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E^N にはめ込まれた多様体と平面との幾何的交点数

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1. Introduction and Main Theorem.

Throughout this article, we work in the C^∞ category. Let E^N be the N-dimensional Euclidean space $\mathbf{R}^N$ with the standard affine structure in the strict sense and the standard orientation. We are interested in a "geometric intersection number" of an immersed manifold and planes in E^N. We based on a general method below ;

In a situation that a set consist of a kind of geometric objects (lines and planes in this article) contains codimension 1 "wall" which decompose the set into some "chamber"s. By giving orientation to the wall, we can associate an number to each chamber. In some special cases, the number could correspond to a geometric phenomenon (a geomtric intersection number here) of the original object.

First, we introduce some spaces and maps with which we are mainly concerned. Let $P(N,n)$ be the set of oriented n-planes (lines if $n = 1$) in E^N. This space $P(N,n)$ admits a structure as an

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$(N-n)(n+1)$-dimensional C^{∞} ORIENTED manifold. In fact, $P(N,n)$ is homeomorphic to the total space of the orthogonal complement vector bundle to the tautological bundle over the corresponding real oriented Grassmannian manifold $G(N,n)$.

Let M^{m_1} be an m_1-dimensional closed connected manifold and $f\colon M \to E^N$ an immersion, where we allow that M is non-orientable. We let m_2 denote the codimension of f : $m_2 = N - m_1$. In section 2, from f, we construct an $((m_2+1)m_1-1)$-dimensional closed ORIENTED manifold $E({L_f}^*(p_1))$ and a map $HL_f\colon E({L_f}^*(p_1)) \to P(N,m_2)$. We regard HL_f as an $((m_2+1)m_1-1)$-cycle in $P(N,m_2)$. Since the codimension of HL_f is 1, the image of HL_f decompose $P(N,m_2)$ into some regions.

For an m_2-plane $x \in P(N,m_2)\backslash \mathrm{Im}HL_f$, We construct an element Γ_x in $H^{(m_2+1)m_1-1}(\,P(N,m_2)\backslash\{x\}\,)$ We have,

THEOREM. *Let $f\colon M \to E^N$ be an immersion and HL_f as above. When we take $x \in P(N,m_2)\backslash \mathrm{Im}HL_f$ and Γ_x corresponding to x,*

(A) $\qquad \#f^{-1}(\{f(M)\cap x\})$ *is finite.*

(B) $\qquad \#f^{-1}(\{f(M)\cap x\}) = \Gamma_x(HL_f)$.

Except "pathological "cases[1], the right hand side $\Gamma_x(HL_f)$ is equal to the number defined by algebraic intersection theory :

After taking a base point $x_0 \in P(N,m_2)$ enough near its end, we can associate an (non-negative) integer to each component of $P(N,m_2)\backslash \mathrm{Im}HL_f$ by algebraic intersection number theory (see [A]) ;

[1]The author knows no way to construct such a pathological example.

$$\mathrm{Int}(\mathrm{Im}HL_f, a_{x_0x}), \qquad (= \Gamma_x(HL_f))$$

where a_{x_0x} is an oriented arc in $P(N, m_2)$ which starts at x_0 and ends at x and which intersects with $\mathrm{Im}HL_f$ transversely. We mean that Γ_x is the Poincare dual of the arc a_{x_0x}. This association depends on neither a_{x_0x} nor x_0 (see [Wh]) (even in the case in which $m_2 = N-1$ and $P(N, N-1) \cong S^{N-1} \times \mathbf{R}$ has two ends).

We adopt $\#f^{-1}(\{f(M) \cap x\})$ as a geometric intersection number of $f(M)$ and x. Note that when f is an embedding, it is equal to the geometric intersection number in the original meaning of the word.

In this story, ORIENTATION plays an impotant role.

