数理解析研究所講究録
1006 巻 1997 年 55-68

VASSILIEV TYPE INVARIANTS OF ORDER ONE OF GENRIC MAPPINGS FROM A SURFACE TO THE PLANE

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ABSTRACT. In this note we give some isotopy invariants of C^∞ stable mappings from a closed surface M to $\mathbb{R}^2$ in the similar way as Vassiliev, Arnol'd and Goryunov [13], [2], [3], [7]. The detailed argument and applications will appear in the forthcoming paper.

§1 INTRODUCTION

V.A.Vassiliev introduced in [13] graded modules of knot invariants (the so-called Vassiliev knot invariants or knot invariants of finite type) by using appropriate startifications of the mapping space from S^1 to $\mathbb{R}^3$. Later, his method was used to produce Arnold's invariants of immersed plane curves, denoted by $J^\pm$ and St (cf. [2], [3]), and Goryunov's invariants of generic mappings from a closed oriented surface into $\mathbb{R}^3$ (cf. [7]). In this note, we will describe in a formal way *Vassiliev type invariants of order one for isotopy classes of C^∞ stable mappings* , that is mostly based on Goryunov's description. When the target manifold of mappings is Euclidean space, we will see that such invariants corresponds to 1-cocycles of the"Vassiliev complex" for $\mathcal{A}$-classes of multi-germs (graded by $\mathcal{A}_e$ codimension). And next, as a concrete example, we will give Vassiliev type invariants for C^∞ stable mappings from a closed surface to the plane. Throughout this paper, we assume that all manifolds and mappings are of class C^∞.

Let N be a closed C^∞ manifold of dimension n and P a C^∞ manifold of dimension p. Recall that f is *C^∞-stable* (simply called *stable*) if there is a neighborhood $\mathcal{U}$ of f in the W^∞ topology on $C^\infty(N,P)$ such that $g \in \mathcal{U}$ implies that there is $h \in \mathrm{Diff}(N)$ and $h' \in \mathrm{Diff}(P)$ such that $g = h' \circ f \circ h$ (i.e., g is $\mathcal{A}$-equivalent to f). In other words, the $\mathcal{A}$-orbit of f is open in $C^\infty(N,P)$. We shall say that two C^∞ stable maps f and g from N to P are C^∞ *stably isotopic* (or simply, *isotopic*) if there exist a C^∞ mapping $F : N \times [0,1] \to P$ such that

(1) for each $0 \leq t \leq 1$, the map $F_t : N \to P$ sending x to $F(x,t)$ is C^∞ stable ;

(2) $F_0 = f$ and $F_1 = g$.

It can be shown that the isotopic relation is an equivalence relation among all C^∞ stable mappings in $C^\infty(N,P)$, and also that any two isotopic C^∞ stable maps are $\mathcal{A}$-equivalent to each other (see §2). We shall often write by $[f]$ the isotopy equivalent class of a C^∞ stable mapping f.

We assume that N and P are connected. Let $\mathcal{M}$ denote the mapping space $C^\infty(N,P)$ and Γ the subset of $\mathcal{M}$ consisting of all C^∞ maps which are not C^∞ stable. The complement $\mathcal{M}-\Gamma$ consists of all C^∞ stable mappings. When $p \leq 2n+1$ and the codimension $\sigma(n,p)$ of moduli spaces of $\mathcal{A}$-orbits is greater than $n+1$ (cf. [9]) , it turns out that Γ can be regarded to have "codimension one in $\mathcal{M}$". In particular, the regular part Γ_{Reg} of Γ consists of C^∞ mappings which have only a (multi-)singularity with codimension one except for C^∞ stable singularities (namely, there is a finite set S of N such that the germ at S, $f : N,S \to P, f(S)$ has $\mathcal{A}_e$-codimension one, and also that $f|_{N-S}$ is proper and C^∞ stable).

We are interested in numerical invariants of C^∞ stable mappings. Let R be a comutative ring with unit. A locally constant function $V : \mathcal{M}-\Gamma \to R$ is said a R valued *isotopy invariant of C^∞ stable mappings* : for any $f,g \in \mathcal{M}-\Gamma$ stably isotopic each other, $V(f) = V(g)$. It may be worthy to note that the 0-th cohomology group $H^0(\mathcal{M}-\Gamma; R)$ can be regarded as the module consisting of all G valued isotopy invariants. Let a C^∞ stable map $f_0 \in \mathcal{M}-\Gamma$ be fixed such as it defines an argumentation $\epsilon : S_0(\mathcal{M}-\Gamma) \to R$ of the singular chain complex $S_*(\mathcal{M}-\Gamma; R)$, and then each element of the reduced 0-th cohomology group $\bar{H}^0(\mathcal{M}-\Gamma; R)$ corresponds to an isotopy invariant which vanishes on the isotopy class of f_0.

