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IMMERSIONS FROM THE 2-SPHERE TO THE 3-SPHERE WITH ONLY TWO TRIPLE POINTS

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ABSTRACT. Let f be an immersion from the 2-sphere S^2 to the 3-sphere S^3. Suppose that the singular set $S(f)$ contains only two triple points, and all components of $S(f)$ contain triple points. In this paper, we list up the neighborhoods of singular sets of immersions up to homeomorphism.

0. INTRODUCTION

Let S^n be the n-dimensional sphere, and $I = [-1, 1]$. In [Y], Yamagata researched about singualr surfaces with only one triple points. In [B], Banchoff showed the following: let F be a closed surface, and f an immersion from F to S^3, then the number of triple points is congruent modulo 2 to the Euler characteristic of F. Therefore an immersion from S^2 to S^3 with only two triple points is the easiest in all immersions from S^2 to S^3 with triple points. Let f be an immersion from S^2 to S^3. In this paper, we list up the following neighborhoods of singular sets of immersions:

(1) the singular set $S(f)$ contains only two triple points, and

(2) all components of $S(f)$ contain triple points.

We will work in the PL category. All submanifolds are assumed to be locally flat.

Put $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1^2 + x_2^2 + x_3^2 \leq 1\}$, $P_i = B \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_i = 0\}$ $(i = 1, 2, 3)$, and $P_1^+ = B \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \geq 0 \text{ and } x_2 = 0\}$. Let P_4 be a cone with a vertex (0,0,0) of a figure eight in ∂B (see Figure 1).

Let F be a compact surface, M a 3-manifold, and $f : F \longrightarrow M$ a map. We say that f is in *general position*, if for each element x of $f(F)$, there exist a regular neighborhood N of x in M and a homeomorphism $h : N \longrightarrow B$ such that N and h satisfy the following three conditions:

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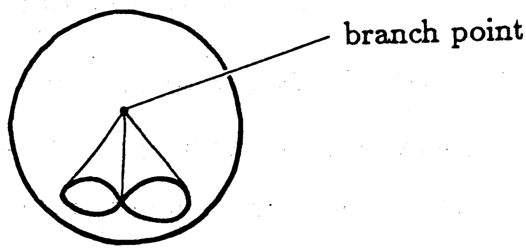


Figure 1

(1) Under h, $(N, N \cap f(F), x)$ is homeomorphic to either $(B, P_1, (0,0,0))$, $(B, P_1 \cup P_2, (0,0,0))$, $(B, P_1 \cup P_2 \cup P_3, (0,0,0))$, $(B, P_1^+, (0,0,0))$ or $(B, P_4, (0,0,0))$.

(2) If $(N, N \cap f(F), x)$ is not homeomorphic to $(B, P_4, (0,0,0))$ and $(B, P_1^+, (0,0,0))$, then for each component R of $f^{-1}(f(F) \cap N)$ there exists an integer i such that $h \circ f|R : R \longrightarrow P_i$ is a homeomorphism.

(3) If $(N, N \cap f(F), x)$ is homeomorphic to $(B, P_4, (0,0,0))$, then $f^{-1}(N \cap f(F))$ is a disk.

Note. If $(N, N \cap f(F), x)$ is homeomorphic to $(B, P_4, (0,0,0))$, then x is called a *branch point* (also known as "Whitney's umbrella" or "a pinch point"). If $(N, N \cap f(F), x)$ is homeomorphic to $(B, P_1 \cup P_2, (0,0,0))$, then x is called a *double point*. If $(N, N \cap f(F), x)$ is homeomorphic to $(B, P_1 \cup P_2 \cup P_3, (0,0,0))$, then x is called a *triple point*.

Let f be a map in general position. Then let $S(f)$ be the set of all double points, triple points and branch points of f. We call $S(f)$ a *singular set* of f. And $\tilde{S}(f) = f^{-1}(S(f))$. Let $T(f)$ be the set of all triple points of f. If $S(f)$ does not contain any branch points, then f is called an *immersion*.

We say that a 3-manifold M is *a cube-with-handles* if M is orientable and M is obtained from a 3-ball by attaching 1-handles.

All homology groups are with coefficients in $\mathbb{Z}_2$.

The paper is organized as follows. In Section 1, we define S-neighborhoods. In Section 2, we consider necessary conditions of singular sets of immersions from closed surfaces. In Section 3, we consider a singular set containing only two triple points. In Section 4, we consider a part of singular sets. In Section 5, we introduce Main Theorem.

1. S-NEIGHBORHOODS

Let F be a compact surface with or without boundary, M a 3-manifold, and f a map from F to M with in general position and $S(f) \cap f(\partial F) = \phi$. Let C be a subset of $S(f)$. We call C *a double curve of* f if there exists an immersion i from S^1 to $S(f)$ with $i(S^1) = C$.

To each component of the double curve, a (2×2)-signed permutation matrix is associated as follows: Choose a double point in the double curve. A disk that is transverse to the double curve at the double point intersects $f(F)$ at a pair of the coordinate arcs. Assign $e_1 = (1,0)$ and $e_2 = (0,1)$ to any two consecutive branches of the coordinate arcs. The opposite branch of e_i is assigned $-e_i$ for $i = 1,2$. Follow the branches e_1 and e_2 around the double curve until they come back to match the branches v_1 and v_2, respectively where $v_1, v_2 = \pm e_1$ or $\pm e_2$. Then the (2×2)-signed permutation matrix (v_1, v_2) is the associated double curve point matrix. We denote by $M(C) = (v_1, v_2)$. The double curve matrix depends on the choice of two consecutive branches. However, it is affected at most by a change of sign when a different choice is made (see [C-K]).

Let $C_1, \dots, C_n$ be double curves of f and $C = \cup_{t=1}^{n} C_t$. Let $N(C)$ be a regular neighborhood of C in M, and $G(C) = N(C) \cap f(F)$. Then we call $\mathfrak{N}(C) = (N(C), G(C))$ *an S-neighborhood of* C.

Remark. If M is orientable, then a neighborhood $N(C)$ of C is a cube-with handles in M, and $\partial N(C) \cap G(C)$ is a 4-valence graph on the oriented closed surface $\partial N(C)$.

