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BIFURCATIONS OF AFFINE INVARIANTS FOR ISOTOPIES OF GENERIC CONVEX PLANE CURVES

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1. Introduction

For a single plane curve in affine plane, we studied some interesting mathematical notions in [7]. We characterized affine vertices and affine inflexions by using the notion of affine distance cubed functions and affine height functions. They are very interesting geometrical notions. We take an interest in the following fact of the geometry: How are affine vertices (affine inflexions) created and destroyed ? In this note, we attempting to answer the question. The basic tools we shall use have are families of affine distance cubed functions and affine heightfunctions.The main result is Theorem III which will given in Section 4.The basic techniques we used in this paper depend heavily on those in the paper of Professor Bruce [3].

All curves and maps considered here are of class C^∞ unless stated otherwise.

2. Basic Notions of Affine Differential Geometry for Plane Curves

In this section we introduce some basic notions of affine differential geometry for plane curves (cf., [2,7,8]).

Let $\mathbb{R}^2$ be an affine plane which adopt the coordinate such that the area of the parallelogram spanned by two vectors $a = (a_1, a_2)$, $b = (b_1, b_2)$ is given by determinant of a and b, that is $|\ a\ b\ | = a_1b_2 - a_2b_1$. Let S^1 be unit circle in $\mathbb{R}^2$, and $\phi : S^1 \longrightarrow \mathbb{R}^2$ be a smooth plane curve with $\left| \ \frac{d\phi}{dt}(t)\ \frac{d^2\phi}{dt^2}(t)\ \right| \neq 0$. If we reparametrize a given curve ϕ by using $s(t) := \int_{t_0}^{t} \left| \ \frac{d\phi}{dt}(t)\ \frac{d^2\phi}{dt^2}(t)\ \right|^{\frac{1}{3}} dt$, then the curve satisfies that $\left| \ \frac{d\phi}{ds}(s)\ \frac{d^2\phi}{ds^2}(s)\ \right| =$ 1. We call such s *an affine (arc-length) parameter.* We call $\frac{d\phi}{ds}(s)$ an *affine tangent vector* and $\frac{d^2\phi}{ds^2}(s)$ an *affine normal vector.* The *affine curvature* is defined to $\kappa(s) =$ $|\ \frac{d^2\phi}{dt^2}(s)\ \frac{d^3\phi}{dt^3}(s)\ |$.

Suppose that $\kappa(s) \neq 0$, then the point $\phi(s) + \frac{1}{\kappa(s)}\frac{d^2\phi}{dt^2}(s)$ is called an *affine center of curvature* of ϕ at s, and its locus is called an *affine evolute.* The curve $\frac{d^2\phi}{dt^2} : S^1 \longrightarrow \mathbb{R}^2$

is called an *affine normal curve* of ϕ.

We say that a point $\phi(s)$ of curve ϕ is an *affine vertex* if $\frac{d\kappa}{ds}(s) = 0$. We also say that a point $\phi(s)$ of curve ϕ is an *affine inflexion* if the affine curvature of ϕ at s is zero.

We assume that ϕ has the following properties, both of which are satisfied generically (cf., [6]).

(A-1) There is no conic having greater than six-point contact with $\phi(S^1)$.

(A-2) The number of points p of $\phi(S^1)$ where the unique non-singular conic touching $\phi(S^1)$ at p with at least five-point contact is a parabola in finite.

(A-3) There is no parabola having six-point contact with $\phi(S^1)$.

In [7], we have shown the following theorem.

Theorem I. [7] *Let $\phi : S^1 \longrightarrow \mathbb{R}^2$ be a smooth plane curve without inflexional points satisfing* (A-1),(A-2),(A-3). *Then We have ;*

(1) *Let p be a point of the affine evolute of ϕ, p being the affine center os curvature at s_0. Then, locally at p, the affine evolute is*

- (a) *diffeomorphic to a line in $\mathbb{R}^2$ if the point $\phi(s_0)$ is not an affine vertex of ϕ ;*
- (b) *diffeomorphic to an ordinary cusp in $\mathbb{R}^2$ if the point $\phi(s_0)$ is an affine vertex of ϕ.*

(2) *Let p be a point of the affine normal curve of ϕ at s_0. Then, locally at p, the affine normal curve is*

- (a) *diffeomorphic to a line in $\mathbb{R}^2$ if the point $\phi(s_0)$ is not an affine inflexional point of ϕ;*
- (b) *diffeomorphic to an ordinary cusp in $\mathbb{R}^2$ if the point $\phi(s_0)$ is an affine inflexional point of ϕ.*

3. Basic Notions of Singularities

In this section we introduce some basic notions of singularity theory (cf., [1,3,4]).

