions for \mathfrak{g} of type (II_p) ($p = 3, 4$) because of the isomorphisms (II₃) \cong (I_{3,1}), (II₄) \cong (IV₆).) There are no solutions for \mathfrak{g} of exceptional types. For the results in this section, see [7], Ch.IV, §§1-5.

5. The solutions over \mathbf{Q}

When one considers solutions over \mathbf{Q} , one may assume that G (or \mathfrak{g}) is defined over \mathbf{Q} and \mathbf{Q} -simple. Then, as is well-known, there exists a totally real number field F of degree l and an absolutely simple Lie algebra \mathfrak{g}_1 of hermitian type such that

$$(10) \quad \mathfrak{g} = R_{F/\mathbf{Q}}(\mathfrak{g}_1) = \sum_{i=1}^l \mathfrak{g}_1^{\sigma_i},$$

σ_i denoting l (distinct) embeddings of F into \mathbf{R} . When \mathfrak{g} is proper, i.e., when all the $\mathfrak{g}_1^{\sigma_i}$ are non-compact, the determination of all ρ_1 defined over \mathbf{Q} is not difficult in view of Theorems 1, 2 (cf. [2] and [7], Ch.IV, §6). However, when \mathfrak{g} is improper, the solution becomes much more complicated involving the combinatorics arising from the representations of the compact factors. The case of $\mathfrak{g}_1 = \mathfrak{sl}_2(\mathbf{R})$, coming from the group of elements of norm 1 in an indefinite quaternion algebra over F , was treated by Kuga and Addington in terms of the so-called "chemistry".

In general, suppose one has an e.h.m.

$$(\rho, \varphi) : (G, \mathcal{D}) \rightarrow (G', \mathcal{D}'),$$

where G and G' have a structure of algebraic groups defined over \mathbf{Q} and ρ is \mathbf{Q} -rational with respect to these \mathbf{Q} -structures. Then, for any arithmetic subgroup Γ' of G' there exists an arithmetic subgroup Γ of G such that $\rho(\Gamma) \subset \Gamma'$. Then φ induces an analytic map $\tilde{\varphi}$ of the arithmetic quotient $\Gamma \backslash \mathcal{D}$ into $\Gamma' \backslash \mathcal{D}'$. It is known ([3]) that the map $\tilde{\varphi}$ can naturally be extended to a morphism of algebraic varieties from the standard compactification $(\Gamma \backslash \mathcal{D})^*$ into $(\Gamma' \backslash \mathcal{D}')^*$. Moreover, for the "canonical automorphy factors" J and J' of G and G' one obtains the relation

$$\rho(J(g, z)) = J'(\rho(g), \varphi(z)) \quad (g \in G, z \in \mathcal{D})$$

(see [4]). Hence the e.h.m. (ρ, φ) gives rise to a pull back of the automorphic forms on \mathcal{D}' to those on \mathcal{D} , which has many applications to the theory of automorphic forms on symmetric domains (e.g., the theory of singular modular forms).

EXAMPLE 2. We consider the case

$$\begin{aligned} G &= Sp(V_1, a_1), & \mathcal{D} &= \mathcal{S}(V_1, a_1), \\ G' &= Sp(V', a'), & \mathcal{D}' &= \mathcal{S}(V', a'), \end{aligned}$$

where the symplectic spaces (V_1, a_1) and (V', a') are both defined over \mathbf{Q} and G and G' are endowed with the \mathbf{Q} -structures defined from them in a natural manner. Then our results (loc.cit.) imply that all (non-trivial) \mathbf{Q} -primary e.h.m (ρ, φ) of (G, \mathcal{D}) into (G', \mathcal{D}') are obtained in the form

$$(11) \quad \begin{aligned} V' &= V_1 \otimes V_2, & \rho(g) &= g \otimes 1 \quad (g \in G), \\ a' &= a_1 \otimes s_2, & \varphi(I_1) &= I_1 \otimes 1 \quad (I_1 \in \mathcal{D}), \end{aligned}$$

where (V_2, s_2) is a real vector space defined over \mathbf{Q} endowed with a positive definite symmetric bilinear form s_2 on $V_2 \times V_2$. Choose \mathbf{Q} -bases of V_1 and V_2 as follows:

$$\begin{aligned} \mathcal{E}_1 &= \{e_1, \dots, e_{2r_1}\}, & (a_1(e_i, e_{j+r_1})) &= \begin{pmatrix} 0 & -B_1 \\ {}^t B_1 & 0 \end{pmatrix}, \\ \mathcal{E}_2 &= \{f_1, \dots, f_{r_2}\}, & (s(f_k, f_l)) &= S_2. \end{aligned}$$