Let $\mathfrak{N}(C) = (N(C), G(C))$ be as above. Then $\partial N(C) \backslash G(C)$ consists of some regions. We say that $\mathfrak{N}(C)$ has *a checkerboard coloring*, if each regions can be colored black or white such that adjacent regions have different colors (i.e. Let $E_1, \dots, E_n$ be the components of $\partial N(C) \setminus G(C)$. Then there exists a map $g : \{E_1, \dots, E_n\} \longrightarrow \{0,1\}$ such that $g(E_i) \neq g(E_j)$ if there exists an arc α in S^3 with $\partial\alpha = a_1 \cup a_2$, $a_1 \in E_i$, $a_2 \in E_j$ and $\alpha \cap f(F) = \{\text{one point}\}$).

Let C be a double curve of f, and $\mathfrak{N}(C)$ an S-neighborhood of C. If C is a simple closed curve, and if M is orientable, then $N(C)$ is a solid torus, and $\mathfrak{N}(C)$ satisfies one of the following conditions:

(C1) $G(C)$ consists of *two* immersed annuli and disjoint meridional disks, (C2) $G(C)$ consists of *one* immersed Möbius band and disjoint meridional disks,

(C3) $G(C)$ consists of *two* immersed Möbius bands and disjoint meridional disks.

Lemma 1.1 ([S1, Lemma 1.1]). *Let f be an immersion from a compact surface F to an oriented 3-manifold M. Let C be a double curve of f, and $\mathfrak{N}(C)$ an S-neighborhood of C. Suppose that C is a simple closed curve and $\mathfrak{N}(C)$ has a checkerboard coloring.*

(1) $\mathfrak{N}(C)$ satisfies the condition (C1) or (C3) if and only if the number of the meridional disks of $G(C)$ is even.

(2) $\mathfrak{N}(C)$ satisfies the condition (C2) if and only if the number of the meridional disks of $G(C)$ is odd.

Lemma 1.2 ([S1, Lemma 1.2]). *Let $F, f, C, \mathfrak{N}(C)$ be as above. If C is a simple closed curve, then*

$$M(C) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \mathfrak{N}(C) \text{ satisfies the condition (C1)} \\ \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \mathfrak{N}(C) \text{ satisfies the condition (C2)} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \mathfrak{N}(C) \text{ satisfies the condition (C3)} \end{cases}.$$

Notes 1.3 ([S1, Lemma 1.3]). Let F be a closed surface, M a 3-manifold, f a map from F into M with in general position, and c a simple closed curve in M such that c is transverse to $f(F)$, $c \cap S(f) = \phi$. If $f_*[F] = 0$ in $H_2(M)$, then

(1) the number of points of $c \cap f(F)$ is even, and

(2) each region of $M \setminus f(F)$ can be colored black or white so that adjacent regions have different colors.

Lemma 1.4 ([S1, Lemma 1.4]). *Let F, f, M be as above. Let $C_1, \dots, C_n$ be double curves of f, $C = \cup_{t=1}^{n} C_t$, and $\mathfrak{N}(C)$ an S-neighborhood of C. If F is closed and $f_*[F] = 0$ in $H_2(M)$, then $\mathfrak{N}(C)$ has a checkerboard coloring.*

Let F be a compact surface, M a 3-manifold, f a map from F into M with in general position and $S(f) \cap f(F) = \phi$. Let C be a double curve of f, $\mathfrak{N}(C)$ an S-neighborhood of

C, and x a point of C. Then we will define two operations at x. Let $r_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation map at (0,0) obtained by

$$r_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Put $X = 0 \times I \cup I \times 0$, $D = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 1\}$, $Q_+ = X \times [1,2]$, $Q_- = X \times [-2,-1]$, $Q = X \times [-2,2]$, $N_+ = D \times [1,2]$, $N_- = D \times [-2,-1]$, and $N = D \times [-2,2]$. Let $\tilde{Q} = Q_- \cup Q_+/(x,-1) \sim (r_\pi(x), 1)$. That is $\tilde{Q}$ is obtained by a half twisting at (0,0,0) to Q. Let $\tilde{N} = N_- \cup N_+/(x,-1) \sim (r_\pi(x), 1)$. We consider $N = \tilde{N}$ and $\tilde{Q} \subset \tilde{N}$.

Suppose that x is a double point of f. Then there exist a regular neighborhood N_x of x in S^3 and a homeomorphism $h : N \longrightarrow N_x$ such that $(N_x \cap N(C), N_x \cap f(F))$ is homeomorphic to (N, Q) under h. Let $\mathfrak{N}_x(C) = (N(C), (G(C) \setminus N_x) \cup h(\tilde{Q}))$, then we say that $\mathfrak{N}_x(C)$ is obtained by *a half twisting at x*.

Let $p_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the map obtained by $p_1(x_1, x_2, x_3) = (x_1, -x_3, x_2)$, and $p_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the map obtained by $p_1(x_1, x_2, x_3) = (x_2, -x_1, x_3)$. Put $N_1 = N$, $N_{k+1} = (p_1 \circ p_3)(N_k)$, $Q_1 = X \times [-2,2] \cup D \times 0$, and $\tilde{Q}_{k+1} = (p_1 \circ p_3)(\tilde{Q}_k)$ for $k = 1, 2$. There exists a homeomorphism $g_k : [-2,2]^3 \longrightarrow [-2,2]^3$ $(k = 1,2,3)$ such that $f_k | \partial[-2,2]^3$ is an identity, and $\tilde{N}_k \cap \tilde{N}_j = \phi$ if $k \neq j$ where $\tilde{N}_k = g_k(N_k)$. Put $\tilde{Q}_k = g_k(Q_k)$, then $\tilde{Q}_k \cap \tilde{Q}_j = \phi$ if $k \neq j$.

Suppose that x is a triple point of f and i. Then there exist a regular neighborhood N_x of x in S^3 and a homeomorphism $h : [-2,2]^3 \longrightarrow N_x$ such that $(N_x \cap N(C), N_x \cap f(F))$ is homeomorphic to $(N_1 \cup N_2 \cup N_3, Q_1 \cup Q_2 \cup Q_3)$ under h. Let $\mathfrak{N}_x(C) = ((N(C) \setminus N_x) \cup (\cup_{k=1}^3 h(\tilde{N}_k)), (G(C) \setminus N_x) \cup (\cup_{k=1}^3 h(\tilde{Q}_k)))$, then we say that $\mathfrak{N}_x(C)$ is obtained by *a decomposition at x* (see Figure 2). We can define a decomposition at x if x is a triple point of f and a double point of i (see Figure 3).