2. A Decomposition of $P(N, m_2)$

In this secion, from a given immersion $f\colon M \to E^N$, we construct an $((m_2+1)m_1 - 1)$-dimensional closed oriented manifold $E(L_f{}^*(p_1))$ and a map $HL_f\colon E(L_f{}^*(p_1)) \to P(N, m_2)$.

First we introduce a canonical double fibration. For $0 < n_1 \leq n_2 < N$, we let $P(N; n_1, n_2)$ denote the set defined as follows :

$$P(N; n_1, n_2) = \{\, (x, X) \in P(N, n_1) \times P(N, n_2) \,|\, x \subset X \,\}.$$

This set admits a structure as an ORIENTED C^∞ manifold whose dimension is $(n_1+1)(N-n_1) + (n_2-n_1)(N-n_2)$. $P(N; n_1, n_2)$ is homeomorphic to the total space of a certain vector bundle over the corresponding real Flag manifold.

We have two canonical fibrations :

$$\begin{array}{rcl} p_1\colon P(N; n_1, n_2) & \to & P(N, n_1) \\ (x, X) & \mapsto & x, \end{array} \quad \text{and} \quad \begin{array}{rcl} p_2\colon P(N; n_1, n_2) & \to & P(N, n_2) \\ (x, X) & \mapsto & X. \end{array}$$

The fiber of p_1 is homeomorphic to the oriented Grassmannian $G(N-n_1, n_2-n_1)$, on the other hand, the fiber of p_2 is homeomorphic to $P(n_2, n_1)$.

REMARK 1.. In the case $n_1 = n_2$, $P(N; n_1, n_2)$ is a disconnected double covering of $P(N; n_1)$, and the anti-diagonal component $\{(x, -x)\}$ has opposite orientation from the one induced from that of $P(N, n_1)$ by p_1.

On the other hand, we define $L_f \colon S(TM) \to P(N, 1)$ as follows, where $S(TM)$ is the spherical bundle associated to TM:

$$S(TM) = (TM \setminus \{\text{0-section}\}) / \sim, \quad \text{where } v_1 \sim v_2 \Leftrightarrow v_1 = \lambda v_2 \text{ for some } \lambda > 0.$$

$S(TM)$ is a $(2m_1 - 1)$-dimensional ORIENTED manifold even if M is non-orientable, because if we change a local orientation of M, the orientation of the fiber over the local base is also changed. We define the orientation of $S(TM)$ as a local orientation of $U(\subset M) \times$ that of the fiber over U. Now we define L_f as ;

$$\begin{aligned} L_f \colon \quad S(TM) &\to P(N, 1) \\ v\ (\in S(T_pM)) &\mapsto \text{The straight line whose orientation is } df(v) \\ &\qquad \text{and which passes through } f(p). \end{aligned}$$

Using L_f and the double fibration p_1, p_2 of $P(N; 1, m_2)$, we construct HL_f by the following diagram : $HL_f = p_2 \circ \tilde{L}_f$.

$$\begin{tikzcd}
E(L_f{}^*(p_1)) \arrow[rr, "\tilde{L}_f"'] \arrow[d, "\tilde{p}_1"] & & P(N;1,m_2) \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & HL_f \\
S(TM) \arrow[r, "L_f"'] & P(N,1) & & P(N,m_2)
\end{tikzcd}$$

We define $E(L_f{}^*(p_1))$ as the total space of the pull-back of the fibration p_1 over $S(TM)$ by L_f. The dimension of the space is $2m_1 - 1 + (m_2 - 1)(N - m_2) = (m_2 + 1)m_1 - 1$.

$$E(L_f{}^*(p_1)) = \{\, (v, (x, X)) \in S(TM) \times P(N; 1, m_2) \,|\, L_f(v) = x \,\}.$$

LEMMA 2.1. *$E(L_f{}^*(p_1))$ is closed and ORIENTED.*

Proof. Since $E(L_f{}^*(p_1))$ is the total space of a fiber bundle over $S(TM)$ whose fiber is $G(N-1, m_2-1)$, it is a closed manifold, since both the basespace and the fiber are closed.