Let $G : \mathbb{R} \times \mathbb{R}^n, 0 \longrightarrow \mathbb{R}, 0$ be a function germ. We call G an unfolding of $g(t) = G(t, 0)$. We say that $g(t)$ has an A_k-singularity at t if $g^{(p)}(t) = 0$ for all $1 \leq p \leq k$, and $g^{(k+1)}(t) \neq 0$. The family G is a versal unfolding of the function g with an A_k-singularity if and only if the truncated Taylor expansions of $\frac{\partial G}{\partial a_i}(t, 0)$, $1 \leq i \leq n$, span the space of polynomials in t of degree at most $k - 1$.

We consider an unfolding $G(t, a, u)$ of the potential function $G(t, 0, 0) = g(t)$. The bifurcation set is defined as

$$\mathfrak{B}(G) := \{ \ (a, u) \in A \times U \mid \frac{\partial G}{\partial t}(a, t) = \frac{\partial^2 G}{\partial t^2}(a, u) = 0 \ \}.$$

We consider the extended unfolding $\widetilde{G}(t, a, u, c) = G(t, a, u) - c$. The discriminant set is defined as

$$\mathfrak{D}(\widetilde{G}) := \{ \ (a, u, c) \in A \times U \times \mathbb{R} \mid G(t, a, u) = c, \frac{\partial G}{\partial t}(a, t) = 0 \ \}.$$

The unfoldings G and $\widetilde{G}$ give rise to families of binfurcation, resp. discriminant sets obtained by fixing the parameter u. We have natural projections $\pi : A\times U \longrightarrow U$ (resp. $\pi_1 : A\times U\times\mathbb{R}\longrightarrow U$).

Theorem II. [4] *Let $G(t,a,u)$ be as above. if* $1, \dfrac{\partial G}{\partial a_i}(t,0,0)$ $(1\le i\le n)$ *and* $\dfrac{\partial G}{\partial u}(t,0,0)$ *span* $\mathbb{R}[t]/\langle t^k\rangle$ *then* $\widetilde{G}$ *(resp. G) is a versal unfolding of the function (resp. potential function) $g(t)$. When this is the case we have the following.*

(a) *If* $1, \dfrac{\partial G}{\partial a_i}(t,0,0)$ *span* $\mathbb{R}[t]/\langle t^k\rangle$ *the projection π (resp. π_1) is equivalent, via a bifurcation (resp. discriminant) preserving diffeomorphism, to the trivial projection onto one factor of a product bifurcation (resp. dicriminant) set.*

(b) *If G is of minimal dimension $k-1$ and* $1, \dfrac{\partial G}{\partial a_i}(t,0,0)$ *span* $\mathbb{R}[t]/\langle t^k\rangle$ *then the projection pi_1 (resp. pi) is equivalent to the projection of the standard discriminant (resp. bifurcation) set of $\widetilde{F}$ (resp. F) above onto the a_1- coordinate,*

where $\widetilde{F}(t,a) = \pm t^{k+1} + a_1t^{k-1}+\cdots+a_{k-1}t+a_k$ *and* $F(t,a)=\pm t^{k+1}+a_1t^{k-1}+\cdots+a_{k-1}t$.

4. One Parameter Family of Plane Curves

Let U be an open interval $(-1,2)$. We consider the following set ;

$$Imm^+(S^1,\mathbb{R}^2) := \{\ i : S^1 \longrightarrow \mathbb{R}^2 \mid i \text{ is } immersion,\ \left| \frac{di}{ds}(s)\ \frac{d^2i}{ds^2}(s)\right| > 0\ \}.$$

We also consider the following set ;

$$\mathfrak{C} := \{\ \Phi : S^1\times U \longrightarrow \mathbb{R}^2 \mid \Phi_u \in Imm^+(S^1,\mathbb{R}^2), for\ any\ u\in U\ \}.$$

In particular, Φ_0 and ϕ_1 satisfy above conditions (A-1), (A-2) and (A-3), and $\Phi_u(s)$ satisfy $\left| \dfrac{d\Phi_u}{ds}(s)\ \dfrac{d^2\Phi_u}{ds^2}(s)\right| = 1.$

Then we have following result.