Lemma 1.5 ([S1, Lemma 1.5]). *Let F be a compact surface, M a 3-manifold, f a map from F into M with in general position and $S(f) \cap f(F) = \phi$. Let C be a double curve of f, $\mathfrak{N}(C)$ an S-neighborhood of C, and x a point of C. If $\mathfrak{N}(C)$ has a checkerboard coloring, then $\mathfrak{N}_x(C)$ has a checkerboard coloring.*

Remark. Suppose that x is a double point of f. Then $\mathfrak{N}(C)$ has a checkerboard coloring if and only if $\mathfrak{N}_x(C)$ has a checkerboard coloring.

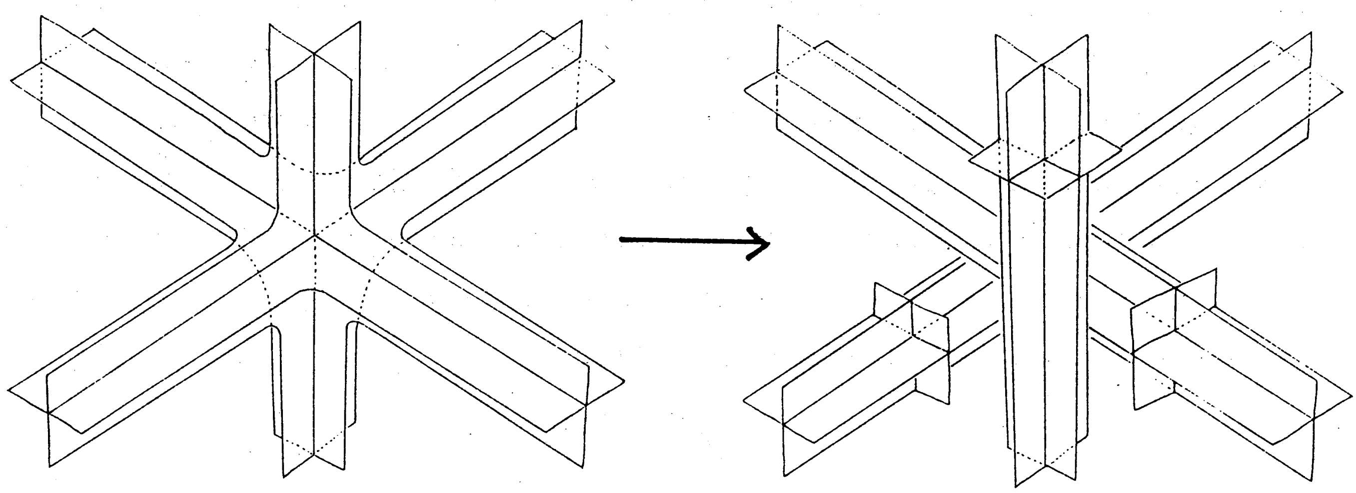


Figure 2

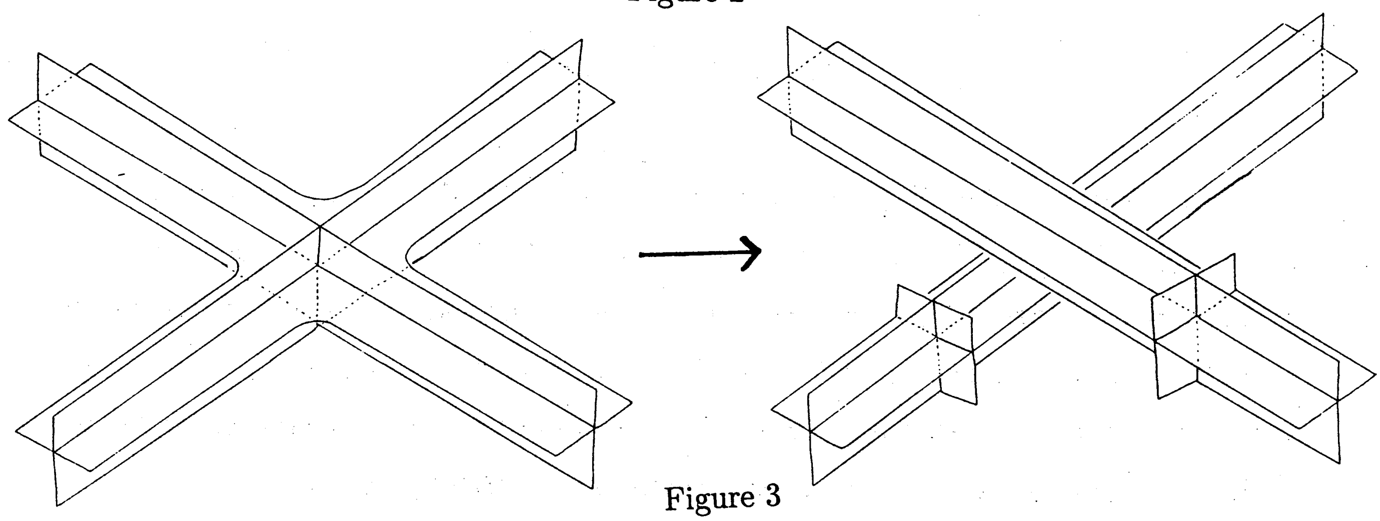


Figure 3

Lemma 1.6 ([S1, Theorem 1.6]). *Let F be an oriented closed surface, and f an immersion from F to an oriented 3-manifold M. Let C_0 be a double curve of f, and C a component of $S(f)$. If $f_*[F] = 0$ in $H_2(M)$, then*

(1) $M(C_0)$ is an identity matrix.

(2) the number of $i^{-1}(T(f))$ is even, where i is an immersion from S^1 to C_0, and

(3) the number of $C \cap S(f)$ is even.

2. Necessary conditions of singular sets

Let F be a compact surface, M a 3-manifold, f an immersion from F into M with $S(f) \cap f(F) = \phi$. In this section we may assume that f is an immersion. Let C be a component of $S(f)$, and $\mathfrak{N}(C)=(N(C), G(C))$ an S-neighborhood of C. Then $G(C)$ is an immersed surface, and $f^{-1}(G(C))$ consists of compact surfaces. Let $c_1, c_2, \dots, c_k$ be simple closed curves of $\partial(f^{-1}(G(C)))$ and $\mathfrak{C} = \{c_1, c_2, \dots, c_k\}$. Let $D_1, D_2, \dots, D_m$ be properly embedded disks in $N(C)$ such that $D_i \cap D_j = \phi$ $(i \neq j)$, D_j is transverse to all simple closed curves of $\mathfrak{C}$, $D_j \cap (\cup\mathfrak{C}) = \{\text{four points}\}$ and $(R, R \cap G(C))$ is homeomorphic to a neighborhood of a triple point, where R is the closure of a component of $N(C) \setminus (\cup_{j=1}^{m} D_j)$. Fix an orientation of $N(C)$. Fix the orientation of ∂D_j. Put $\{x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}\} = D_j \cap (\cup\mathfrak{C})$ in the order in which they appear on ∂D_j (see Figure 4).