Definition 1.1. Assume that R has no elements of order 2. An isotopy invariant $V : \mathcal{M}-\Gamma \to R$ is called *Vassiliev type of order one* if V can be extended to a function $\mathcal{M} \to R$ satisfying the following condition : there is a locally finite partition $\mathcal{G}$ of Γ_{Reg} consisting of some cooriented strata $\{\Xi_i\}$ and non-coorientable strata such that

(i) V is constant on each stratum of $\mathcal{G}$, and especially, constantly zero over non-coorientable strata ;

(ii) V is constantly zero over $\Gamma - \Gamma_{Reg}$;

(iii) (the difference equation) for each cooriented stratum Ξ_i and for any family of C^∞ maps $\phi = \phi_t : (-a,a) \to \mathcal{M}$, $\phi_0 \in \Xi_i$, which is transversal to Ξ_i with the positive direction compatible to the coorientation, it holds that

$$V(\Xi_i) = V([\phi_{+\epsilon}]) - V([\phi_{-\epsilon}]), \qquad (\epsilon > 0)$$

(iv) (normalization condition) V is constantly zero on the isotopy class of the distinguished map f_0.

In particular, according to Goryunov's terminology [7], we state one more definition :

Definition 1.2. A Vassiliev type invariant V of order one is called *local* if each stratum of the partition of Γ_{Reg} corresponds to a singularity type (i.e., $\mathcal{A}$-equivalent class of germs) with codimension one, and the coorientation of a stratum is determined by the coorientation of the corresponding singularity type (that is the coorientation of the parameter space of its versal deformation, see §2).

In §3 we will introduce *Vassiliev cycle of order one for $\mathcal{A}$-classes of multi-germs* , and we will see in Proposition 4.2 in §4 that for the case of $P = \mathbb{R}^p$ there is one-to-one correspondence between order one local invariants and Vassiliev cycles.

Remark 1.3. (1) We can also define $\mathbb{Z}_2$ valued invariants of order one, by ignoring the coorientability of strata in the above definition. (2) Given any Vassiliev type invariants V and V' of order one, by taking a refinement of both of accosiated partitions of Γ_{Reg}, any linear combination $aV + bV'$ ($a, b \in R$) also becomes an invariant of order one. Thus all Vassiliev type invariants of order one form a submodule of $\bar{H}^0(\mathcal{M} - \Gamma; R)$. (3) As in [2], [3], [14], there may be several way to coorient strata by using the data of configurations of singular point sets of maps in N.

Remark 1.4. In the above, as the mapping space $\mathcal{M}$, we consider the space of all C^∞ mappings, but it is also possible to consider the space of C^∞ mappings with several constraint as $\mathcal{M}$ (for example, the space of immersed plane curves with a fixed winding number [2], the space of plane fronts with a fixed Maslov index [3], the space of algebraic projective plane curves [15], etc).

Now let us consider a special case where N is a connected closed surface and P is the 2-plane $\mathbb{R}^2$. Elements of $\mathcal{M} - \Gamma$, i.e., C^∞ stable maps f, can be characterized as follows : f admits singularities only of type (1) fold, (2) cusp (3) double fold (bi-germ of fold types whose contours are transverse to each other). Besides, generic 1-parameter local bifurcations of multi-singularities, $N \times \mathbb{R}, S \times \{0\} \to \mathbb{R}^2, 0$, S being a finite set, can be also classified. The classification (for uni-germs, the case where S is a single point) is due to Arnold [1], Rieger [10], and Rieger-Ruas [11]. These bifurcations of apparent contours and images are dipicted in Figure 1 below, and normal forms are given in Table 1 on the end of §5.

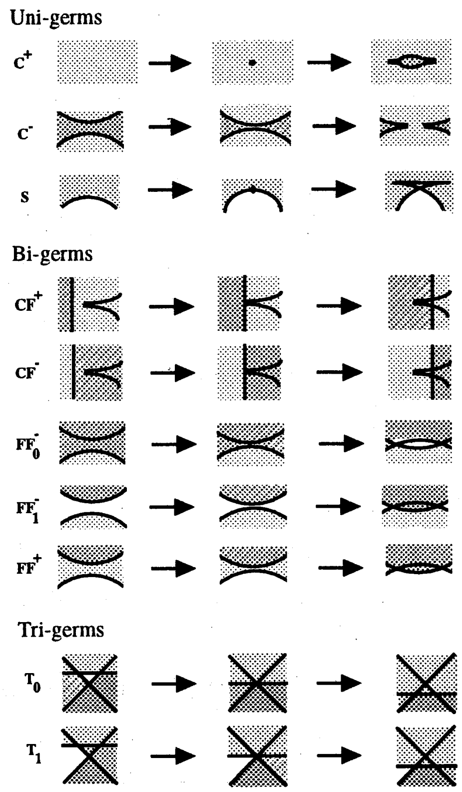

Figure 1
Generic 1-parameter bifurcations of map-germs of the plane to the plane
The brack lines are the apparent contours and the darked areas are the images.