Then $\mathcal{E}' = \{e_i \otimes f_k\}$ is a \mathbf{Q} -basis of V' for which one has

$$(a'(e_i \otimes f_k, e_j \otimes f_l)) = \begin{pmatrix} 0 & -B_1 \otimes S_2 \\ {}^t B_1 \otimes S_2 & 0 \end{pmatrix}.$$

If $I_1 \in \mathcal{D}$ corresponds to $Z_1 \in \mathcal{H}_{r_1}$ in the sense explained in Ex. 1, then $I' = \varphi(I_1) = I_1 \otimes 1 \in \mathcal{D}'$ corresponds to

$$V'_-(I') = V_-(I_1) \otimes V_{2C} = \{(e_i \otimes f_k) \begin{pmatrix} Z_1 \otimes S_2^{-1} \\ B_1^{-1} \otimes S_2^{-1} \end{pmatrix}\}_C.$$

Hence the corresponding e.h.m. of \mathcal{H}_{r_1} into $\mathcal{H}_{r'}$ ($r' = r_1 r_2$) is given in the form

$$(12) \quad \varphi : Z_1 \rightarrow Z_1 \otimes S_2^{-1}.$$

Let L_1 and L' be the lattices in V_1 and V' spanned by \mathcal{E}_1 and \mathcal{E}' , respectively, and let, for instance, Γ and Γ' be the arithmetic subgroups of G and G' consisting of those elements leaving fixed L_1 and L' , respectively. Then clearly the condition $\rho(\Gamma) \subset \Gamma'$ is satisfied.

6. Siegel domains

It will be convenient to give here the definitions of Siegel domains (in the sense of Piatetski-Shapiro). Let U and V be real vector spaces of finite dimension and let A be an alternation bilinear map $V \times V \rightarrow U$. Let \mathcal{C} be an open convex cone in U (with vertex at 0). We suppose that there exists a complex structure I on V such that $A(v, Iv')$ ($v, v' \in V$) is symmetric and " \mathcal{C} -positive" in the sense that one has

$$A(v, Iv) \in \bar{\mathcal{C}} - \{0\} \quad \text{for } \forall v \in V, v \neq 0.$$

We denote by $\mathcal{S}(V, A, \mathcal{C})$ the set of all complex structures on V satisfying these conditions. We put

$$(13) \quad Sp(V, A) = \{g \in GL(V) \mid A(gv, gv') = A(v, v') \ (\forall v, v' \in V)\}.$$

Then it is known ([6]) that the pair $(Sp(V, A)^\circ, \mathcal{S}(V, A, \mathcal{C}))$ has a natural structure of h.s.p. (of type (I), (II), (III)). For $I \in \mathcal{S}(V, A, \mathcal{C})$, we define the "Siegel domain (of the second kind)" by

$$(14) \quad \mathcal{S}(U, V, A, \mathcal{C}, I) = \{(u, w) \in U_{\mathcal{C}} \times V_+(I) \mid \text{Im } u - \frac{i}{2}A(\bar{w}, w) \in \mathcal{C}\},$$

where $V_+(I) = \{v \in V_{\mathcal{C}} \mid Iv = iv\}$. We also define the (universal) Siegel domain of the third kind by

$$\mathcal{S}(U, V, A, \mathcal{C}) = \{(u, w, I) \mid (u, w) \in \mathcal{S}(U, V, A, \mathcal{C}, I), I \in \mathcal{S}(V, A, \mathcal{C})\},$$

which can be realized as a domain in $U_{\mathcal{C}} \times V_+(I) \times \mathcal{S}(V, A, \mathcal{C})$ with a fixed I . For more about Siegel domains, especially on their \mathcal{Q} -structures, see [10].

