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AFFINE DIFFERENTIAL GEOMETRY OF SPACE CURVES AND SINGULARITY THEORY

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ABSTRACT. We study affine invariants of space curves from the view point of singularity theory of smooth functions. By the aid of singularity theory we define a new equi-affine frame for space curves. We also introduce two surfaces associated with this equi-affine frame and give a generic classification of the singularities of those surfaces.

1. INTRODUCTION

There are several articles which study about "generic differential geometry" in the Euclidian space ([2,3,4,5,6,7,11,12 etc.]). The main tools in these articles are the distance-squared functions and the height functions on submanifolds. The classical invariants of extrinsic differential geometry can be treated as "singularities" of these functions (i.e.,this is a philosophy of R. Thom), however, as Fidal[7] pointed out, the geometric interpretation of sextactic points of a convex plane curve is quite complicated from this point of view. It has been classically known that a sextactic point is an equi-affine invariant. Affine differential geometry is a classical area in differential geometry [1,14,15], however, it has been revived recently [13]. In [10] we introduced the notion of affine distance-cubed functions and affine height functions on convex plane curves. These functions are quite useful for the study of generic properties of invariants of extrinsic affine differential geometry on convex plane curves. As a consequence, we applied ordinary techniques of singularity theory for these functions and described equi-affine invariants for convex plane curves. Moreover, we can say that the theory of generic affine differential geometry of convex plane curves is completely analogous to that of generic euclidian differential geometry. Giblin and Sapiro[8,9] also studied affine invariant symmetry sets for convex plane curves. They have given two different definitions of affine invariant symmetry sets. One of these is defined by using the affine distance-cubed function.

In this paper we proceed the similar arguments on space curves, however the situation is quite different. There are still some vagueness in the classical affine differential geometry of space curves. For example, the classical notion of affine torsion is not a torsion in the geometric sense (cf., §2). In this paper, by the aid of the method of the singularity theory, we define the new notions of the affine torsion and the affine binormal (i.e., *the intrinsic affine torsion* and *the intrisic affine binormal*). We also introduce equi-affine invariant surfaces (i.e., *the intrinsic affine binormal developable* and *the affine rectifying Gaussian surface*) associated

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with space curves which are singular surfaces in §4. The main result is Theorem 4.1 which gives local classifications of these singular surfaces. The proof of Theorem 4.1 is given in §5 and §6. The basic techniques we used in this paper depend heavily on those in the attractive book of Bruce and Giblin [5], so that the authors are grateful to both of them.

All curves and maps considered here are of class C^∞ unless stated otherwise.

2. Basic notions

We now present basic concepts on affine differential geometry of space curves. The classical theory has been established in [1,14,15], however, there are still some vagueness compared with the theory for plane curves. Let $\mathbb{R}^3$ be the affine space which adopt the coordinate such that the volume of the parallelepiped spanned by three vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ is given by the determinant

$$|a\ b\ c| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

We fix the above coordinate in this paper. Let $\gamma : I \to \mathbb{R}^3$ be a curve with $|\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)| > 0$, where $\dot\gamma(t) = \frac{d\gamma}{dt}(t)$. *The affine arc-length* of a curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{\frac{1}{6}}\, dt.$$

Then a parameter s is determined such that $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$, where $\gamma'(s) = \frac{d\gamma}{ds}(s)$. So we say that a curve γ is *parameterized by the affine arc-length* if it satisfies that

$$|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1.$$

We say that a quantity or property of the curve γ is *an equi-affine invariant* if it is invariant under the action of the special linear group and the group of parallel transportations. Affine differential geometry is a geometry which studies the equi-affine invariants in the sense of Klein's program.