The main result is the following theorem :

Theorem 1.5. *The submodule of $\bar{H}^0(\mathcal{M}-\Gamma,\mathbb{Z})$ consisting of local Vassiliev type invariants of order one are generated by the following three invariants :*

$$
\begin{aligned}
I_C &:= C+S,\\
I_D &:= S+2CF+2FF^{+}+2FF^{-},\\
I_F &:= 2FF^{-}+CF.
\end{aligned}
$$

Theorem 1.6. *The submodule of $\bar{H}^0(\mathcal{M}-\Gamma;\mathbb{Z}_2)$ consisting of local Vassiliev type invariants of order one are generated by the following three invariants :*

$$I_{C;2}:=C, I_{D;2}:=S, I_{F;2}:=CF.$$

Remark 1.7. (1) The choise of f_0 is of course not unique, and there is no standard way to choose it. (2) The value of the invariant I_C is equal to a half of the difference between the (geometric) number of cusps of f and one of the distinguished map f_0. Also the value of the invariant I_S is equal to the difference between the (geometric) number of transverse double folds points of f and one of f_0.

§2 Preliminary : Multi-germs and $\mathcal{A}$-equivalence

In this section, we quickly review the most fundamental notions in Singularity Theory, which will be used later. For the detail, see, e.g., [16], [6], [8], [9], [4], [5].

Multi-germs, deformations and $\mathcal{A}$-equivalences.

Two maps f and g between N and P is said to define the same *germ* at a compact subset S of N if there is a neighborhood of S on which f coincides to g. Usually we are concerned with the case when S consists of finitely many points and $f(S)$ is one point, and we shall simply write the germ of f at S like as $f : N, S \to P, y$. In particular, we often say it *a multi-germ* if S is not one point. A deformation of a multi-germ $f : N, S \to P, y$ with a parameter space $\mathbb{R}^s$ centered at 0 means a germ $F : N \times \mathbb{R}^s, S \times \{0\} \to P, y$ satisfying that $F(x, 0) = f(x)$. We often write $F_p(x)$ to be $F(x,p)$. Let $\pi : N \times \mathbb{R}^s \to \mathbb{R}^s$ denote the projection onto the parameter space.

Map-germs $f : N, S \to P, y$ and $g : N', S' \to P', y'$ are called *$\mathcal{A}$-equivalent* if there exist germs of diffeomorphisms $\sigma : N, S \to N', S'$ and $\varphi : P, y \to P', y'$ such that $g \circ \sigma = \varphi \circ f$. Deformations F of f and G of g with the same dimension of parameters are called *$\mathcal{A}$-equivalent* if F and G are $\mathcal{A}$-equivalent as map-germs by the diffeomorphism-germs letting the following diagram commute :

$$\begin{array}{ccccc}
(N \times \mathbb{R}^s, S \times \{0\}) & \xrightarrow{(F,\pi)} & (P \times \mathbb{R}^s, (y, 0)) & \xrightarrow{\pi} & (\mathbb{R}^s, 0) \\
\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle \phi} \\
(N' \times \mathbb{R}^s, S' \times \{0\}) & \xrightarrow{(G,\pi)} & (P' \times \mathbb{R}^s, (y', 0)) & \xrightarrow{\pi} & (\mathbb{R}^s, 0).
\end{array}$$

Two deformations F and G of f with the same dimension of parameter spaces are called *f-isomorphic* if F and G are $\mathcal{A}$-equivalent by a triplet (R, L, ϕ), where R and L are deformations of identity maps id_N and id_P, respectively.

Let $F : N \times \mathbb{R}^s, S \times \{0\} \to P, y$ be a deformation of f and $g : \mathbb{R}^t, 0 \to \mathbb{R}^s, 0$ a map-germ, then we define the indeced deformation $g^*F : N \times \mathbb{R}^t, S \times \{0\} \to P, y$, by $g^*F(x, w) = F(x, g(w))$. A deformations F of f is called *versal* if any deformation G of f is isomorphic to a deformation induced from F. An versal deformation of a germ f is called *miniversal* if the parameter space has the minimal dimension in all versal deformations of f.

For a germ $f : N, S \to P, y$, let $\theta(f)_S$ denote the set of C^∞ vector fields along f, i.e., germs of C^∞ maps $\zeta : N, S \to TP$ such that $\zeta(x) \in TP_{f(x)}(x \in N)$. We set $\theta(N)_S = \theta(1_N)_S$, $\theta(P)_y = \theta(1_P)_{\{y\}}$ and let $tf : \theta(N)_S \to \theta(f)_S$ and $\omega f : \theta(P)_y \to \theta(f)_S$ be defined as $tf(\xi) = Tf \circ \xi$ and $\omega f(\eta) = \eta \circ f$. *The extended tangent space* $T\mathcal{A}_e f$ is given by

$$T\mathcal{A}_e f := tf[\theta(N)_S] + \omega f[\theta(P)_y] \subset \theta(f)_S,$$

and the diminsion of the quotient space $\theta(f)_S/T\mathcal{A}_e f$ is called *$\mathcal{A}_e$-codimension* of f.

