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Extrapolation Spaces for Semigroups

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Abstract. *To a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *we will associate semigroups* $(T_n(t))_{t\geq 0}$ *on new Banach spaces* X_n *for each* $n \in \mathbb{Z}$. *This construction is inspired by the classical Sobolev spaces and, due to its simplicity, of great help in understanding abstract and concrete semigroups.*

1. Sobolev Towers

We start with a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X for which we assume that its growth bound ω_0 is negative. Therefore, the generator $(A, D(A))$ is invertible and $A^{-1} \in \mathcal{L}(X)$. In addition, we assume, after renorming X if necessary, that $\|\lambda R(\lambda, A)\| \leq 1$ for all $\lambda > 0$. On the domains $D(A^n)$ of A^n, $n \in \mathbb{N}$, we now introduce new norms $\|\cdot\|_n$.

1.1 Definition. *For each* $n \in \mathbb{N}$ *and* $x \in D(A^n)$ *we define the* n*-norm*

$$\|x\|_n := \|A^n x\|$$

and call

$$X_n := (D(A^n), \|\cdot\|_n)$$

the n*-th* **Sobolev space** *associated to* $(T(t))_{t\geq 0}$. *The operators* $T(t)$ *restricted to* X_n *will be denoted by*

$$T_n(t) := T(t)_{|X_n}.$$

It turns out that the restrictions $T_n(t)$ behave surprisingly well on X_n.

1.2 Proposition. *With the above definitions the following holds.*

(i) Each X_n *is a Banach space.*

(ii) The operators $T_n(t)$ *form a strongly continuous semigroup* $(T_n(t))_{t\geq 0}$ *on* X_n.

(iii) The generator A_n *of* $(T_n(t))_{t\geq 0}$ *is given by the part of* A *in* X_n, *i.e.,*

$$A_n x = Ax \quad \text{for } x \in D(A_n) := \{x \in X_n : Ax \in X_n\}.$$

Proof. It suffices to prove the assertions for $n = 1$ only. Assertion (i) follows since A is a closed operator and $\|\cdot\|_1$ is equivalent to the graph norm as can be seen from the estimate

$$\|x\|_A = \|A^{-1}Ax\| + \|Ax\| \leq \left(\|A^{-1}\| + 1\right) \cdot \|x\|_1 \leq \left(\|A^{-1}\| + 1\right) \cdot \|x\|_A$$

for $x \in X_1$. From elementary semigroup properties it follows that $T(t)$ maps X_1 into X_1. Each $T_1(t)$ is bounded since

$$\|T_1(t)x\|_1 = \|T(t)Ax\| \leq \|T(t)\| \cdot \|x\|_1 \qquad \text{for } x \in X_1,$$

so $\left(T_1(t)\right)_{t\geq 0}$ is a semigroup on X_1. The strong continuity follows from

$$\|T_1(t)x - x\|_1 = \|T(t)Ax - Ax\| \to 0 \quad \text{for } t \downarrow 0 \quad \text{and } x \in X_1.$$

Finally, (iii) follows since

$$\|\cdot\|_1 \text{-} \lim_{h\downarrow 0} \frac{1}{h}\left(T_1(h)x - x\right)$$

exists in X_1 if and only if

$$\|\cdot\| \text{-} \lim_{h\downarrow 0} \frac{1}{h}\left(T(h)Ax - Ax\right)$$

exists in X, i.e., if and only if $x \in D(A^2)$. $\square$

We suggest to visualize the above spaces and semigroups in form of a diagram. Before doing so we point out that, by definition, A_n is an isometry (with inverse A_n^{-1}) from X_{n+1} onto X_n. Moreover, we include the case $n = 0$ and write $X_0 := X$, $T_0(t) := T(t)$ and $A_0 := A$.

$$\begin{tikzcd}
X_0 \arrow[r, "T_0(t)"] & X_0 \arrow[d, "A_0^{-1}"] \\
X_1 \arrow[u, "A_0"] \arrow[r, "T_1(t)"] & X_1 = D(A_0) \arrow[d, "A_1^{-1}"] \\
X_2 \arrow[u, "A_1"] \arrow[r, "T_2(t)"] & X_2 = D(A_1) = D(A_0^2) \arrow[d] \\
\vdots \arrow[u] & \vdots
\end{tikzcd}$$

Observe that each X_{n+1} is densely embedded in X_n but also, via A_n, isometrically isomorphic to X_n. In addition, the semigroup $\left(T_{n+1}(t)\right)$ is the restriction of $\left(T_n(t)\right)_{t\geq 0}$, but also similar to $\left(T_n(t)\right)_{t\geq 0}$. We state this important property explicitly.