Let us denote $T(s) = \gamma'(s), N(s) = \gamma''(s)$ and $B(s) = \gamma'''(s)$ and call $T(s)$ *an affine tangent, $N(s)$ an affine principal normal and $B(s)$ an affine binormal.* We also define *the affine curvature* by $\kappa_a(s) = |\gamma'(s)\ \gamma'''(s)\ \gamma^{(4)}(s)| = |T(s)\ B(s)\ B'(s)|$ and *the affine torsion* by $\tau_a(s) = -|\gamma''(s)\ \gamma'''(s)\ \gamma^{(4)}(s)| = -|N(s)\ B(s)\ B'(s)|$. By the similar arguments as those of in euclidian differential geometry, we have the following *Frenet-Serret type formula*:

$$\begin{cases} T'(s) = N(s) \\ N'(s) = B(s) \\ B'(s) = -\kappa_a(s)N(s) - \tau_a(s)T(s). \end{cases} \tag{2.1}$$

It is known that this frame is not a good frame that the functions $\kappa_a(s)$ and $\tau_a(s)$ do not give the natural equation [1,14]. We can also say that the function $\tau_a(s)$ is not the torsion in the following sense: Remember the original Frenet-Serret formula in euclidian differential geometry:

$$
\left\{\begin{aligned} \mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) &= -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) &= -\tau(s)\mathbf{n}(s), \end{aligned}\right. \tag{2.2}
$$

where s is the euclidian arc-length parameter, $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}$ is the principal normal vector and $\mathbf{b}$ is the binormal vector. Under this notation, suppose that the Euclidian torsion $\tau(s)$ is identically zero, then the binormal vector $\mathbf{b}$ is constant along s. So the meaning of the torsion function is to estimate the torsion of the binormal direction along $\gamma(s)$. For the affine case, even if the affine torsion $\tau_a(s)$ is identically zero the affine binormal is not constant.

On the other hand, the above affine frame has the property that $|T(s)\ N(s)\ B(s)| = 1$. We call the frame which has this property *an equi-affine frame.* There are infinitely many equi-affine frames. For example, $(T(s), N(s), B(s)+\lambda(s)T(s))$ is an equi-affine frame for each function $\lambda(s)$. We call such a family *a general equi-affine frame* and each member is called *a special equi-affine frame.*

We remark that $(T(s), N(s), B(s) + \mu(s)N(s))$ is also an equi-affine frame for each μ. If we adopt this frame, we have

$$
\begin{aligned}
(B(s) + \mu(s)N(s))' =& B'(s) + \mu'(s)N(s) + \mu(s)N'(s) \\
=& - \kappa_a(s)N(s) - \tau_a(s)T(s) + \mu'(s)N(s) + \mu(s)B(s) \\
=& (-\kappa_(s) + \mu(s) - \mu^2(s))N(s) - \tau_a(s)T(s) + \mu(s)(B(s) + \mu(s)N(s)).
\end{aligned}
$$

This means that the derivative of the new binormal $B(s) + \mu(s)N(s)$ in the Frenet-Serret type formula depends on itself, so that this frame is not good. In the following sections we try to choose a good special equi-affine frame by the aid of the singularity theory.

3. Affine invariant functions

In this section we introduce two different families of functions on a non-degenerate space curve which are useful for the study of the equi-affine invariants of the curve. Let $\gamma : I \to \mathbb{R}^3$ be a space curve with $|\gamma(s)'\ \gamma''(s)\ \gamma'''(s)| = 1$.

3-1) Affine distance functions. We now define a three parameter family of smooth functions on I

$$F : I \times \mathbb{R}^3 \to \mathbb{R}$$

by

$$F(s, u) = |\gamma'(s)\ \gamma''(s)\ \gamma(s) - u|.$$

We call F *an affine distance (6-th powered) function on* γ.

We denote where $f_u(s) = F(s,u)$ for any $u \in \mathbb{R}^2$ and $M_{ij}(s) = |\gamma^{(i)}(s)\ \gamma^{(j)}(s)\ \gamma(s) - u|$. Differentiating $f_u(s)$ with respect to s, we have