When $\mathcal{A}_e$-codimension of f is finite, letting $\{g_i\}$ be a $\mathbb{R}$-basis of $\theta(f)_S/T\mathcal{A}_e f$ and set $F := f + \sum_i u_i g_i$ by using a local coordinate systems of P. Then the deformation F becomes a versal deformation of f. Besides, it also holds that for any versal deformation F of f, the set of the derivatives $\partial_i F(:= \frac{\partial F}{\partial u_i}(x, 0))$ with respect to the parameter coordinates form a basis of $\theta(f)_S/T\mathcal{A}_e f$. A germ $f : N, S \to P, y$ is called *$\mathcal{A}_e$-finite* if $\dim \theta(f)_S/T\mathcal{A}_e f < \infty$. It should be noted that every $\mathcal{A}_e$-finite multi-germ is *finitely determined,* that is its $\mathcal{A}$ eqauivalent class is determined by its jet of finite order, and hence it is represented as polynomial map-germs whose images are in general position.

Coorientability.

Definition 2.1. An $\mathcal{A}_e$-finite germ $f : N, S \to P, y$ is said to be *non-coorientable* if for any miniversal deformation F of f there is a triplet (R, L, ϕ) which makes an f-isomorphism from F to itself where ϕ is a germ of an orientation-reversing diffeomorphism of the parameter space.

Note that the (non) coorientability of $\mathcal{A}$-finite germs are preserved under $\mathcal{A}$-equivalence, thus we can say that an $\mathcal{A}$-class is coorientable or non-coorentable.

Multi-jets, Transversality and Stability.

Let $N^{(r)}$ be the set of ordered r-tuples of distinct elements of N, denoted by $\mathbf{x} =< x_1, \cdots, x_r >$ with $x_i \neq x_j$ for $i \neq j$. Let $\pi_N : J^l(N, P) \to N$ denote the projection, where $J^l(N, P)$ is the bundle of l-jets. Define ${}_rJ^l(N, P) = (\pi_N^r)^{-1}[N^{(r)}]$, where $\pi_N^r : J^l(N, P)^r \to N^r$ is the r fold Cartesian product of π_N with itself. A C^∞ mapping $f : N \to P$ defines a C^∞ section ${}_rj^l f : N^{(r)} \to {}_rJ^l(N, P)$ sending $< x_1, \cdots, x_r >$ to $< j^l f(x_1), \cdots, j^l f(x_r) >$, which is called *the multi-l-jet extension of* f . Here are various characterizations of C^∞ stability of mappings :

Theorem 2.2. [Mather; V] *Let $r \geq p+1$ and $l \geq p$, where p is the dimension of P. Let f be a proper C^∞ mapping from N to P. Then the following conditions are equivalent :*

(1) *f is C^∞ stable;*
(2) *f is infinitesimally stable, i.e., $tf[\theta(N)] + \omega f[\theta(P)] = \theta(f)$;*
(3) *${}_rj^l f$ is transversal to every $\mathcal{A}$-orbit in ${}_rJ^l(N,P)$;*
(4) *For any point $y \in P$ and any multi-germ f_S of f at any finite subset S of $f^{-1}(y)$ consisting of r or less than r points, we have*

$$\theta(f)_S = T\mathcal{A}_e f_S + \mathrm{m}_S^{l+1}\theta(f)_S.$$

Let $F : N \times W \to P$ a C^∞ mappings, which is considered as a family of C^∞ maps from N to P with a manifold W of parameters. Such a family F defines a family of C^∞ sections

$$ {}_rJ_1^l F : N^{(r)} \times W \to {}_rJ^l(N,P), \quad {}_rJ_1^l F(\mathbf{x},p) := {}_rJ^l F_p(\mathbf{x}).$$

Theorem 2.3. cf. [Mather, V] *Let $F : N \times W \to P$ a smooth family with a parameter manifold W of dimension s. Then the following conditions are equivalent :*

(1) *${}_rj_1^l F$ is transversal to every $\mathcal{A}$-orbit in ${}_rJ^l(N,P)$;*
(2) *For every $p \in W$ and every finite subset S of N consisting of r or less than r points, such that $F_p(S)$ is a single point, we have*

$$\theta(f)_S = T\mathcal{A}_e F_p + \{\partial_1 F|_{u=p}, \cdots, \partial_s F|_{u=p}\}_{\mathbb{R}}.$$

A parametrized version of Thom's multi-transversality theorem are stated as follows :