Theorem III. *There exists a dence subset $\mathcal{O}\subset\mathfrak{C}$ such that for any $\Phi\in\mathcal{O}$ we have the following ;*

(1)*Let p be a point of the family of affine normal curve of Φ_u at s, then locally at p, the family of affine normal curve is ;*

(a) *diffeomorphic to the plane in $\mathbb{R}^3$ and projection is equivalent to the trivial projection if Φ_u has A_1-singularity at s.*

(b) *diffeomorphic to the cuspidal edge in $\mathbb{R}^3$ and projection is equivalent to the trivial projection if Φ_u has A_2-singularity at s.*

(c) *diffeomorphic to the swallowtail in $\mathbb{R}^3$ and projection is equivalent to the projection : $\mathfrak{B}(F)\longrightarrow\mathbb{R}$; $(u,a)\mapsto a_1$ if Φ_u has A_3-singularity at s.*

(2) *Let p be a point of the family of affine evolute of Φ_u at s, then locally at p, the family of affine evolute is ;*

(a) *diffeomorphic to the plane in $\mathbb{R}^3$ and projection is equivalent to the trivial projection if Φ_u has A_2-singularity at s.*

(b) *diffeomorphic to the cuspidal edge in $\mathbb{R}^3$ and projection is equivalent to the trivial projection if Φ_u has A_3-singularity at s.*

(c) *diffeomorphic to the swallowtail in $\mathbb{R}^3$ and projection is equivalent to the projection : $\mathfrak{D}(\widetilde{F}) \longrightarrow \mathbb{R}$; $(u,a,c) \mapsto a_1$ if Φ_u has A_4-singularity at s.*

Here the cuspidal edge is given as $\mathbb{R} \times \{ (x_1,x_2) \mid x_1^2 = x_2^3 \}$ *and the swallowtail is given as* $\{ (x_1,x_2,x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v \}$.

We present ideas of the proof in outline as follows.

4-1. Family of Affine Height Functions. When we study a single plane curve in affine plane, we define the affine height function as follows. For $\phi : S^1 \longrightarrow \mathbb{R}^2$ as above, $h : S^1 \times S^1 \longrightarrow \mathbb{R}$; $h(s,a) := \left| \frac{d\phi}{ds}(s)\ a \right|$. Similarly we now define the one parameter family of affine height function

$$H : S^1 \times U \times S^1 \longrightarrow \mathbb{R}$$

by

$$H(s,u,a) := \left| \frac{\partial \Phi}{\partial s}(s,u)\ a \right|.$$

We also define a function $\widetilde{H} : S^1 \times U \times S^1 \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\widetilde{H}(s,u,a,v) := H(s,u,a) - v.$$

The discriminant set of $\widetilde{H}$ is as follows ;

$$\mathfrak{D}^{\pm}(\widetilde{H}) = \{ (u, \lambda(s,u)\frac{\partial^2 \Phi}{\partial s^2}(s,u), \lambda(s,u)) \in U \times S^1 \times \mathbb{R} \mid s \in S^1 \},$$

where $\lambda(s,u) = \pm \dfrac{1}{\sqrt{\sum_{i=1}^{2} (\frac{\partial^2 \Phi^i}{\partial s^2}(s,u))^2}}$, $\Phi(s,u) = (\Phi^1(s,u), \Phi^2(s,u))$.

We now consider $\mathfrak{D}^{\pm}(\widetilde{H})$. We define a map $\Psi : U \times (\mathbb{R}^2 - \{o\}) \longrightarrow U \times S^1 \times \mathbb{R}$ by

$$\Psi(u,x_1,x_2) := (u, (\pm\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \pm\frac{x_2}{\sqrt{x_1^2+x_2^2}},), \pm\frac{1}{\sqrt{x_1^2+x_2^2}}).$$

It is clear that Ψ is a diffeomorphism and $\Psi(N_\Phi) = \mathfrak{D}^{\pm}(\widetilde{H})$,
where $N_\Phi := \{ (u, \frac{\partial^2 \Phi}{\partial s^2}(s,u)) \mid s \in S^1 \}$, that is the family of affine normal curves.