In the case $m_2 = 1$, $E(L_f{}^*(p_1))$ is the double covering of $S(TM)$ stated in Remark 1, thus it is oriented. The case $N = 2$ $(m_1 = m_2 = 1)$ is included in this case.

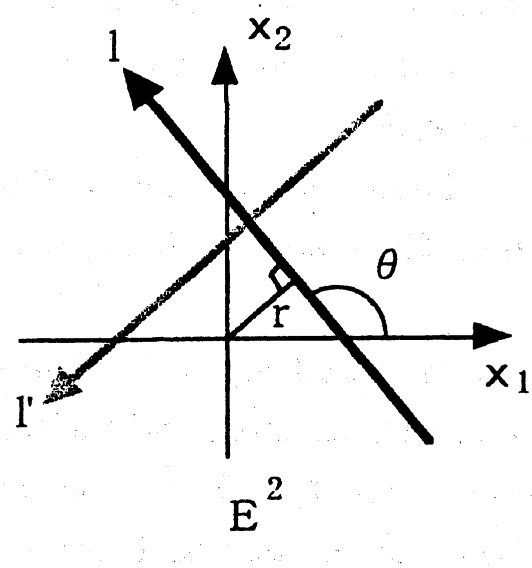
Next, we study the other cases, $N > 2$ and $m_2 > 1$. The base space $P(N,1)$ and every fiber of p_1 are both oriented, and the bundle p_1 is orientable, because the base space $P(N,1) \cong TS^{N-1}$ is simply connected in this case [MS]. $E(L_f{}^*(p_1))$ is the pull-back of p_1 over the oriented manifold $S(TM)$, thus it is orientable. We define the orientation of the total space by the local orientation of the base space $\times$ that of the fiber. We have the lemma. $\square$

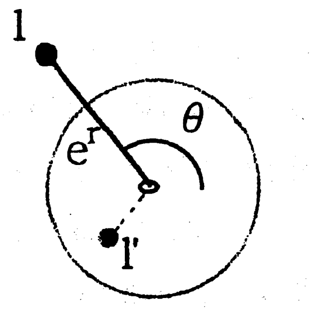
3. The Lowest Dimensional Case.

In this section, we study the case in which $m_1 = 1$ and $N = 2$, i.e., the case of immeresd plane curves. When we fix an affine coordinate (the origin o and a frame x_1, x_2-axes), $P(2,1)$ (= the set of oriented lines in E^2) can be identified with $\mathbf{C}\backslash\{0\}$ as follows ([T,IUN]). Here we identify E^2 and $\mathbf{C}$ by $(x_1, x_2) \leftrightarrow x_1 + ix_2$.

$$\begin{array}{ccc} P(2,1) & \longrightarrow & \mathbf{C}\backslash\{0\} \\ l = \{ta - ira | t \in \mathbf{R}\} & \mapsto & e^r a, \end{array}$$

where $a \in S^1$ (unit vector pointing the orientation of the line) and $r \in \mathbf{R}$. $|r|$ is the Euclidean distance between the origin o of E^2 and the line l ($r < 0$ if the origin is in the right hand side of the line).





EASY OBSERVATION..
$\mathrm{dist}(o, l) = |\log|z_l||$. $\{l \,|\, o \in l\} = \{z \in \mathbf{C} \,|\, |z| = 1\}$.
If a line $l \in P(2,1)$ is corresponding to z, $-l$ (= l with the orientation reversed) is corresponding to $-\bar{z}^{-1}$.