Theorem 2.4. cf. [Mather, V] *Let Θ be a $\mathcal{A}$-invariant subset of ${}_rJ^l(N,P)$, and $F : N \times W \to P$ a C^∞ mapping as a family of C^∞ maps from N to P. Then F can be approximated by those families $G : N \times W \to P$ that the parametrized jet extension ${}_rj_1^l G : N^{(r)} \times W \to {}_rJ^l(N,P)$ is transversal to Θ.*

§3 Vassiliev cycles of order one for $\mathcal{A}$-classes

In this section, we describe a formal set-up of the first degree part of the so-called Vassiliev complex for simple $\mathcal{A}$-equivalent classes of multi-germs of C^∞ mappings (

cf. [12], [4]). We assume that the pair of dimensions (n, p) satisfies that there are finitely many $\mathcal{A}$-classes with $\mathcal{A}_e$-codimension less than or equal to 2.

For each coorientable $\mathcal{A}$-equivalent classes of multi-germs with $\mathcal{A}_e$-codimension 1, we take a miniversal deformation of a multi-germ representing the class :

$$F_i : \mathbb{R}^n \times \mathbb{R}, S_i \times \{0\} \to \mathbb{R}^p, 0, \quad (i = 1, \cdots, l).$$

For each $\mathcal{A}$-class of multi-germs with $\mathcal{A}_e$-codimension 2, we also take a miniversal deformation

$$G_j : \mathbb{R}^n \times \mathbb{R}^2, S_j' \times \{0\} \to \mathbb{R}^p, 0, \quad (j = 1, \cdots, l').$$

We can assume that every F_i (resp. G_j) is presented at each point of S_i (resp. S_j') as a polynomial map-germ. We fix the orientation of the parameter space $\mathbb{R}$ of each germ F_i, by which the corresponding class are cooriented. We also fix the orientation of the parameter space $\mathbb{R}^2$ of each germ $G_{j(k)}$, although the corresponding class is not nacessarily coorientable. We simply write $(F_i)_t(x) = F_i(x,t)$ and $(G_j)_p(x) = G_j(x,p)$.

Then we set as a formal way

$$C^1(\mathcal{A}_{n,p}^{ori}) := \text{ the } R\text{-module generated by } \{F_1, \cdots, F_l\},$$
$$C^2(\mathcal{A}_{n,p}) := \text{ the } R\text{-module generated by } \{G_1, \cdots, G_{l'}\} \ ,$$

We should remark that for each F_i the $\mathcal{A}$-class of the induced deformation $\iota^* F_i$, where $\iota : \mathbb{R}, 0 \to \mathbb{R}, 0$ is a germ of an orientation-reversing diffeomorphism, is identified with $-F_i$ as an element in $C^1(\mathcal{A}_{n,p}^{ori})$. We don't require such identification for G_j.

Next we shall define an operator $\delta : C^1(\mathcal{A}_{n,p}^{ori}) \to C^2(\mathcal{A}_{n,p})$. To do this, for every pairs of F_i and G_j we define an integer $[F_i : G_j]$ as follows. Simply, we write F and G instead of F_i and G_j. Let $\tilde{G} : U \times W \to \mathbb{R}^p$ be a representative of the germ G, U an open neighborhood of the source points $S \subset \mathbb{R}^n$ and W an open neighborhood of the origin in $\mathbb{R}^2$. We let $W_F(\tilde{G})$ denote the set consisting of $p \in W$ satisfying that there is a point $y \in \mathbb{R}^p$ near 0 and a subset $S_p \subset \tilde{G}_p^{-1}(y)$ such that the multi-germ $\tilde{G}_p : U, S_p \to \mathbb{R}^p, y$ is equivalent to F. If $W_F(\tilde{G})$ is empty, define $[F : G]$ to be zero. Otherwise, by the multi-transversality theorem, taking U and W sufficiently small if nessacary, the closure of $W_F(\tilde{G})$ is one dimensional semialgebraic set in W whose closure contains the origin (since the closure of a $\mathcal{A}$-finite orbit in a multi-jet space becomes a semi-algebraic set). In particular, it turns out that there is $\epsilon > 0$ such that for any $0 < \epsilon' \leq \epsilon$, the circle $S^1_{\epsilon'}$ centered at the origin with radius ϵ' is transverse to $W_F(\tilde{G})$. According to the fixed orientation of the parameter space of G, we let the circles be anti-clockwise oriented. Since the class equivalent to F_i is oriented, the stratum $W_F(\tilde{G})$ has cooriented. Thus an intersection index of S^1_ϵ and $W_F(\tilde{G})$ is well-defined, and we denote it by $[F : G]$. Obviously the integer is independent of the choice of the representative $\tilde{G}$, and if we take another orientation of the parameter space of G, the index has opposite sign.