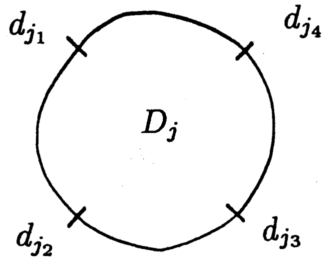
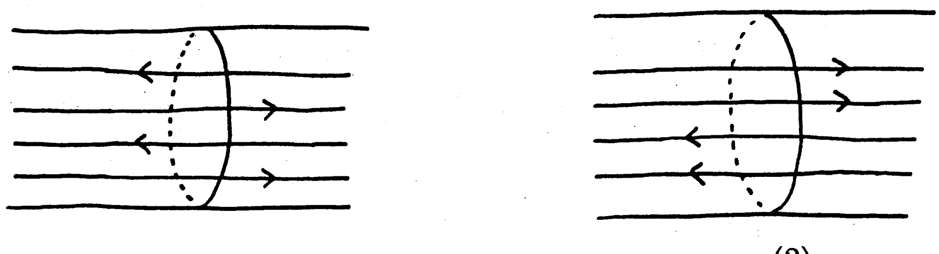


Figure 4

Fix an orientation of each c_i. Let x_{j_s} be a point of $D_j \cap (\cup\mathfrak{C})$, c_i a simple closed curve of $\mathfrak{C}$ with $x_{j_s} \subset D_j \cap c_i$. We can find an embedding $h : I^2 \longrightarrow \partial N(C)$ such that $h(0) = x_{j_s}$, $h^{-1}(D_j) = I \times 0$ and $h^{-1}(c_i) = 0 \times I$. Then the orientation of h is determined by the orientations of $h|I \times 0$ and $h|0 \times I$. Therefore we can define the sign of x_{j_s}, $\epsilon(x_{j_s}) = \pm 1$, as follows. Choose h so that $h|I \times 0$ and $h|0 \times I$ are in the given orientation for D_j and c_i. Then $\epsilon(x_{j_s}) = +1$ if h is in the given orientation for $\partial N(C)$ and -1 if not.



(1) Figure 5 (2)

We say that $\mathfrak{C}$ has an *orientation for colorings* if there exists an orientation of c_i such that $(\epsilon(x_{j_1}), \epsilon(x_{j_2}), \epsilon(x_{j_3}), \epsilon(x_{j_4})) = (+1, -1, +1, -1)$ or $(-1, +1, -1, +1)$ for all disks D_j (see Figure 5 (1)). We say that $\mathfrak{C}$ has a *good orientation* if there exists an orientation of c_i such that $(\epsilon(x_{j_1}), \epsilon(x_{j_2}), \epsilon(x_{j_3}), \epsilon(x_{j_4})) = (+1, +1, -1, -1)$, $(+1, -1, -1, +1)$, $(-1, -1, +1, +1)$ or $(-1, +1, +1, -1)$ for all disks D_j (see Figure 5 (2)).

Note. If $\mathfrak{C}$ has an orientation for colorings, there are only two cases of the orientations for colorings.

Lemma 2.1 ([S1, Theorem 2.1]). *Let F be a compact surface, M an oriented 3-manifold, f an immersion from F into M. Let C be a component of $S(f)$, and $\mathfrak{N}(C) = (N(C), G(C))$ an S-neighborhood of C. Then*

(1) $G(C)$ is an orientable immersed surface if and only if $\mathfrak{C}$ has a good orientation.

(2) $\mathfrak{N}(C)$ has a checkerboard coloring if and only if $\mathfrak{C}$ has an orientation for colorings.

Corollary 2.2 ([S1, Corollary 2.2]). *Let $C, \mathfrak{N}(C), N(C), G(C), \mathfrak{C}$ be as above. Let D_j, c_i be as above.*

(1) If $G(C)$ is an orientable immersed surface, and if $\mathfrak{N}(C)$ has a checkerboard coloring, then $D_j \cap c_i$ is at most two points for all D_j and c_i.

(2) The map $f|f^{-1}(G(C))$ is extended to an immersion from closed surfaces to M without changing $G(C)$ if and only if $\mathfrak{C}$ has an orientation for colorings.

Let f be an immersion from a compact orientable surface F to a 3-manifold M with $S(f) \cap f(F) = \phi$. Fix an orientation of each double curve of f. And fix the orientation of $\tilde{S}(f)$ induced from the orientation of $S(f)$. Let $\tilde{C}$ be a simple closed curve in $\tilde{S}(f)$, and $N(\tilde{C})$ a regular neighborhood of $\tilde{C}$ in F. Then $N(\tilde{C}) \cap \tilde{S}(f)$ consists of oriented immersed arcs. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the arcs in $N(\tilde{C}) \cap \tilde{S}(f)$ such that $(N(\tilde{C}), \cup_{j=1}^{k} \beta_j)$ is homeomorphic to $(\{t_1, t_2, \ldots, t_k\}, S^1) \times I$. We can define define the sign of β_j, $\epsilon'(\beta_j) = \pm 1$ in a similar way above Lemma 2.1. And we define $I(\tilde{C}) = \Sigma_{j=1}^{k} \epsilon'(\beta_j)$.

Let C and C' be double curves of immersions f and f', respectively. We define two relations of double curves and S-neighborhoods. Let $\mathfrak{N}(C)$ and $\mathfrak{N}(C')$ be S-neighborhoods of C and C', respectively. If $\mathfrak{N}(C)$ is homeomorphic to $\mathfrak{N}(C')$ or $-\mathfrak{N}(C')$ where $-\mathfrak{N}(C')$ is a mirror image of $\mathfrak{N}(C')$, then we say $\mathfrak{N}(C)$ is *equivalent* to $\mathfrak{N}(C')$ and C is *equivalent* to C'.