Let $c\colon S^1 \to E^2$ be an immersion. Let $L_c\colon S(TS^1) \to P(2,1)$ be the following map. L_c is regarded as a 1-cycle by the canonical orientation of $ST(S^1)$.

$$\begin{aligned} L_c\colon S(TS^1) &\to P(2,1) \\ v_{\text{at } x} &\mapsto \text{The straight line whose orientation is } dc(v) \\ &\qquad \text{and which passes through } c(x). \end{aligned}$$

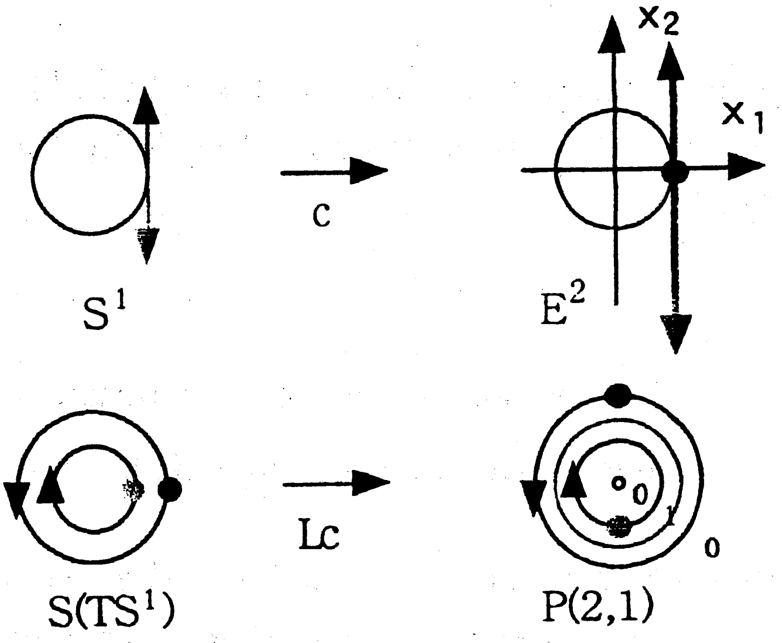
$\mathrm{Im}L_c$ decomposes $P(2,1)$ into some regions. After we take a base point $x_0 \in P(2,1)$ enough near its one end (0 or ∞, each of which can be chosen), we associate an integer to the each region by algebraic intersection number of L_c and a transverse arc from x_0 to x. This is well-defined ([A,Wh]) and essentially this number is equal to $\Gamma_x(L_c)$. In this case, our theorem claims,

THEOREM (in the case in which $N = 2$). *When we take $x \in P(2,1)\backslash \mathrm{Im}\, L_c$,*

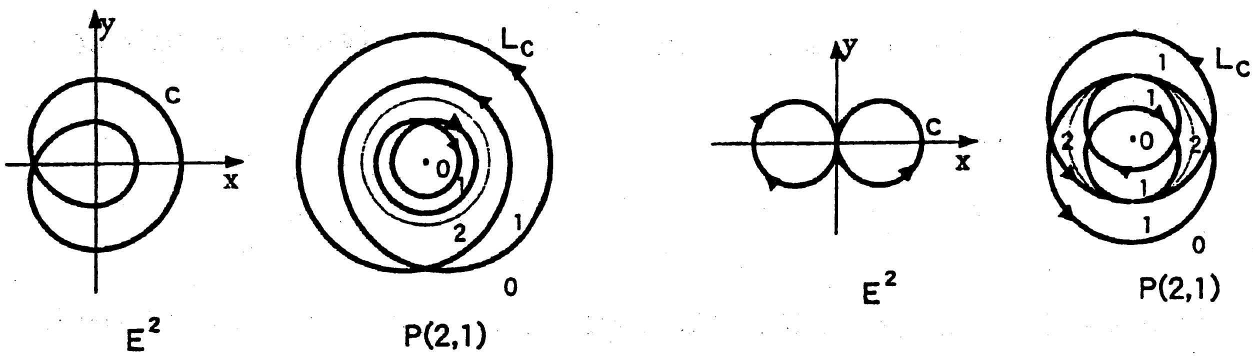
$$\# c^{-1}\{\mathrm{Im}c \cap x\} = 2\,\mathrm{Int}(L_c, a_{x_0 x}) \quad < \infty.$$

REMARK 2. Here the 2 in the right hand side appears, because we do not take the double covering stated in Remark 1.