Now we can define a R-homomorphism

$$\delta : C^1(\mathcal{A}^{ori}_{n,p}; R) \to C^2(\mathcal{A}_{n,p}; R), \quad \text{by} \quad \delta F_i := \sum_{j=1}^{l'} [F_i : G_j] G_j.$$

Definition 3.1. Let c be a non-trivial element of $C^1(\mathcal{A}^{ori}_{n,p})$ such that $\delta c = 0$, then we call c a *Vassiliev cycle of order one for $\mathcal{A}$-equivalent classes of multi-germs with the pair of dimensions (n,p)* .

In the next section, for such a Vassiliev cycle we will define an invariants of isotopy classes of generic maps.

§4 Invariants of isotopy classes of C^∞ stable mappings to Euclidean space

In this section we treat with the case that $P = \mathbb{R}^p$. As in the previous section we here assume the pair (n,p) to satisfy that there are finitely many $\mathcal{A}$-classes with $\mathcal{A}_e$-codiension less than or equal to 2. As in §1, we let $\mathcal{M}$ denote the mapping space $C^\infty(N, \mathbb{R}^p)$, Γ the subset of all non-generic (C^∞ unstable) mappings, and f_0 a fixed generic mapping in $\mathcal{M} - \Gamma$.

First, since the target space is a linear space $\mathbb{R}^p$, it is easily seen that the mapping space $\mathcal{M}(= C^\infty(N, \mathbb{R}^p))$ is contractible. In particular, any generic mapping f can be joined to f_0 by a smooth homotopy $\tau : N \times I \to \mathbb{R}^p$ with $\tau(x,0) = f_0(x)$, $\tau(x,1) = f(x)$, for instance, which can be acheived by $f_0 + t(f - f_0)$. For $t \in I$ we simply set $\tau_t : N \to \mathbb{R}^p$ to be the map sending x to $\tau(x,t)$. It is convenient to regard a smooth homotopy as a continuous path in the mapping space $\mathcal{M}$ with Whitney C^∞ topology, and when we distinguish them, we will often write $\bar{\tau} : I \to \mathcal{M}$ (i.e.,$\bar{\tau}(t) := \tau_t$).

By using the parametrized transversality theorem, we can assume τ to satisfy that there is a finite subset A of I such that

(1) at each point t outside A the map τ_t is a C^∞ stable mpping ;
(2) at each point t of A there is a point y of $\mathbb{R}^p$ and $S \subset {\tau_t}^{-1}(y)$ so that the germ $\tau_i : N, S \to \mathbb{R}^p, y$ is $\mathcal{A}$-equivalent to an oriented class in $C^1(\mathcal{A}^{ori}_{n,p})$.

For a smooth homotopy τ satisfying the property, we say roughly that the path $\bar{\tau}$ is *transverse to the discriminant* Γ . For such a path τ, we define an integer $\epsilon_i(\tau)$ to be the number (taking accounts of sign) of events of local bifurcations of type F_i moving along the path $\bar{\tau}$. Namely, if the germ τ at $S \times \{t\}$ is equivalent to the normal form of the class F_i compatibly on the orientation of parameter lines, we count $+1$, and otherwise -1. Summing up the signs at all events, the amount is just $\epsilon_i(\tau)$. It is reasonable to regard $\epsilon_i(\tau)$ as the intersection index of the strata of type F_i in Γ and the path $\bar{\tau}$.

Let $c \in \ker \delta$, a Vassiliev cycle of order one, and assume that c is written as a linear form $\sum_{i=1}^{s} \lambda_i F_i$ where F_i are generators of $C^1(\mathcal{A}^{ori}_{n,p})$ and $\lambda_i \in R$. For c, f and τ, we define an integer $I_c(f;\tau)$ by

$$I_c(f;\tau) := \sum_{i=1}^{s} \lambda_i \epsilon_i(\tau).$$

Lemma 4.1. *The value $I_c(f;\tau)$ is independent of the choice of τ.*

Proof of Lemma. Take another path $\bar{\tau}' : I \to \mathcal{M}$ from f_0 to f transverse to the discriminant Γ. Then we have a continuous homotopy $\eta : N \times I \to \mathbb{R}^p$ which is defined by $\eta(x,t) = \tau(x,2t)$ for $0 \le t \le 1/2$ and $\eta(x,t) = \tau'(x, 2-2t)$ for $1/2 \le t \le 1$. The homotopy η is smooth off $t = 0$ and 1, and we can slightly modify η to be a C^∞ mapping over $N \times I$ using the partition of unity if nessesary. Since η defines a continuous loop in $\mathcal{M}$ and $\mathcal{M}$ is contractible, there is a C^∞ mapping $\Xi : N \times D^2 \to \mathbb{R}^p$, where D^2 is the unit closed disc in $\mathbb{C}$ centered at the origin satifying that $\Xi(x, e^{2\pi t}) = \eta(x,t)$.