Lemma 2.3 ([S1, Lemma 2.3]). *Let $f, \tilde{C}$ be as above. If f is an immersion from the 2-sphere S^2, then $I(\tilde{C}) = 0$.*

Proof. Let D be a disk in S^2 with $\partial D = \tilde{C}$. The set $D \cap \tilde{S}(f)$ consists of oriented immersed arcs and oriented immersed closed circles. Therefore $I(\tilde{C}) =$ (the number of arcs into D) $-$ (the number of arcs out of D) $= 0$. □

3. A singular set $S(f)$ containing only two triple points

Let F be an oriented closed surface, and f an immersion from F to S^3. Let C be a double curve of f, and i an immersion from S^1 to C. Let k_j be the number of $i^{-1}(t_j)$ where t_j is a triple point of f. Let $C \cap T(f) = \{t_1, \ldots, t_n\}$. Then we say that C is the double curve of *type* $(k_1, \ldots, k_n)$. We may assume $k_1 \geq k_2 \geq \cdots \geq k_n$. Suppose that $T(f)$ consists of only 2 points. If C contains triple points of f, then C is type (k_1) or (k_1, k_2). If C is type (k_1), then $k_1 = 2$ by Lemma 1.5. If C is type of (k_1, k_2), then $k_1 + k_2 = 0$ (mod 2) by Lemma 1.5. Therefore C is type $(1,1)$, $(2,2)$ $(3,1)$ or $(3,3)$.

Let C be a double curve of f, i an immersion from S^1 to C. Let α be a subarc in S^1 such that $\alpha \cap i^{-1}(T(f)) = \{t_1, t_2\} \subset int\alpha$, and $i(t_1) = i(t_2)$. Then we call $i(\alpha)$ *a loop* in $S(f)$.

Lemma 3.1. *Let F be an oriented closed surface, f an immersion from F to S^3, and C a double curve. Suppose that $T(f)$ consists of only two points. If the double curve C contains triple points of f, then C is equivalent to one of figures as in Figure 6.*

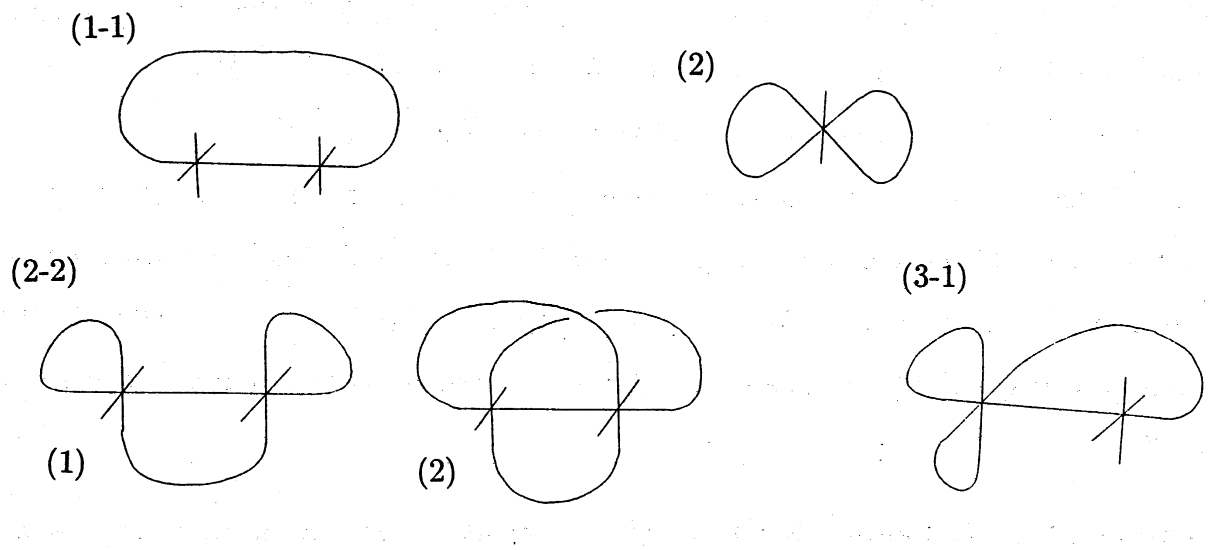


Figure 6

(3-3)

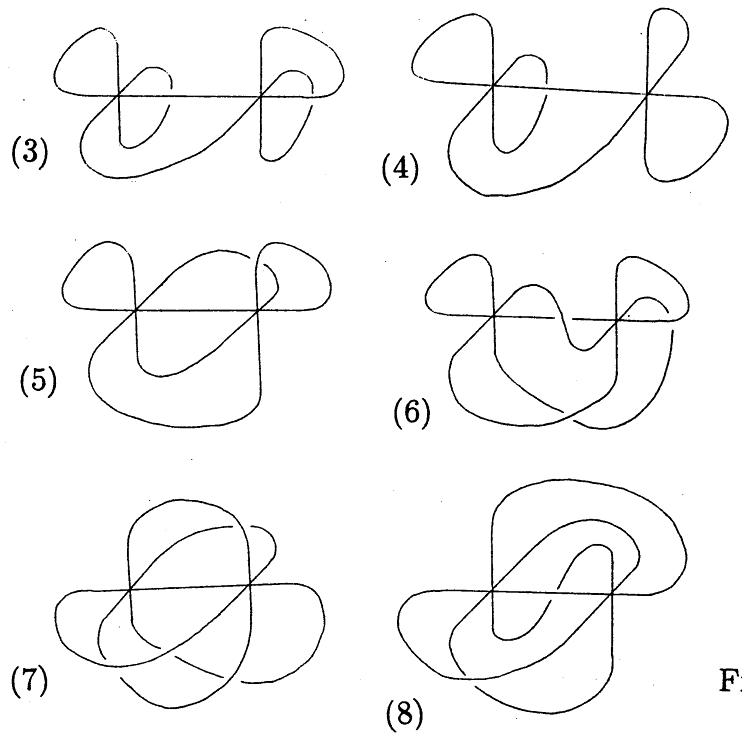


Figure 6

Lemma 3.2. *Let F, f be as above. Suppose that the triple points set $T(f)$ consists of only two points. If C is a component of $S(f)$ with triple points, then C is equivalent to one of figures as in Figure 7.*