EXAMPLES. First, we show the easiest case in which c is the unit circle. $P(2,1)\backslash\mathrm{Im}L_c$ is devided into 3 parts. It is easy to see that our theorem works in this case.



We show two more examples by drawings. Each of the left figures are images of immersed circles c and the right figures are images of the corresponding L_c in $P(2,1)$.



A double point of L_c corresponds to a double tangent line for c. An inflection point of c corresponds to a critical point of $p \circ L_c$, where $p\colon P(2,1) \to S^1$ is the standard natural projection ([IUN]).

4. A Proof (in the lowest dimensional case) .

In this section, we prove our main theorem in the lowest dimensional special case.

We study a geometric intersection number $\# c^{-1}\{\mathrm{Im} c \cap x\}$ of immersed plane curve c and a line x. After changing and fixing an affine coordinate (x_1, x_2) if needed, we may assume that $x = (+)x_1\text{-axis} = \{x_2 = 0\}$, which is corresponding to 1 in $P(2,1) = \mathbf{C}\backslash\{0\}$.

Let $x_i \colon E^2 \to \mathbf{R}$ ($(x_1, x_2) \mapsto x_i$) be the i-th factor projection and c_i the composition $x_i \circ c$ $(i = 1, 2)$. We have

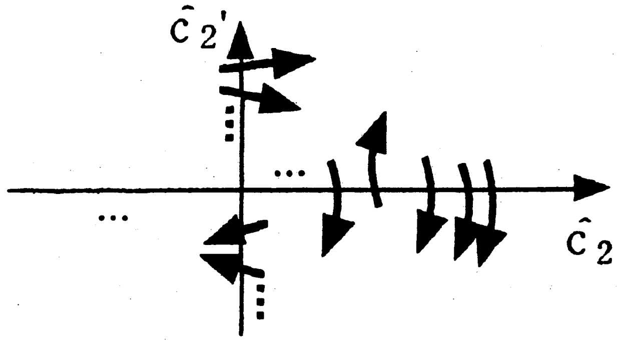
$$\# c^{-1}\{\mathrm{Im} c \cap x\} = \#\{{c_2}^{-1}(0)\}.$$

[Finiteness] By the assumption $x \in P(2,1)\backslash \mathrm{Im}\, L_c$, x is not a tangent line for c, which means that 0 is a regular value of c_2. Thus ${c_2}^{-1}(0)$ consists of discrete points.

[The formula] Let $\hat{c_i}$ be the composition $c_i \circ \pi \colon S(TS^1) \to \mathbf{R}$ and $\hat{c_i}' \colon S(TS^1) \to \mathbf{R}$ its differential. By the regularity, $({c_2}^{-1}(0)) \cap ({c_2'}^{-1}(0)) = \phi$ and $(\hat{c_2}^{-1}(0)) \cap ((\hat{c_2}')^{-1}(0)) = \phi$. Thus,

$$\begin{aligned}
\#\{{c_2}^{-1}(0)\} &= \frac{1}{2}\#\{\hat{c_2}^{-1}(0)\} \\
&= \frac{1}{2}\#\{v \in ST(S^1) \mid \hat{c_2}(v) = 0\} \\
&= \frac{1}{2}[\#\{v \in ST(S^1) \mid \hat{c_2}(v) = 0, \hat{c_2}'(v) > 0\} \\
&\qquad + \#\{v \in ST(S^1) \mid \hat{c_2}(v) = 0, \hat{c_2}'(v) < 0\}] \\
&= \#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0\} \\
&\qquad \text{(when it can be defined)}.
\end{aligned}$$

The last equality comes from the fact $\hat{c_2} > 0$ (or < 0) if and if only $\hat{c_2}'$ is increasing (or decreasing) and standard homotopy theory.