7. The problem (P2) (The Wolf-Koranyi theory)

We consider the h.s.p. $(SL_2(\mathbf{R}), \mathcal{H}_1)$. One has

$$\mathfrak{sl}_2(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}_{\mathbf{R}}.$$

In what follows, we set $\mathfrak{g}^1 = \mathfrak{sl}_2(\mathbf{R})$ and fix an H -element $H^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in \mathfrak{g}^1 corresponding to $\sigma^1 = \sqrt{-1} \in \mathcal{H}_1$. Suppose there is given a h.s.p. (G, \mathcal{D}) with an H -element H_o in \mathfrak{g} corresponding to $\sigma \in \mathcal{D}$. For a (Lie algebra) homomorphism $\kappa : \mathfrak{g}^1 \rightarrow \mathfrak{g} = \text{Lie } G$ set

$$(15) \quad X_\kappa = \kappa \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \quad e_\kappa = \kappa \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right), \quad e_\kappa^* = \kappa \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$$

and

$$(16) \quad H_{o,\kappa}^{(1)} = H_o - \frac{1}{2}(e_\kappa - e_\kappa^*).$$

Then it is clear that (κ, o^1, o) is admissible, i.e., κ satisfies the condition (H_1) with respect to (H^1, H_o) if and only if one has

$$(17) \quad [H_{o,\kappa}^{(1)}, \kappa(\mathfrak{g}^1)] = \{0\}.$$

We denote by $\mathcal{K} = \mathcal{K}(G, \mathcal{D})$ the set of all non-trivial homomorphisms $\kappa : \mathfrak{g}^1 \rightarrow \mathfrak{g}$ satisfying the condition (H_1) for some H_o .

THEOREM 3. The notation being as above, let $\kappa \in \mathcal{K}$. Then the set of eigen values of $\text{ad } X_\kappa$ is given by $\{0, \pm 1, \pm 2\}$ or $\{0, \pm 2\}$.

For $\kappa \in \mathcal{K}$, we set

$$\mathfrak{g}(X_\kappa; \nu) = \{x \in \mathfrak{g} \mid [X_\kappa, x] = \nu x\} = \mathfrak{z}(X_\kappa), V_\kappa, \text{ and } U_\kappa,$$

according as $\nu = 0, 1$, and 2 , and

$$(18) \quad \mathfrak{g}_+(X_\kappa) = \mathfrak{z}(X_\kappa) + V_\kappa + U_\kappa.$$

Then $\mathfrak{g}_+(X_\kappa)$ is a parabolic subalgebra of \mathfrak{g} .

We now consider \mathcal{D} in a realization as a bounded symmetric domain in \mathbf{C}^n . We say two points z, z' in the closure $\bar{\mathcal{D}}$ of \mathcal{D} in \mathbf{C}^n are "equivalent" if there exists a sequence of points z_i ($0 \leq i \leq s$) in $\bar{\mathcal{D}}$ with $z_0 = z, z_s = z'$ and a sequence of holomorphic maps $\varphi_i : \mathcal{H}_1 \rightarrow \mathbf{C}^n$ ($1 \leq i \leq s$) with $z_{i-1}, z_i \in \varphi_i(\mathcal{H}_1) \subset \bar{\mathcal{D}}$. A "boundary component" (b.c. for short) of \mathcal{D} is by definition an equivalence class in $\bar{\mathcal{D}}$ in this sense. \mathcal{D} itself is an (improper) boundary component; all other b.c., contained in the boundary of \mathcal{D} , is called a "proper" b.c. It is known that a proper b.c. of \mathcal{D} is holomorphically equivalent to a bounded symmetric domain of lower dimension.

In what follows, we fix a proper b.c. \mathcal{F} of \mathcal{D} and set

$$(19) \quad N(\mathcal{F}) = \{x \in G \mid g\mathcal{F} = \mathcal{F}\}, \quad \mathfrak{n}(\mathcal{F}) = \text{Lie } N(\mathcal{F}).$$

Then it is known that the following two conditions for $\kappa \in \mathcal{K}$ are equivalent.