$$
\begin{aligned}
&(3.1) && f'_u(s) = M_{13}(s),\\
&(3.2) && f''_u(s) = M_{23}(s) - \kappa_a(s)M_{12}(s),\\
&(3.3) && f'''_u(s) = (\tau_a(s) - \kappa_a(s))M_{12}(s) + (\tau_a(s) - 2\kappa'_a(s))M_{13}(s) - \kappa_a(s)M_{23}(s),\\
&(3.4) && f^{(4)}_u(s) = (\tau''_a(s) - \kappa''_a(s) + \kappa^2_a(s))M_{12}(s)\\
& && \qquad +(\tau_a(s) - 2\kappa'_a(s))M_{13}(s) - \kappa_a(s)M_{23}(s)(s),\\
&(3.5) && f^{(5)}_u(s) = (\tau''_a(s) - \kappa'''_a(s) + 4\kappa_a(s)\kappa'_a(s) - 2\kappa_a(s)\tau_a(s))M_{12}\\
& && \qquad +(2\tau'_a(s) - 3\kappa'_a(s) + \kappa^2_a(s))M_{13}(s) + (\tau_a(s) - 3\kappa'_a(s))M_{23}(s) - \kappa_a(s).
\end{aligned}
$$

It follows from these formulae that we have the following proposition.

Proposition 3.1. *Let $\gamma : I \to \mathbb{R}^3$ be a space curve with $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$. Then*
(1) $f'_u(s_0) = 0$ if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \lambda\gamma'(s_0) - \mu\gamma'''(s_0)$.
(2) $f'_u(s_0) = f''_u(s_0) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$.
(3) $f'_u(s_0) = f''_u(s_0) = f'''_u(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa'_a(s_0) \neq 0$ and $u = \gamma(s_0) + \frac{1}{\tau-a(s_0)-\kappa'_a(s_0))}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0)$.
(4) $f'_u(s_0) = f''_u(s_0) = f'''_u(s_0) = f^{(4)}(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa'_a(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\tau_a(s_0)-\kappa'_a(s_0)}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $\tau'_a(s_0) - \kappa''_a(s_0) = 0$.
(5) $f'_u(s_0) = f''_u(s_0) = f'''_u(s_0) = f^{(4)}(s_0) = f^{(5)}(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa'_a(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\tau_a(s_0)-\kappa'_a(s_0)}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $\tau'_a(s_0) - \kappa''_a(s_0) = \tau''_a(s_0) - \kappa'''_a(s_0) = 0$.

Proof. (1) By the formula (3.1), $f'_u(s_0) = 0$ if and only if $\gamma'(s_0)$, $\gamma'''(s_0)$ and $\gamma(s_0) - u$ are linearly dependent, and $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$ implies that $\gamma'(s_0)$ and $\gamma'''(s_0)$ are linearly independent.
(2) It follows from the formula (1) and (3.2) that $f'_u(s_0) = f''_u(s_0) = 0$ if and only if $\lambda, \mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \lambda\gamma'(s_0) - \mu\gamma'''(s_0)$ and $0 = M_{23}(s_0) - \kappa_a(s_0)M_{12}(s_0) = \lambda - \mu\kappa_a(s_0)$.
(3) By the consequence (2) and (3.3), $f'_u(s_0) = f''_u(s_0) = f'''_u(s_0) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau_a(s_0) - \kappa'_a(s_0))M_{12}(s_0) - \kappa_a(s_0)M_{13}(s_0) + 1 = (\tau_a(s_0) - \kappa'_a(s_0))\mu + 1$. The last condition is equivalent to the condition that $\tau_a(s_0) - \kappa'_a(s_0) \neq 0$ and $\mu = -\frac{1}{\tau_a(s_0)-\kappa'_a(s_0)}$.
(4) By the consequence (3) and (3.4), $f'_u(s_0) = f''_u(s_0) = f'''_u(s_0) = f^{(4)}_u(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa'_a(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\tau_a(s_0)-\kappa'_a(s_0)}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0)$ and

$$
\begin{aligned}
0 &= (\tau'_a(s_0) - \kappa''_a(s_0) + \kappa^2_a(s_0))M_{12}(s_0) + (\tau_a(s_0) - 2\kappa'_a(s_0))M_{13}(s_0) - \kappa'_a(s_0)M_{23}(s_0)\\
&= -\frac{1}{\tau_a(s_0) - \kappa'_a(s_0))}(\tau'_a(s_0) - \kappa''_a(s_0) + \kappa^2_a(s_0)) + \frac{\kappa^2_a(s_0)}{\tau_a(s_0) - \kappa'_a(s_0)}.
\end{aligned}
$$

The last condition is equivalent to the condition that $\tau'_a(s_0) - \kappa''_a(s_0) = 0$.