We consider the following set Σ and the canonical projection P_1 ;

$$\Sigma = \{\ (s,u,a,v) \in S^1 \times U \times S^1 \times \mathbb{R} \mid a = \lambda(s,u)\frac{\partial^2 \Phi}{\partial s^2}(s,u)\ \},$$

that is singular set of $\widetilde{H}$,

$$P_1 : \Sigma \longrightarrow U \times S^1 \times \mathbb{R}.$$

Then we have $\mathfrak{D}^{\pm}(\widetilde{H}) = C_{P_1}$, where is C_{P_1} is the critical value set of P_1.

Without loss of generality we shall work at $u = 0$ and $t = 0$. We write $\Phi(t,u) = (t, c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6 + \mathbf{O}(7))$ where $\mathbf{O}(k)$ denotes a smooth function on $\mathbb{R}$ vanishing at $t = 0$ to order $k-1$. In particular, we assume that $c_2 \neq 0$. If $a = (\mp\frac{c_3}{\sqrt{c_3^2+4c_2^4}}, \pm\frac{2c_2^2}{\sqrt{c_3^2+4c_2^4}})$, the condition for $\widetilde{H}_a$ to have a A_k-singularity at $t = 0$ is as follows ;

(1) $\widetilde{H}_a$ has a A_1-singularity $t = 0$ if and only if $5c_3^2 - 4c_2c_4 \neq 0$,
(2) $\widetilde{H}_a$ has a A_2-singularity $t = 0$ if and only if $5c_3^2 - 4c_2c_4 = 0$, $7c_3c_4 - 5c_2c_5 \neq 0$,
(3) $\widetilde{H}_a$ has a A_3-singularity $t = 0$ if and only if $5c_3^2 - 4c_2c_4 = 0$, $7c_3c_4 - 5c_2c_5 = 0$, $21c_3^2c_4 - 10c_2^2c_6 \neq 0$,
(4) $\widetilde{H}_a$ has a $A_{\geq 4}$-singularity $t = 0$ if and only if $5c_3^2 - 4c_2c_4 = 0$, $7c_3c_4 - 5c_2c_5 = 0$, $21c_3^2c_4 - 10c_2^2c_6 = 0$,

where $\widetilde{H}_a(t,u) = \widetilde{H}(t,u,a)$ for any $a \in S^1$.

We now define the function $F : S^1 \times U \times \mathbb{R} \longrightarrow \mathbb{R}$;

$$F(s,u,x) = \frac{\partial \Phi_1}{\partial s}(s,u)\sin x_1 - \frac{\partial \Phi_2}{\partial s}(s,u) - x_2,$$

where $x = (x_1, x_2)$. This is considered as a local representation of $\widetilde{H}$. We may use F instead of $\widetilde{H}$. We obtain ;

$$\frac{\partial F_{x_1}}{\partial x_1}(s,0) = \pm(\frac{(2c_2)^{-\frac{1}{3}}}{\sqrt{c_3^2+4c_2^4}})(-c_3 + 4c_2^3t + 6c_2^2c_3t^2 + \cdots),$$
$$\frac{\partial F_{x_1}}{\partial x_2}(s,0) = -1,$$
$$\frac{\partial F_{x_1}}{\partial u}(s,0) = \pm(\frac{(2c_2)^{-\frac{1}{3}}}{\sqrt{c_3^2+4c_2^4}})((2c_2^2d_1 - c_3e_1)t + (2c_2^2d_2 - c_3e_2)t^2 + \cdots),$$

where $\frac{\partial^2 \Phi_1}{\partial u \partial t}(t,0) = d_1t + d_2t^2 + \mathbf{O}(3)$, $\frac{\partial^2 \Phi_2}{\partial u \partial t}(t,0) = e_1t + e_2t^2 + \mathbf{O}(3)$.

(i) A_1 and A_2-singularity

Since $1, \frac{\partial F_{x_1}}{\partial x_1}(s,0)$, $\frac{\partial F_{x_1}}{\partial x_2}(s,0)$ span $\mathbb{R}[t]/\langle t^3\rangle$, the projection is the teivial one, by Theorem II. And F is always versal unfolding.