- (i) $\mathfrak{g}_+(X_\kappa) = \mathfrak{n}(\mathcal{F}),$
- (ii) $o_\kappa = \lim_{\lambda \rightarrow \infty} \exp(\lambda X_\kappa) o \in \mathcal{F}.$

When these conditions are satisfied, we say that κ belongs to the b.c. \mathcal{F} ; the set of all $\kappa \in \mathcal{K}$ belonging to \mathcal{F} is denoted by $\mathcal{K}(\mathcal{F})$. One has $\mathcal{K}(\mathcal{F}) \neq \emptyset$.

Let $\kappa \in \mathcal{K}(\mathcal{F})$. One denotes the unipotent radical of $N(\mathcal{F})$ by $N(\mathcal{F})_u$ and writes $\mathfrak{n}(\mathcal{F})_u = \text{Lie } N(\mathcal{F})_u$. If one sets

$$(20) \quad U_F = \text{center of } \mathfrak{n}(\mathcal{F})_u, \quad V_F = \mathfrak{n}(\mathcal{F})_u / U_F,$$

then $U_\kappa = U_F$ and one has a canonical isomorphism $V_\kappa \cong V_F$. The bracket product in \mathfrak{g} defines an alternating bilinear map

$$A_F : V_F \times V_F \rightarrow U_F.$$

The centralizer $Z(X_\kappa)$ of X_κ in G is Zariski connected and reductive, and is canonically isomorphic to $G_F = N(\mathcal{F})/N(\mathcal{F})_u$. By the adjoint action, one has representations of $Z(X_\kappa)$ (or G_F) on $U_\kappa (= U_F)$ and $V_\kappa (\cong V_F)$, which we denote by ρ_U and ρ_V . We denote by $G_\kappa^{(1)}$ (resp. $G_F^{(1)}$) the identity connected component of the kernel of ρ_U in $Z(X_\kappa)$ (resp. G_F). Then one has an almost direct product decomposition

$$(21) \quad Z(X_\kappa)^\circ = G_\kappa^{(1)} \cdot G_\kappa^{(2)}, \quad G_F^\circ = G_F^{(1)} \cdot G_F^{(2)}$$

with connected reductive \mathbf{R} -subgroups $G_\kappa^{(i)} \cong G_F^{(i)}$ ($i = 1, 2$).

8. The canonical decomposition of \mathcal{D}

We put

$$(22) \quad \mathcal{X}_F = \{X_\kappa \mid \kappa \in \mathcal{K}(\mathcal{F})\},$$

$$(23) \quad \mathcal{C}_F = \{e_\kappa \mid \kappa \in \mathcal{K}(\mathcal{F})\},$$

$$(24) \quad \mathcal{D}_\kappa = \{o \in \mathcal{D} \mid (\kappa, o^1, o) \text{ is admissible}\}.$$

Then we obtain the following theorems.

THEOREM 4. The map $X_\kappa (\in \mathcal{X}_F) \rightarrow Z(X_\kappa)^\circ$ gives a one-to-one correspondence between \mathcal{X}_F and the set of all maximal connected reductive \mathbf{R} -subgroups of $N(\mathcal{F})$. Thus \mathcal{X}_F has a natural structure of a principal homogeneous space of $N(\mathcal{F})_u$.

THEOREM 5. \mathcal{C}_F is an open convex cone in U_F and through ρ_U the reductive \mathbf{R} -group $G_F^{(2)}$ acts transitively on \mathcal{C}_F . Thus \mathcal{C}_F is a self-dual homogeneous cone, and one has a natural isogeny $\rho_U : G_F^{(2)} \rightarrow \text{Aut}(U_F, \mathcal{C}_F)^\circ$.