The assertion (5) follows from the similar arguments as the proof of the assertion (4). This completes the proof. □

3-2) Affine height functions. Let $S^2 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We also define a family of smooth functions on γ parameterized by S^2

$$H : I \times S^2 \to \mathbb{R}$$

by

$$H(s, u) = |\gamma'(s)\ \gamma''(s)\ u|.$$

We call H *an affine height function on* γ.

We denote that $h_u(s) = H(s, u)$ for any $u \in S^2$ and $N_{ij}(s) = |\gamma^{(i)}(s)\ \gamma^{(j)}(s)\ u|$.

Differentiating $h_u(s)$ with respect to s and applying Frenet-Serret type formula (2.1), we have the followings:

$$h'_u(s) = N_{13}(s), \tag{3.6}$$

$$h''_u(s) = N_{23}(s) - \kappa_a(s) N_{12}(s), \tag{3.7}$$

$$h'''_u(s) = (\tau_a(s) - \kappa'_a(s)) N_{12}(s) - \kappa_a(s) N_{13}(s), \tag{3.8}$$

$$\begin{aligned} h^{(4)}(s) =& (\tau'_a(s) - \kappa''_a(s) + \kappa^2(s)) N_{12}(s) \\ &+ (\tau_a(s) - 2\kappa'_a(s)) N_{13}(s) - \kappa_a(s) N_{23}(s), \end{aligned} \tag{3.9}$$

$$\begin{aligned} h^{(5)}(s) =& (\tau''_a(s) - \kappa'''_a(s) + 4\kappa_a(s)\kappa'_a(s) - 2\kappa_a(s)) N_{12}(s) \\ &+ (2\tau'_a(s) - 3\kappa''_a(s) + \kappa_a^2(s)) N_{13}(s) + (\tau_a(s) - 3\kappa'_a(s)) N_{23}(s). \end{aligned} \tag{3.10}$$

It follows from these formulae that we have the following proposition.

Proposition 3.2. *Let* $\gamma : I \to \mathbb{R}^2$ *be a space curve with* $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$. *Then*
(1) $h'_u(s_0) = 0$ *if and only if there exist* $\lambda, \mu \in \mathbb{R}$ *such that* $u = \lambda\gamma'(s_0) + \mu\gamma'''(s_0)$.
(2) $h'_u(s_0) = h''_u(s_0) = 0$ *if and only if there exists* $\mu \in \mathbb{R}$ *such that* $\mu \neq 0$ *and* $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$.
(3) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = 0$ *if and only if there exists*$\mu \in \mathbb{R}$ *such that* $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ *and* $\tau_a(s_0) - \kappa'_a(s_0) = 0$.
(4) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h_u^{(4)} = 0$ *if and only if there exists*$\lambda \in \mathbb{R}$ *such that* $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ *and* $\tau_a(s_0) - \kappa'_a(s_0) = \tau'_a(s_0) - \kappa''_a(s_0) = 0$.
(5) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h_u^{(4)} = h_u^{(5)} = 0$ *if and only if there exists*$\lambda \in \mathbb{R}$ *such that* $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ *and* $\tau_a(s_0) - \kappa'_a(s_0) = \tau'_a(s_0) - \kappa''_a(s_0) = \tau''_a(s_0) - \kappa'''_a(s_0) = 0$.

Proof. (1) By the formula (3.6), $h'_u(s_0) = 0$ if and only if $u\ \gamma'(s_0)$, $\gamma'''(s_0)\ u$ are linearly dependent, and $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$ implies that $\gamma'(s_0)$, $\gamma'''(s_0)$ are linearly independent.
(2) It follows from (1) and the formula (4.7) that $h'_u(s_0) = h''_u(s_0) = 0$ if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that $u = \lambda\gamma''(s_0) + \mu\gamma'''(s_0)$, $\lambda^2 + \mu^2 \neq 0$ and $0 = N_{23}(s_0) - \kappa_a(s_0)N_{12}(s_0) = \lambda - \mu\kappa_a(s_0)$.
(3) By (2) and (3.8), $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = 0$ if and only if there exists$\mu \in \mathbb{R}$ such that $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau_a(s_0) - \kappa'_a(s_0))N_{12}(s_0) - \kappa_a(s_0)N_{13}(s_0) = (\tau_a(s_0) - \kappa'_a(s_0))\mu$. Because $\mu \neq 0$, we have $\tau_a(s_0) - \kappa'_a(s_0) = 0$.