By the transversality theorem, it can be assumed that the parametrized jet extension of Ξ is transversal to all $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension less than or equal to two. Hence there is a Whitney stratification $\mathcal{W}$ of D^2 satisfying the following properties :

(1) for any point p in the top strata of dimension 2, $\Xi_p : N \to \mathbb{R}^p$ is C^∞ stable;

(2) each 1-dimensional stratum consists of such points $p \in D^2$ which satisfy that there is a point y of $\mathbb{R}^p$ such that the germ $\Xi_p : N, S \to \mathbb{R}^p, \{y\}$, $\quad S \subset {\Xi_p}^{-1}(y)$ is $\mathcal{A}$-equivalent to a class $(F_i)_0$ in $C^1(\mathcal{A}^{ori}_{n,p})$, and such a stratum is denoted by W_{F_i} ;

(3) for each point of the 0-dimensional strata $\{p_1, \cdots, p_s\}$, $\Xi_{p_k} : N \times p_k \to \mathbb{R}^p$ has a (multi-) singularity equivalent to a class in $C^2(\mathcal{A}_{n,p})$, denoted by $G_{j(k)}$ ($k = 1, \cdots, s$).

We take small disjoint k discs $B(p_k)$ ($k = 1, \cdots, s$) centered at p_k in the interior $\mathrm{int}D^2$ transverse to the stratification $\mathcal{W}$. Let ∂D^2 and every $\partial B(p_k)$ be anti-clockwise oriented. It can be easily verified that the intersection index of ∂D^2 and W_{F_i} is equal to the sum of the intersection indices $\partial B(p_k)$ and W_{F_i} over all $k = 1, \cdots, s$. Hence, by definitions, we have that $\epsilon_i(\eta) = \sum_k \pm[F_i : G_{j(k)}]$ where the sign $\pm$ depends on the fixed orientation of the parameters of $G_{j(k)}$. Thus,

$$\begin{aligned} I_c(f;\tau) - I_c(f;\tau') &= \sum_i \lambda_i(\epsilon_i(\tau) - \epsilon_i(\tau')) \\ &= \sum_i \lambda_i \epsilon_i(\eta) \\ &= \sum_i \lambda_i \sum_k \pm[F_i : G_{j(k)}] \\ &= \sum_k \pm \sum_i \lambda_i [F_i : G_{j(k)}] \\ &= \sum_k \pm \text{ (the coefficient of } \delta c \text{ with respect to } G_{j(k)}) = 0. \end{aligned}$$

This completes the proof.

In the same way of the above proof, we can see that the integer $I_c(f;\tau)$ depends only on the isotopy classes of f and f_0. So we shall write it by $I_c(f;f_0)$ or simply $I_c(f)$. This defines a homomorphism $I : \ker\delta \to \bar{H}^0(\mathcal{M} - \Gamma; R)$. In particular, we can show that the following proposition :

Proposition 4.2. *For each cycle $c \in \ker\delta$, I_c is an isotopy invariant of local Vassiliev type of order one described as in §1. Furthermore, when $\dim N$ is greater than 1, every order one local inariant can be expressed as I_c for some $c \in \ker\delta$.*

The second assertion comes from the fact that the subset of $\mathcal{M}$ consisting of C^∞ maps of N to $\mathbb{R}^p$ which have singularity of type F_i (the closure of the strata of Γ_{Reg} corresponding to the class F_i) is connented.

§5 $\mathcal{A}$-classes for mappings from the plane to the plane and Theorems

From now on we treat with C^∞ mappings from a closed surface N to 2-plane $\mathbb{R}^2$. The lists at the end of this section show all $\mathcal{A}$-equivalent classes of multi-germs from the plane to the plane with $\mathcal{A}_e$-codimension less than or equal to 2. The clssification of uni-germs is due to Rieger [10] and Rieger-Ruas [11], and we use their notation for uni-germs. For 1-parameter deformations, we consider $\mathcal{A}$-equivalent classes of oriented deformations. In the list, every multi-germ $N, S \to P, y$, $S = \{p_k\}_k$, is described as the set consisting of k germs $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ taking local coordinate systems of N centered at p_k and a local coordinate system of P centered at y.

Coorientation.

As to the coorientation, we define the orientation of the parameter line as the direction such that the number of cusp points and double fold points increase for uni-germs and bi-germs, and the number of sheets covering the "vanishing triangle" increases for triple fold points, T_0 and T_1. Figure 1 in §1 depicts local bifurcations of apparent contours and shadows (the image) of the map in these direction.

Vassiliev complex.