THEOREM 6. Suppose $\kappa \in \mathcal{K}(\mathcal{F})$. Then:

- 1) \mathcal{D}_κ is a complex submanifold of \mathcal{D} on which $G_\kappa^{(1)}$ acts transitively. The pairs $((G_\kappa^{(1)})_s, \mathcal{D}_\kappa)$ and $((G_F^{(1)})_s, \mathcal{F})$ have a natural structure of h.s.p., $()_s$ denoting the semisimple part, and the canonical isomorphism $(G_\kappa^{(1)})_s \rightarrow (G_F^{(1)})_s$ together with the map $o \rightarrow o_\kappa$ gives an e.h. isomorphism of $((G_\kappa^{(1)})_s, \mathcal{D}_\kappa)$ onto $((G_F^{(1)})_s, \mathcal{F})$. (Note that $G_\kappa^{(1)}$ itself is a reductive \mathbf{R} -group of hermitian type with an H -element $H_{o_\kappa}^{(1)}$.)
- 2) One has an e.h.m.

$$(25) \quad (\rho_V, \psi_F) : ((G_F^{(1)})_s, \mathcal{F}) \rightarrow (Sp(V_F, A_F), \mathcal{S}(V_F, A_F, \mathcal{C}_F)),$$

where ψ_F is given by $\psi_F(o_\kappa) = \text{ad}(2H_{o_\kappa}^{(1)})|V_F$.

THEOREM 7 ([8]). Fix a b.c. \mathcal{F} of \mathcal{D} . Then:

- 1) The map $\kappa \rightarrow (X_\kappa, e_\kappa)$ gives a one-to-one correspondence between $\mathcal{K}(\mathcal{F})$ and $\mathcal{X}_F \times \mathcal{C}_F$.

2) \mathcal{D} is a disjoint union of \mathcal{D}_κ ($\kappa \in \mathcal{K}(\mathcal{F})$). Hence as C^∞ -manifolds one has

$$(26) \quad \mathcal{D} \cong \mathcal{K}(\mathcal{F}) \times \mathcal{F} \cong \mathcal{X}_F \times \mathcal{C}_F \times \mathcal{F}$$

by the correspondence

$$o \rightarrow (\kappa, o_\kappa) \rightarrow (X_\kappa, e_\kappa, o_\kappa).$$

COROLLARY. For a fixed $o \in \mathcal{D}$, the set $\mathcal{K} = \mathcal{K}(G, \mathcal{D})$ is in one-to-one correspondence with the set of all proper b.c. \mathcal{F} of \mathcal{D} by the relations $o \in \mathcal{D}_\kappa$, $\kappa \in \mathcal{K}(\mathcal{F})$.

For these results, see [7], Ch.III, §§1-4 and [8]. The decomposition (26) of \mathcal{D} is called the "canonical decomposition" of \mathcal{D} with respect to \mathcal{F} . This is an algebraic analogue of the Siegel domain realization of \mathcal{D} given in the Wolf-Koranyi theory. Actually, from the above results it is not difficult to see that the manifold \mathcal{D} has a structure of a fiber space over \mathcal{F} whose fiber through a point $o \in \mathcal{D}_\kappa$ is the union of all geodesics passing through o and tending to points in \mathcal{F} , and this fiber can naturally be identified with the Siegel domain

$$\mathcal{S}(U_F, V_F, A_F, \mathcal{C}_F, \psi_F(o_\kappa)).$$

Thus one obtains the expression of \mathcal{D} as a Siegel domain of the third kind, which is a pull back of the universal one $\mathcal{S}(U_F, V_F, A_F, \mathcal{C}_F)$ by the e.h.m. of the base space

$$\psi_F: \mathcal{F} \rightarrow \mathcal{S}(V_F, A_F, \mathcal{C}_F).$$

9. One further obtains the following theorems.

THEOREM 8. For $\kappa \in \mathcal{K}(\mathcal{F})$, the following conditions are equivalent.

- (i) κ satisfies the condition (H_2) (w.r.t. H^1).
- (ii) $H_o^{(1)} = 0$.
- (iii) $V_F = \{0\}$.

When these conditions are satisfied, \mathcal{F} reduces to a point and one has

$$(27) \quad \mathcal{D} \cong U_F + i\mathcal{C}_F.$$

(When such a κ exists, \mathcal{D} is called "of tube type".)