(4) By (3) and (3.9), $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h_u^{(4)} = 0$ if and only if there exists$\lambda \in \mathbb{R}$ such that $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau'_a(s) - \kappa''_a(s) + \kappa^2(s))N_{12}(s) + (\tau_a(s) - 2\kappa'_a(s))N_{13}(s) - \kappa_a(s)N_{23}(s) = (\tau'_a(s_0) - \kappa''_a(s_0))\mu$. By the fact that $\mu \neq 0$, we have $\tau'_a(s_0) - \kappa''_a(s_0) = 0$.

The asserion (5) follows the similar arguments as the proof of the case (4). This completes the proof. □

4. Equi-affine invariants of space curves

By the propositions in the last section, we can recognize that the function $\tau_a(s) - \kappa'_a(s)$ and the direction $\kappa_a(s)\gamma'(s) + \gamma'''(s)$ have special meanings. In fact, we define a new special equi-affine frame as follows:

$$\begin{aligned}
\tilde{T}(s) &= \gamma'(s),\\
\tilde{N}(s) &= \gamma''(s),\\
\tilde{B}(s) &= \kappa_a(s)\gamma'(s) + \gamma'''(s).
\end{aligned}$$

Differentiating each vector with respect to s, we have

$$\tilde{T}'(s) = \gamma''(s) = \tilde{N}(s) \tag{4.1}$$

$$\begin{aligned}
\tilde{N}'(s) &= \gamma'''(s) = \tilde{B}(s) - \kappa_a(s)\gamma'(s)\\
&= -\kappa_a(s)\tilde{T}(s) + \tilde{B}(s)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\tilde{B}'(s) &= \kappa'_a(s)\gamma'(s) + \kappa_a(s)\gamma''(s) + \gamma^{(4)}(s)\\
&= \kappa'_a(s)\gamma'(s) - \tau_a(s)\gamma'(s)\\
&= -(\tau_a(s) - \kappa'_a(s))\gamma'(s) = -(\tau_a(s) - \kappa'_a(s))\tilde{T}(s).
\end{aligned} \tag{4.3}$$

Hence, if we put $\sigma_a(s) = \tau_a(s) - \kappa'_a(s)$, then we have the following new Frenet-Serret type formula:

$$\begin{cases}
\tilde{T}'(s) = \tilde{N}(s)\\
\tilde{N}'(s) = -\kappa_a(s)\tilde{T}(s) + \tilde{B}(s)\\
\tilde{B}'(s) = -\sigma_a(s)\tilde{T}(s).
\end{cases}$$

We remark that if $\sigma_a(s)$ is constantly equal to zero, then $\tilde{B}(s)$ is a constant vector along $\gamma(s)$. We may call $\tilde{B}(s)$ *an intrinsic affine binormal* and $\sigma_a(s)$ *an intrinsic affine torsion* of the curve $\gamma(s)$. It is clear that the frame $(\tilde{T}(s), \tilde{N}(s), \tilde{B}(s))$ is an equi-affine frame, so that we call it *an intrinsic special affine frame* (briefly, *an IS-frame*). Accordingly the assertion (2) of Proposition 3.1, we define a ruled surface generated by the intrinsic affine binormal of $\gamma(s)$:

$$RS(\gamma) = \{x \in \mathbb{R}^3 | x = \gamma(s) + \eta\tilde{B}(s)\ \eta \in \mathbb{R}\}.$$

On the other hand, we define a one-parameter family of functions $R : I \times \mathbb{R}^3 \to \mathbb{R}$ by

$$R(s,u) = |\gamma'(s)\ \gamma'''(s)\ \gamma(s) - u| = f'_u(s).$$

For a fixed point $s_0 \in I$, the plane $P_{s_0} = \{u \in \mathbb{R}^3 | R(s_0, u) = 0\}$ is *the affine rectifying plane* of $\gamma(s)$ at $\gamma(s_0)$,which is the affine plane generated by the intrinsic affine tangent $\tilde{T}(s_0)$ and the intrinsic affine binormal $\tilde{B}(s_0)$.