(ii) A_3-singularity

The condition for a versal unfolding is that $1, \dfrac{\partial F_{x_1}}{\partial x_1}(s,0),\ \dfrac{\partial F_{x_1}}{\partial x_2}(s,0),\ \dfrac{\partial F_{x_1}}{\partial u}(s,0)$ span $\mathbb{R}[t]/\langle t^3\rangle$, that is $c_3 \neq 0$. And the condition that the projection is automatic by Theorem II.

Applying the transversality results of Bruce [4] as Theorem II, we have Theorem III (1).

4-2. Family of Affine Distance Cubed Functions. When we study a single plane curve in affine plane, we define the affine distance cubed function as follows. For $\phi : S^1 \longrightarrow \mathbb{R}^2$ as above, $d : S^1 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$; $d(s,a) := \left| \dfrac{d\phi}{ds}(s)\ \phi(s) - a \right|$. Similarly we now define the one parameter family of affine distance cubed function

$$D : S^1 \times U \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

by

$$D(s,u,a) := \left| \frac{\partial \Phi}{\partial s}(s,u)\ \Phi(s,u) - a \right|.$$

The bifurcation set of D is as follows ;

$$\mathfrak{B}(D) = \{\ (u, \Phi(s,u) + \frac{1}{\kappa(s,u)}\frac{\partial^2 \Phi}{\partial s^2}(s,u)) \in U \times \mathbb{R}^2 \mid \kappa(s,u) \neq 0, s \in S^1\ \}.$$

$\mathfrak{B}(D)$ is the family of affine evolute. We now condsider the following set Σ and the natural projection P ;

$$\Sigma = \{\ (s,u,a) \in S^1 \times U \times \mathbb{R}^2 \mid a = \Phi(s,u) - \lambda \frac{\partial^2 \Phi}{\partial s^2}(s,u), \lambda \in \mathbb{R}\ \},$$

that is singular set of D,

$$P : \Sigma \longrightarrow U \times \mathbb{R}^2.$$

Then we have $\mathfrak{B}(D) = C_P$, C_P is the critical value set of P.

Without loss of generality we shall work at $u = 0$ and $t = 0$. We write $\Phi(t,u) = (t, c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6 + c_7t^7 + \mathbf{O}(8))$. In particular, we assume that $c_2 \neq 0$. If $a = (-\dfrac{c_2}{2c_4 - 3c_3}, \dfrac{2c_2^3}{c_3(2c_4 - 3c_3)})$ the condition for D_a to have a A_k-singularty at $t = 0$ is as follows ;

(1) D_a has a A_2-singularity $t = 0$ if and only if $3c_2c_3c_4 - 2c_3^3 - c_2^2c_5 \neq 0$,

(2) D_a has a A_3-singularity $t = 0$ if and only if $3c_2c_3c_4 - 2c_3^3 - c_2^2c_5 = 0$, $148c_2c_3^2c_4 - 160c_3^4 + 128c_2^2c_4^2 - 60c_2^3c_6 - 5c_2^3c_4 + 4c_2c_3c_4^2 \neq 0$,