THEOREM 9. Let (G, \mathcal{D}) be an irreducible h.s.p. with \mathbf{R} -rank $\mathfrak{g} = r$ and let $o \in \mathcal{D}$. Then there exist r mutually commutative homomorphism $\kappa_i: \mathfrak{g}^1 \rightarrow \mathfrak{g}$ such that (κ_i, o^1, o) is admissible. Let $\kappa^{(i)} = \kappa_1 + \dots + \kappa_i$ ($1 \leq i \leq r$). Then, $\kappa^{(i)}$ is a homomorphism of \mathfrak{g}^1 into \mathfrak{g} such that $(\kappa^{(i)}, o^1, o)$ is admissible and $\{\kappa^{(1)}, \dots, \kappa^{(r)}\}$ is a complete set of representatives of the conjugacy classes (w.r.t. $\text{Ad}(G)$) of homomorphisms $\kappa: \mathfrak{g}^1 \rightarrow \mathfrak{g}$ satisfying the condition (H_1) (w.r.t. H^1). Moreover, if \mathcal{F}_i is the b.c. such that $\kappa^{(i)} \in \mathcal{K}(\mathcal{F}_i)$, then \mathcal{F}_{i+1} is a b.c. of \mathcal{F}_i for $1 \leq i \leq r-1$, and $\{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ is a complete set of representatives of the G -equivalence classes of proper b.c. of \mathcal{D} .

EXAMPLE 3. Consider a h.s.p.

$$G = SO(n, 2), \quad \mathcal{D} = (IV_n) \quad (n \geq 2),$$

which is irreducible for $n > 2$. (Note that $SO(1, 2)$ and $SO(2, 2)$ are isogenous to $SL_2(\mathbf{R})$ and $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$, respectively.) In this case, \mathbf{R} -rank $\mathfrak{g} = 2$. One fixes a point $o^{(n)} \in \mathcal{D}$ corresponding to $K = SO(n) \times SO(2)$ and the H -element $H_{o^{(n)}} = (0, J)$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which defines the Hodge structure of G (as well as the one in the corresponding spin group). One denotes by ι_n the natural injection $SO(2, 2) \rightarrow SO(n, 2)$. Since one has $\iota_n(H_{o^{(2)}}) = H_{o^{(n)}}$, ι_n gives rise to an e.h.m. $(IV_2) \rightarrow (IV_n)$ satisfying (H_2) .

There exist mutually commutative homomorphisms $\kappa_i : \mathfrak{g}^1 \rightarrow \mathfrak{g} = \mathfrak{o}(n, 2)$ ($i = 1, 2$) given by

$$\begin{aligned} X_{\kappa_1} &= \iota_n \left(\begin{pmatrix} 0 & F' \\ F & 0 \end{pmatrix} \right), \quad e_{\kappa_1} = \frac{1}{2} \iota_n \left(\begin{pmatrix} J & E' \\ E' & J \end{pmatrix} \right), \quad e_{\kappa_1}^* = \frac{1}{2} \iota_n \left(\begin{pmatrix} -J & E' \\ E' & -J \end{pmatrix} \right), \\ X_{\kappa_2} &= \iota_n \left(\begin{pmatrix} 0 & F' \\ -F' & 0 \end{pmatrix} \right), \quad e_{\kappa_2} = \frac{1}{2} \iota_n \left(\begin{pmatrix} -J & E \\ E & J \end{pmatrix} \right), \quad e_{\kappa_2}^* = \frac{1}{2} \iota_n \left(\begin{pmatrix} J & E \\ E & -J \end{pmatrix} \right), \end{aligned}$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

One can check easily that for $n = 2$ the map $(x_1, x_2) \rightarrow \kappa_1(x_1) + \kappa_2(x_2)$ gives an isomorphism $\mathfrak{g}^1 \oplus \mathfrak{g}^1 \cong \mathfrak{o}(2, 2)$ with $\kappa_1(H^1) + \kappa_2(H^1) = H_{o^{(2)}}$, whence follows that $(\kappa_i, H^1, H_{o^{(n)}})$ ($n \geq 2$) is admissible.