By the assertion (2) of Proposition 3.1, $R(s,u) = \frac{\partial R}{\partial s}(s,u) = 0$ if and only if $u \in RS(\gamma)$. It follows that the surface $RS(\gamma)$ is the envelope of the family of affine rectifying planes along $\gamma(s)$, so that it is called *affine rectifying surface.* Since the affine rectifying surface is the envelope of the one-parameter family of planes in $\mathbb{R}^3$, it is classically known that such a surface is a developable. By this reason, we also call it *an intrinsic affine binormal developable.* By definition and the Frenet-Serret type formula, the intrinsic affine developable is a cylindrical surface if and only if $\sigma_a(s) \equiv 0$.

By the assertion (1) of Proposition 3.2, the point $u = \lambda\gamma'(s) + \mu\gamma'''(s)$ seems to have special meanings for the curve $\gamma(s)$. If $\mu = 0$, then u is on the tangent line of $\gamma(s)$. so that this case is rather a critical case. Assume that $\mu \neq 0$, then we have $\dfrac{u}{\mu} = \dfrac{\lambda}{\mu}\gamma'(s) + \gamma'''(s)$, so that $\dfrac{u}{\mu}$ is on the following surface associated with the curve $\gamma(s)$.

$$GS(\gamma) = \{\nu\gamma'(s) + \gamma'''(s) | \nu \in \mathbb{R}, s \in I\}.$$

We call $GS(\gamma)$ *an affine rectifying Gaussian surface.* We do not know that these surfaces have classically been studied or not. We only describe the generic singularities appearing these surfaces.

In order to formulate the main results, we consider that a curve is an immersion $\gamma : S^1 \to \mathbb{R}^3$. Let $\mathrm{Imm}(S^1, \mathbb{R}^3)$ be the space of immersions equipped with the Whitney C^∞-topology. We consider an open subset

$$\mathrm{Imm}_a^+(S^1, \mathbb{R}^3) = \{\gamma \in \mathrm{Imm}(S^1, \mathbb{R}^3) || \dot{\gamma}(t)\ \ddot{\gamma}(t)\ \dddot{\gamma}(t)| > 0\}.$$

Theorem 4.1. *There exists a dense subset $\mathcal{O} \subset \mathrm{Imm}_a^+(S^1, \mathbb{R}^3)$ such that for any $\gamma \in \mathcal{O}$ we have the following:*
(1) *Let p be a point of the intrinsic affine binormal developable of γ at s_0, then, locally at p, the intrinsic affine binormal developable is*
(i) *diffeomorphic to the plane in $\mathbb{R}^3$ if $\sigma_a'(s_0) \neq 0$*
(ii) *diffeomorphic to the cuspidal edge in $\mathbb{R}^3$ if $\sigma_a'(s_0) = 0$ and $\sigma_a''(s_0) \neq 0$.*
(iii) *diffeomorphic to the swallow tail in $\mathbb{R}^3$ if $\sigma_a'(s_0) = \sigma_a''(s_0) = 0$ and $\sigma_a'''(s_0) \neq 0$.*
(2) *Let p be a point on the affine rectifying Gaussian surface of γ at s_0, then locally at p, the affine rectifying Gaussian surface is*
(i) *diffeomorphic to the plane in $\mathbb{R}^3$ if $\sigma_a(s_0) \neq 0$.*
(ii) *diffeomorphic to the cuspidal edge in $\mathbb{R}^3$ if $\sigma_a(s_0) = 0$ and $\sigma_a'(s_0) \neq 0$.*
(iii) *diffeomorphic to the swallow tail in $\mathbb{R}^3$ if $\sigma_a(s_0) = \sigma_a'(s_0) = 0$ and $\sigma_a''(s_0) \neq 0$.*
Here, the the cuspidal edge is given as $\mathbb{R} \times \{(x_1, x_2) | x_1^2 = x_2^3\}$ and the swallow tail is given as $\{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

In this note, we do not give the proof of Throrem 4.1. The detailed version will be appeared in elswhere soon.

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