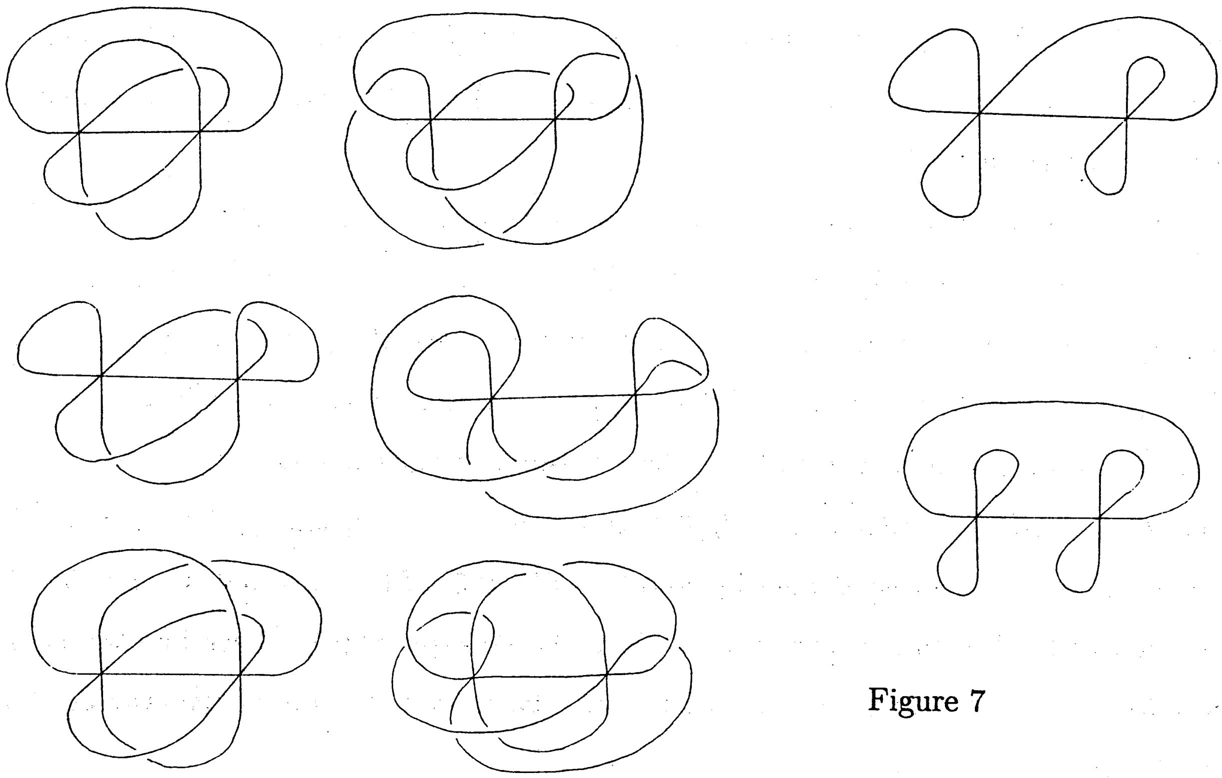


Figure 7

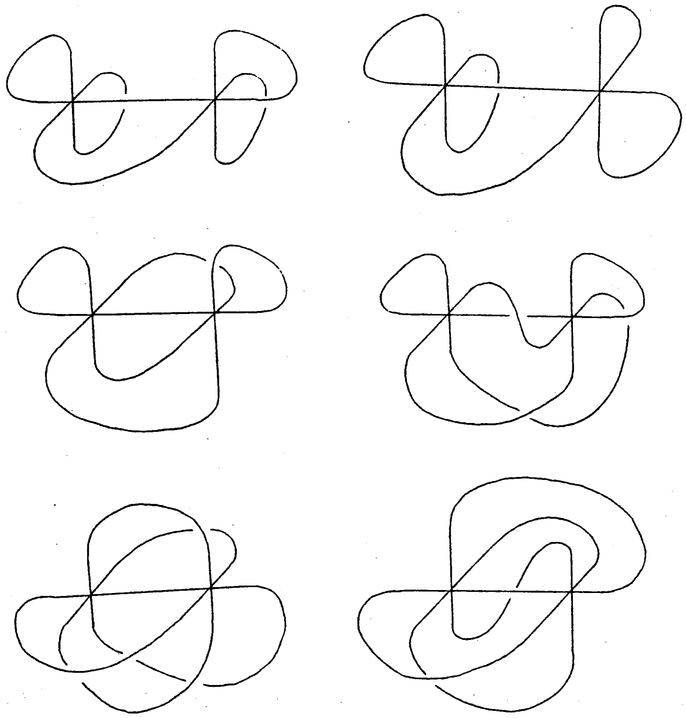


Figure 7

4. A PART OF A SINGULAR SET

Let F be a compact surface, f a map from F to a 3-manifold M with in general position and $S(f) \cap f(\partial F) = \phi$. Let C be a double curve of f, i an immersion from S^1 to C. Let α be a subarc in S^1 such that $\alpha \cap i^{-1}(T(f)) = \{t_1, t_2\} \subset int\alpha$, and $i(t_1) = i(t_2)$. Then we call $i(\alpha)$ *a loop* in $S(f)$.

We use the following notation about an S-neighborhood of a singular set. Let C be a double curve of f, and $\mathfrak{N}(C) = (N(C), G(C))$ an S-neighborhood of C. Let $\alpha_1, \ldots, \alpha_n$ be the subarcs of C such that α_i connects triple points of f, $int\alpha_i \cap T(f) = \phi$, and $\cup_{i=1}^{n} \alpha_i = C$. Put $D = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 1\}$ and $X = \{(x, y) \in \mathbb{R}^2 | \ (x^2 + y^2 \leq 1)$ and $(x = 0$ or $y = 0)\}$. For each α_i, there exists an immersion $f_i : D \times I \longrightarrow M$ such that $Imf_i \cap G(C) = f_i(X \times I) \cup f_i(D \times \partial I)$, $Imf_i \subset N(C)$, and $f_i(0 \times I) = \alpha_i$. Let $v = (0, 1) \in D$. Then we denote by $Sk(C) = C \cup (\cup_{i=1}^{n} f_i(v \times I))$ and we call $Sk(C)$ a *skeleton* of C. We can construct $\mathfrak{N}(C)$ from the skeleton $Sk(C)$. Therefore we use the notation $Sk(C)$. We can define a skeleton of subarcs in $S(f)$ in a similar way as above (see Figure 8).