Since c is an immmersion, $\hat{c_1}'(v) \neq 0$ if $\hat{c_2}'(v) = 0$. Thus,

$$\begin{aligned}
&\#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0\} \\
&\quad = \#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0, \hat{c_1}'(v) > 0\} \\
&\quad\quad + \#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0, \hat{c_1}'(v) < 0\} \\
&\quad = 2\,\#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0, \hat{c_1}'(v) > 0\},
\end{aligned}$$

because the involution $-: ST(S^1) \to ST(S^1)$ carries the first set on the to the second bijectively.

We note that

$$\begin{aligned}
&\#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0, \hat{c_1}'(v) > 0\} \\
&\quad = \#\{\text{Maximal points of } c_2\} - \#\{\text{minimal points of } c_2\},
\end{aligned}$$

in the case in which they can be defined.

When $\hat{c}$ is locally represented as $(\hat{c_1}(t), \hat{c_2}(t))$ with $\hat{c_1}' > 0$, L_c is represented as

$$t \mapsto x_2 = \frac{\hat{c_2}'(t)}{\hat{c_1}'(t)} x_1 + \frac{\hat{c_1}'(t)\hat{c_2}(t) - \hat{c_1}(t)\hat{c_2}'(t)}{\hat{c_1}'(t)} \text{ (Oriented line)} .$$

Thus it is not hard to see that

$$\begin{aligned}
&\#_{alg}\{v \in ST(S^1) \mid \hat{c_2}(v) > 0, \hat{c_2}'(v) = 0, \hat{c_1}'(v) > 0\} \\
&\quad = \mathrm{Int}(L_c, a_{[01]}),
\end{aligned}$$

where $a_{[01]}$ is an arc in $P(2,1) = \mathbf{C}\backslash\{0\}$ from an end "0" to 1. Here we note that $t \in [0,1]$ is corresponding to the oriented line $x_2 = \log \frac{1}{t}$.

In the argument above, we may need to perturb the arc homotopically to define the numbers, but here we omit such a detailed part of our proof. □

5. Questions.

(5-1) Let γ be a local arc (or a germ) in $P(2,1)$. What is the condition that $\gamma = \mathrm{Im} L_c$ for an arc c in E^2 ?

[Obserbation] Around x_1-axis $= 1$ in $P(2,1) = \mathbf{C}\backslash\{0\}$,

(1) e^t $(-\epsilon < t < \epsilon)$ (family of parallel lines)

(2) $e^{R\cos\theta} e^{i\theta} (-\epsilon < \theta < \epsilon)$ (family of lines passing through a point $(R,0)$)

can not be $\mathrm{Im} L_c$ for any arc c.

(5-2) Does our idea work in generalized cases of the set of circles, the set of ovals or the set of embedded closed curves (∞-dimensional space) instead of $P(2,1)$?

[Observation] Curvature and such a differential-geometric method may be concerned with this question.

(5-3) Generalize our formula in the case in which $m_2 > N - m_1$, i.e., the case in which higher dimensional manifolds appear as an intersection of an immersed manifold and planes.

6. After the author's talk, it was pointed out by some professors that our formula might be known in other area of mathematics. The author is trying some more information again (Integral geometry [S and its references,T], Radon transformation,...). Any information from you would be gratefully appreciated.

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参考文献

[A] J.Alexander, 'Topological invariants of knots and links', *Trans.Amer. Math. Soc.* 30 (1928), 275–306.

[Wh] H.Whitney, 'On regular closed curves in the plane', *compositio Math.* 4 (1937), 276–284.

[MS] J.W.Milnor and J.D.Stasheff, 'Characteristic Classes', *Ann. Math. Studies 76, Princeton Univ. Press* (1974).

[S] L.A.Santalo, 'Integral geometry and geometric probability', Encyclopedia of Mathematics and its Applications 1 *Addison-Wesley, London* (1976).

[T] 田崎博之, '積分幾何学 ', 上智大学数学講究録 No.37 (1994).

[IUN] 飯高茂、上野健爾、浪川幸彦, 'デカルトの精神と代数幾何 ', 日本評論社 (1993).