The module $C^1(\mathcal{A}^{ori}_{2,2};\mathbb{Z})$ (and $C^1(\mathcal{A}^{ori}_{2,2};\mathbb{Z}_2)$) is generated by ten elements

$$C_\pm, S, CF^\pm, FF^+, FF_0^-, FF_1^-, T_\pm,$$

and $C^2(\mathcal{A}_{2,2};\mathbb{Z})$ (and $C^2(\mathcal{A}_{2,2};\mathbb{Z}_2)$) is generated by

$$[6^\pm], [4_3^\pm], [11_5], I^{1,1}_{2,2}, II^{1,1}_{2,2}, \tilde{C}_\pm, \tilde{S}, Q_\pm, \tilde{FF}^+, \tilde{FF_0}^-, \tilde{FF_1}^-, CC, FC, \tilde{CF}_{\pm,\pm}, \tilde{T}_\pm.$$

Proposition 5.1. *The coboundary operation $\delta : C^1(\mathcal{A}^{ori}_{2,2};\mathbb{Z}) \to C^2(\mathcal{A}_{2,2};\mathbb{Z})$ is determined as follows (in the case coefficients in $\mathbb{Z}_2$, these equalities valid modulo 2)*

$$\begin{aligned}
&\delta C_+ = [4_3^+] + [4_3^-], \quad \delta C_- = -[4_3^+] - [4_3^-] - 2[11_5], \quad \delta S = 2[11_5],\\
&\delta FF^+ = -[11_5] + FC, \quad \delta FF_0^- = -Q_-, \quad \delta FF_1^- = Q_- + FC,\\
&\delta CF^+ = \tilde{C}_+ + \tilde{C}_- + \tilde{S} - FC, \quad \delta CF^- = -\tilde{C}_+ - \tilde{C}_- - \tilde{S} - FC,\\
&\delta T_+ = -\tilde{FF_0}^- - \tilde{FF_1}^- + \tilde{CF}_2 + \tilde{CF}_3,\\
&\delta T_- = -\tilde{S} + \tilde{FF_0}^- + \tilde{FF_1}^- + \tilde{CF}_2 + \tilde{CF}_3.
\end{aligned}$$

This proposition follows from direct computation. Solving the equation $\delta c = 0$, we have Theorem 1.5 which is introduced in §1. In the case of coefficients in $\mathbb{Z}_2$, considering the equalities in (2) of the above Proposition modulo 2, we get Theorem 1.6.

Table of the Classification

Stable-germs

Type	normal form $f(x,y)$
regular	(x, y)
fold	(x, y^2)
cusp	$(x, y^3 + xy)$
doublefold	$(x, y^2), \quad (x'^2, y')$

1-parameter deformations

Type	versal deformation $F(x,y,a)$
$C^{\pm}(4_2)$	$(x, y^3 \pm y(x^2 - a))$
$S, (5)$	$(x, y^4 + xy - ay^2)$
$CF^{\pm}$	$(x, y^3 + xy), \quad (\pm y'^2 - a, x')$
FF^{+}	$(x, y^2 + a), \quad (x', x'^2 + y'^2)$
FF_0^-, FF_1^-	$(x, \mp y^2 + a), \quad (x', x'^2 \pm y'^2)$
T_0, T_1	$(x + y^2, x - y^2 + a), (x', y'^2), (\mp x''^2, y'')$

2-parameter deformations

Type	versal deformation $G(x,y,a,b)$
$4_3^{\pm}$	$(x, y^3 \pm x^3 y + ax^2 y + bxy)$
$6^{\pm}$	$(x, xy + y^5 \pm y^7 + ay^3 + by^2)$
11_5	$(x, xy^3 + y^4 + y^5 + axy + by)$
$I_{2,2}^{1,1}$	$(x^2 + y^3 + ay, y^2 + x^3 + bx)$
$II_{2,2}^{1,1}$	$(x^2 - y^2 + x^3 + ay, xy + bx)$
$\tilde{C}^{\pm}$	$C^{\pm}$ and $(x'^2 + b, y')$
$\tilde{S}$	S and $(x'^2 + b, y')$
$Q_{\pm}$	$(x, x^3 - ax + y^2), \quad (x', \pm y'^2 + b)$
$\tilde{FF}^{+}$	$(x, x^2 + y^2), (x', y'^2 + a), (x''^2 + b, y'')$
$\tilde{FF}_0^-, \tilde{FF}_1^-$	$(x, x^2 \pm y^2), (x', \mp y'^2 + a), (x''^2 + b, y'')$
CC	$(x + a, y^3 + xy), (y'^3 + x'y', x' + b)$
FC	$(x + a, y^3 + xy), \quad (x', y^2 + ax + b)$
$\tilde{CF}_{\epsilon_1,\epsilon_2}$	$(x, y^3 + xy), \quad (x' + \epsilon_1 y'^2, x' - \epsilon_1 y'^2 + a),$
$(\epsilon_1, \epsilon_2 = \pm 1)$	$(x'' + \epsilon_2 y''^2, -x'' + \epsilon_2 y''^2 + b)$
$\tilde{T}_0, \tilde{T}_1$	$T_{0,1}$ and $(x''' + y'''^2, -x''' + y'''^2 + b)$

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