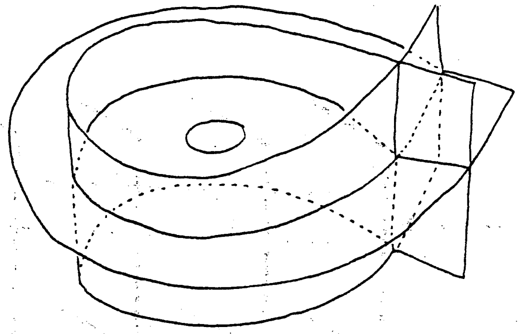
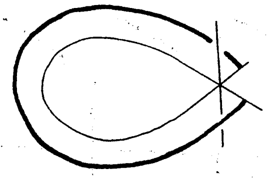



Figure 8

Lemma 4.1 ([S1, Lemma 3.1]). *Let F be a closed oriented surface, and f an immersion from F to S^3. Let α be a loop in $S(f)$. Then $Sk(\alpha)$ is equivalent to a figure as in Figure 9.*

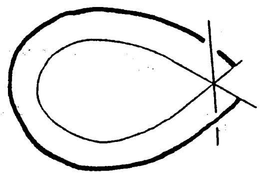


Figure 9

Lemma 4.2 ([S1, Lemma 3.2]). *Let F, f be as above. Let C be a double curve of f. If C is the double curve of type (2), then the genus of F is greater than 1.*

5. Main Theorem

We define an immersed surface with arcs. Let G be a closed surface, g an immersion from G to S^3. Let $\alpha_1, \dots, \alpha_n$ be pairwise disjoint arcs in S^3 which satisfy the following conditions.

(1) $S(g) \cap \alpha_i = \phi$ for all i $(1 \leq i \leq k)$.

(2) $\partial\alpha_i \subset f(G)$, and $int\alpha_i$ is transverse to $g(G)$ for all i $(1 \leq i \leq k)$.

Then we call $(g(G), \cup_{i=1}^{n} \alpha_i)$ an *immersed surface with arcs*.

Let $\mathfrak{G} = (g(G), \cup_{i=1}^{n} \alpha_i)$ be an immersed surface with arcs. We construct an immersed surface $F(\mathfrak{G})$ in S^3 as follows. Let D^2 be a disk. Let $N(\alpha_i)$ be a small product neighborhood of α_i in S^3 such that $N(\alpha_i)$ has a parametrization as $\alpha_i \times D^2$ with $\alpha_i = \alpha_i \times \{0\}$ and $N(\alpha_i) \cap f(G) = (\alpha_i \cap g(G)) \times D^2$ (see Figure 10). Set $G' = g(G) \setminus (\cup_{i=1}^{k} \partial\alpha_i \times D^2)$. Let $\gamma_1^*, \dots, \gamma_m^*$ be the components of $intN(\alpha_i) \cap g(G)$. An immersed surface $F(\mathfrak{G})$ in S^3 satisfies $F(\mathfrak{G}) = G' \cup (\cup_{i=1}^{k} \alpha_i \times \partial D^2)$.

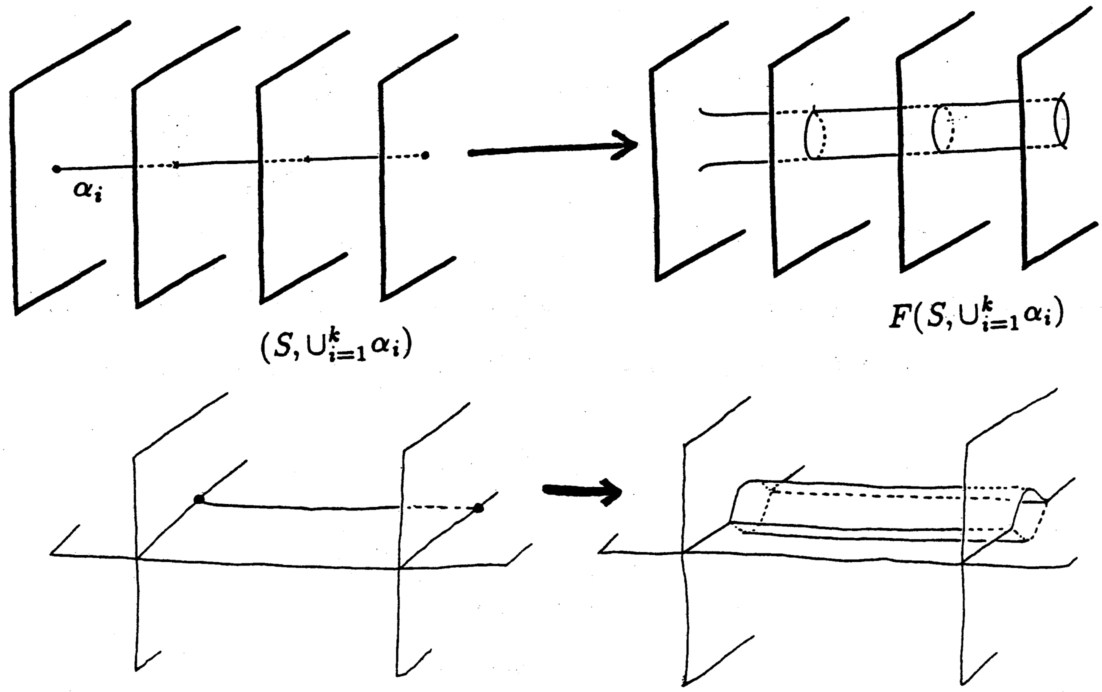


Figure 10

Main Theorem ([S2]). *Let f be an immersion from S^2 to S^3. Suppose that $T(f)$ consist only of two points, and each component of $S(f)$ contains triple points. Then an S-neighborhood of $S(f)$ is equivalent to one of immersions as in Figure 11.*