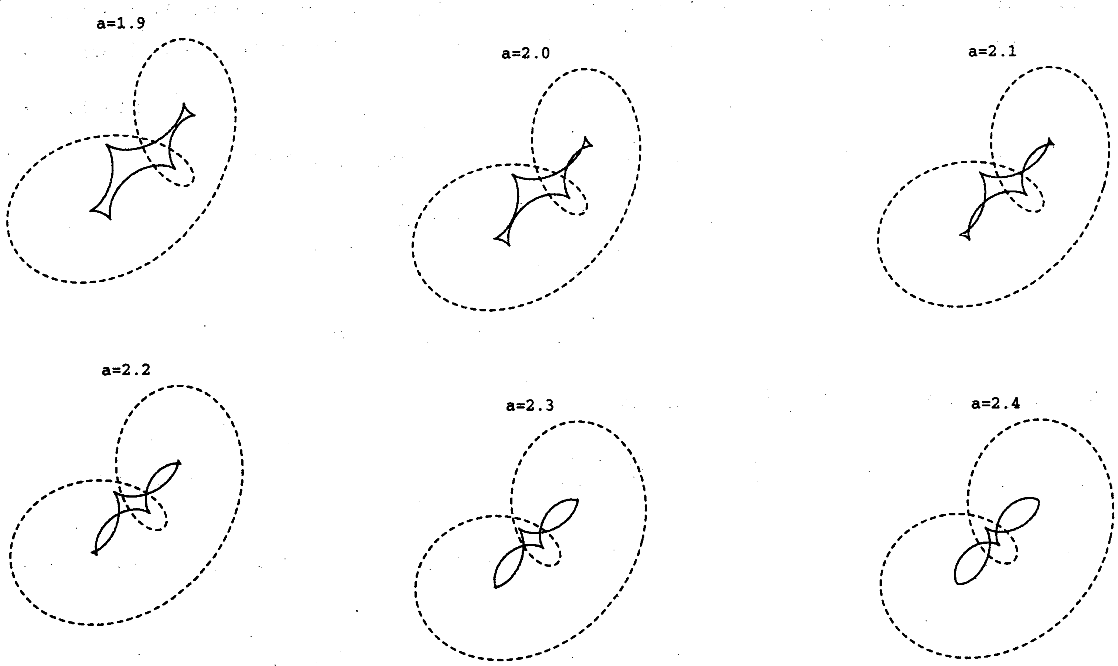
(3) D_a has a A_4-singularity $t = 0$ if and only if $3c_2c_3c_4 - 2c_3^3 - c_2^2c_5 = 0$, $148c_2c_3^2c_4 - 160c_3^4 + 128c_2^2c_4^2 - 60c_2^3c_6 - 5c_2^3c_4 + 4c_2c_3c_4^2 = 0$, $16060c_2^3c_3c_6 - 84360c_2c_3^3c_4 + 168499c_3^5 - 2080c_2^2c_3c_4^2 - 3360c_2^4c_7 + 5520c_2c_3^4c_4 - 3840c_2^2c_3^2c_4 - 8280c_2c_3^5 + 5760c_2^2c_3^3c_4 \neq 0$,

(4) D_a has a $A_{\geq 5}$-singularity $t = 0$ if and only if $3c_2c_3c_4 - 2c_3^3 - c_2^2c_5 = 0$, $148c_2c_3^2c_4 - 160c_3^4 + 128c_2^2c_4^2 - 60c_2^3c_6 - 5c_2^3c_4 + 4c_2c_3c_4^2 = 0$, $16060c_2^3c_3c_6 - 84360c_2c_3^3c_4 + 168499c_3^5 - 2080c_2^2c_3c_4^2 - 3360c_2^4c_7 + 5520c_2c_3^4c_4 - 3840c_2^2c_3^2c_4 - 8280c_2c_3^5 + 5760c_2^2c_3^3c_4 = 0$,

where $D_a(t,u) = D(t,u,a)$ for any $a \in \mathbb{R}^2$.

By similar argument to 4-., we have Theorem III (2).

The situation described in Theorem III (2) is depicted as follows. The dotted line is the curve $\phi(t) = (\cos 2t - \cos(t+a), \sin 2t + \sin t)$. The real line is the affine evolute of ϕ. The parameter a is 1.9, 2.0, 2.1, 2.2, 2.3, 2.4.



REFERENCES

1. V. I. Arnol'd, *Wavefront evolution and equivariant Morse lemma*, Comm. Pure Appl. Math. **29** (1976), 557–582.
2. W. Blaschke, *Vorlesungen uber Differentialgeometrie II*, Springer, 1923.
3. J. W. Bruce and P. J. Giblin, *Curves and singularities*, Cambridge University press, 1984.
4. J. W. Bruce, *Isotopies of generic plane curves*, Glasgow Math. J. **24** (1983), 195–206.
5. D.L.Fidal and P.J.Giblin, *Generic 1-parameter families of caustics by reflexion in the plane*, Math. Proc. Camb. Phil. Soc. **96** (1984), 425–432.
6. D.L.Fidal, *The existence of sextactic points*, Math. Proc. Camb. Phil. Soc. **96** (1984), 433–436.
7. S. Izumiya and T. Sano, *Generic affine differential geometry of plane curves*, to appear in Proceeding of the Edinbugh Mathematical Society.
8. K. Nomizu and T. Sasaki, *Affine differential geometry – Geometry of affine immersions –*, Cambridge University press, 1994.