For $\kappa = \kappa^{(1)} = \kappa_1$, one has $\mathcal{F}_1 \cong \mathcal{H}_1$, $H_{o, \kappa}^{(1)} = \frac{1}{2} \iota_n \left(\begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \right)$ and

$$G_{\kappa}^{(1)} \cong SL_2(\mathbf{R}) \times SO(n-2), \quad G_{\kappa}^{(2)} \cong \mathbf{R}_+,$$

$$V_{\kappa} \cong \mathbf{R}^{2(n-1)}, \quad U_{\kappa} \cong \mathbf{R}.$$

For $\kappa = \kappa^{(2)} = \kappa_1 + \kappa_2$, one has $\mathcal{F}_2 = (\text{a point})$, $H_{o, \kappa}^{(1)} = 0$ and

$$G_{\kappa}^{(1)} = \{1\}, \quad G_{\kappa}^{(2)} \cong \mathbf{R}_+ \times SO(n-1, 1)^{\circ},$$

$$V_{\kappa} = \{0\}, \quad U_{\kappa} \cong \mathbf{R}^n.$$

In this case, one has a tube domain expression $\mathcal{D} \cong \mathbf{R}^n + i\mathcal{P}(n-1, 1)$, where one denotes

$$\mathcal{P}(n-1, 1) = \{u = (u_i) \in \mathbf{R}^n \mid \sum_{i=1}^{n-1} u_i^2 - u_n^2 < 0, u_n > 0\}.$$

When (G, \mathcal{D}) has a \mathbf{Q} -structure, a b.c. \mathcal{F} is called "rational" if $\mathfrak{n}(\mathcal{F})$ is defined over \mathbf{Q} or, equivalently, if there exists $\kappa \in \mathcal{K}(\mathcal{F})$ which is defined over \mathbf{Q} . When \mathfrak{g} is \mathbf{Q} -simple, one can prove an analogue of Theorem 9 over \mathbf{Q} , replacing r by $r_0 = \mathbf{Q}$ -rank \mathfrak{g} . The canonical decomposition of \mathcal{D} is useful in discussing the \mathbf{Q} -structures of G , especially in the determination of the "rational points" in \mathcal{D} (cf. [8], [9]).

References

- [1] I. Satake, Holomorphic imbeddings of symmetric domains into a Siegel space, Amer. J. Math. **87**(1965), 425-461.
- [2] —, Symplectic representations of algebraic groups satisfying a certain analyticity condition, Acta Math. **117**(1967), 215-279.
- [3] —, A note on holomorphic imbeddings and compactifications of symmetric domains, Amer. J. Math. **90**(1968), 231-247.
- [4] —, On some properties of holomorphic imbeddings of symmetric domains, Amer. J. Math. **91**(1969), 289-305.
- [5] —, On modular imbeddings of a symmetric domain of type (IV), in "Global Analysis, Papers in Honor of K. Kodaira" (ed. Spencer and Iyanaga), Univ. of Tokyo Press and Princeton Univ. Press, 1970, pp. 341- 354.
- [6] —, Le déformation des formes hermitiennes et son application aux domaines de Siegel, Ann. Sci. l'Ecole Norm. Sup. **11**(1978), 445-449.
- [7] —, Algebraic Structures of Symmetric Domains, Publ. Math. Soc. Japan **14**, Iwanami Shoten and Princeton Univ. Press, 1980.
- [8] —. On the rational structures of symmetric domains, I, in "Intern. Symp. in Memory of Hua Loo Keng" (Beijing, 1988, ed. Gong Sheng et al.), Vol. II Analysis, Springer-Verlag and Chinese Sci. Press, 1991, pp. 231- 259.
- [9] —, On the rational structures of symmetric domains, II, Determination of rational points of classical domains, Tohoku Math. J. **43**(1991), 401-424.
- [10] —, On \mathbf{Q} -structures of quasisymmetric domains, Tohoku Math. J. **47**(1995), 357-390; Addendum, *ibid.* 613- 614.