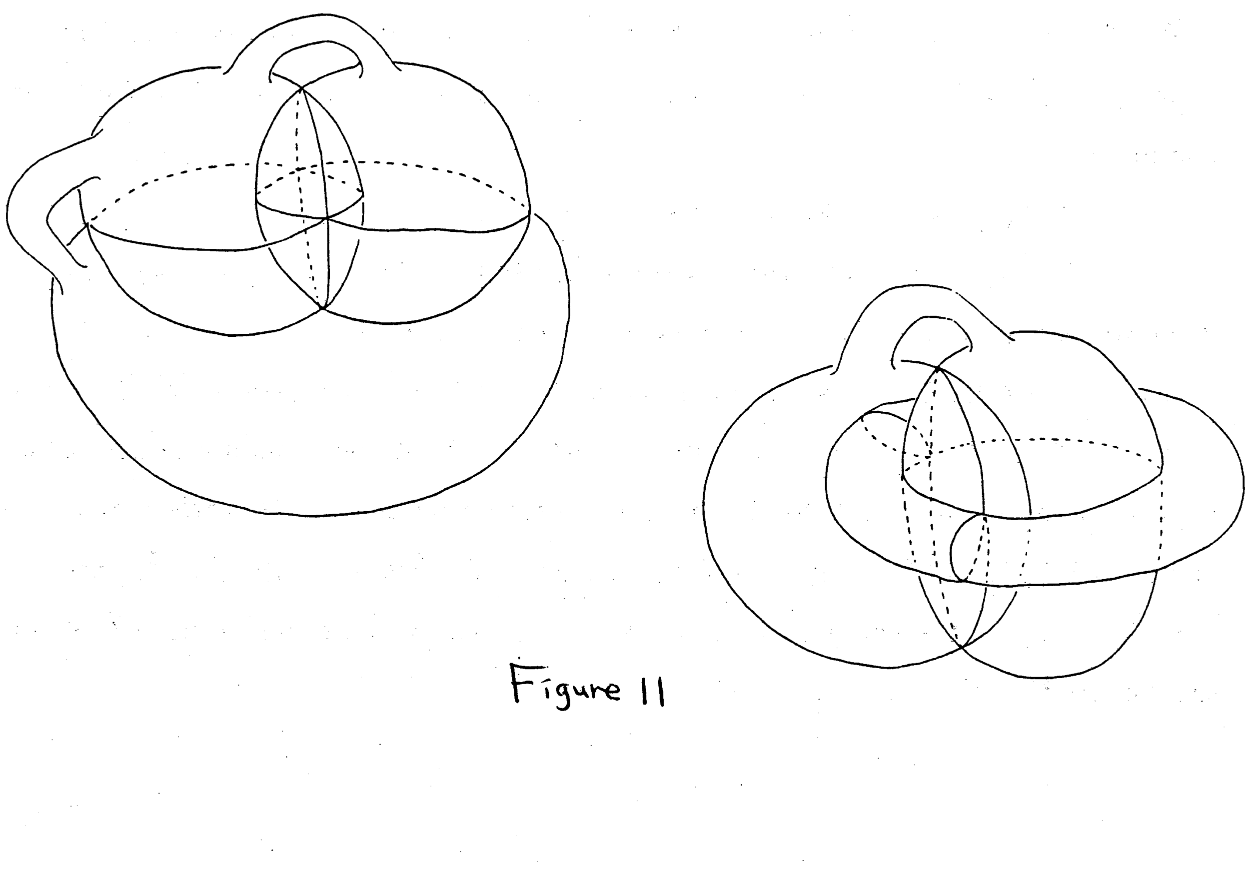


Figure 11

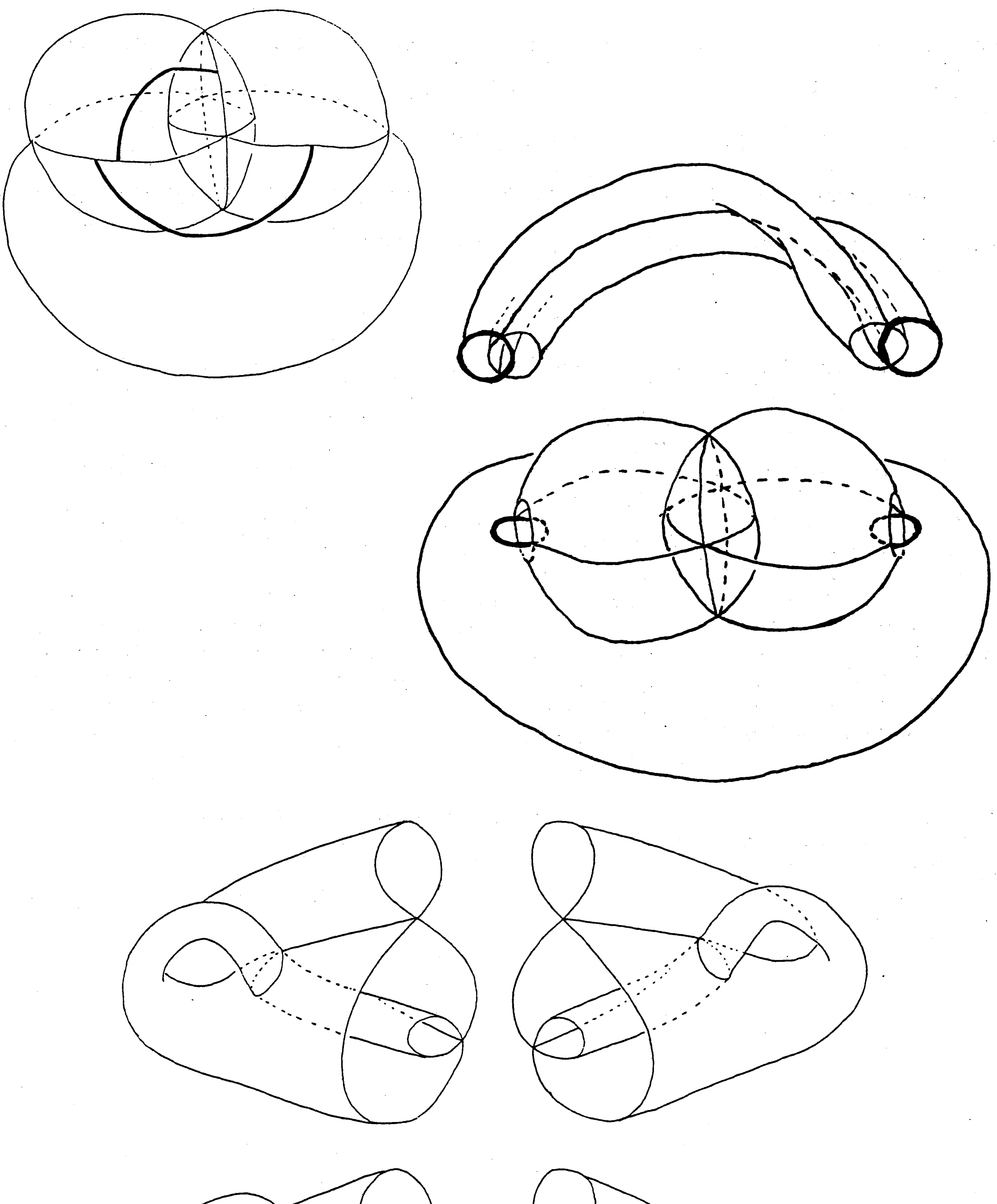


Figure 11

Remark. As above immersion f can be lifted to an embedding to S^4 (i.e. there exists an embedding $\tilde{f} : S^2 \longrightarrow S^4 \setminus \{\infty\}$ with $p \circ \tilde{f} = f$ where p is the projection map from $S^4 \setminus \{\infty\}$ to $S^3 \setminus \{\infty\}$) (for a definition of liftings, see [C-S2]).

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