1.3 Corollary. *All the strongly continuous semigroups $(T_n(t))_{t\geq 0}$ on the spaces X_n are similar. More precisely,*

$$\begin{aligned} T_{n+1}(t) &= A_n^{-1} T_n(t) A_n \\ &= T_n(t)_{|X_{n+1}} \qquad \text{for } n \geq 0. \end{aligned}$$

This similarity has the consequence that properties like spectrum, spectral bound, growth bound etc. coincide for all the semigroups $(T_n(t))_{t\geq 0}$.

In our construction we obtained the $(n+1)$-st Sobolev space from the n-th Sobolev space. However, X_{n+1} being a dense subspace of X_n, it is possible to invert this procedure and obtain X_n from X_{n+1} as the completion for the norm

$$\|x\|_n := \left\|A_{n+1}^{-1} x\right\|_{n+1}.$$

This observation permits to extend the above diagram to the negative integers and to define Sobolev spaces of negative order.

1.4 Definition. *For each $n \in \mathbb{N}$ and $x \in X_0$ we define the norm*

$$\|x\|_{-n} := \left\|A_0^{-n} x\right\|$$

and call the completion

$$X_{-n} := (X_0, \|\cdot\|_{-n})^{\sim}$$

the **Sobolev space of order $-n$** *associated to $(T_0(t))_{t\geq 0}$. The continuous extensions of the operators $T_0(t)$ to the space X_{-n} will be denoted by*

$$T_{-n}(t) \qquad \text{for } t \geq 0.$$

The extended operators $T_{-n}(t)$ on the *extrapolated spaces* X_{-n} have properties analogous to Proposition 1.2, so our previous results hold for all $n \in \mathbb{Z}$.

1.5 Theorem. *With the above definitions the following holds for all $n \in \mathbb{Z}$.*

(i) All X_n are Banach spaces with X_n densely contained in X_m for $m \leq n$.

(ii) The operators $T_n(t)$ form strongly continuous semigroups $(T_n(t))_{t\geq 0}$ on X_n.

(iii) The generator A_n of $(T_n(t))_{t\geq 0}$ has domain $D(A_n) = X_{n+1}$ and is the unique continuous extension of $A_m : X_{m+1} \to X_m$ for $m \leq n$ to an isometry from X_{n+1} onto X_n.

Proof. It suffices to prove the assertions for $n = 0$ and $m = -1$ only. Then (i) is true by definition. From

$$\|T_0(t)x\|_{-1} = \left\|T_0(t)A_0^{-1}x\right\|_0 \le \|T_0(t)\| \cdot \|x\|_{-1}$$

we see that $T_0(t)$ extends continuously to X_{-1}. The semigroup property holds on X_0, hence for $\left(T_{-1}(t)\right)_{t\ge 0}$. Similarly, the strong continuity follows since it holds on the dense subset X_0 (even for the stronger norm $\|\cdot\|_0$). To prove (iii) we observe first that A_{-1} extends A_0 since $T_{-1}(t)$ extends $T_0(t)$, so $D(A_0) \subset D(A_{-1})$. Since $D(A_0)$ is dense in X_0, hence in X_{-1} and is $\left(T_{-1}(t)\right)_{t\ge 0}$-invariant it is a core for A_{-1}. This means that $D(A_{-1})$ is the closure of $D(A_0)$ for the graph norm

$$\|x\|_{A_{-1}} := \|x\|_{-1} + \|A_{-1}x\|_{-1}\,.$$

This norm is equivalent to $\|\cdot\|_0$, hence $D(A_{-1}) = X_0$. The rest follows from the fact that $A_0 : D(A_0) \subset X_0 \to X_{-1}$ is, by definition of the norms, an isometry. $\square$

So we have constructed a two-sided infinite sequence of Banach spaces and strongly continuous semigroups and will again visualize this **Sobolev tower** associated to the semigroup $\left(T_0(t)\right)_{t\ge 0}$ by a diagram. Note that Corollary 1.3 now holds for all $n \in \mathbb{Z}$. In addition, if we start this construction from any level, i.e., from the semigroup $\left(T_k(t)\right)_{t\ge 0}$ on the space X_k for some $k \in \mathbb{Z}$, we will obtain the same scale of spaces and semigroups.

1.6 Diagram.

$$\begin{tikzcd}
\vdots & \vdots \arrow[d] & \\
X_{-1} \arrow[u] \arrow[r, "T_{-1}(t)"] & X_{-1} \arrow[d, "A_{-1}^{-1}"] & = \left(X_0,\ \|\cdot\|_{-1}\right)^{\sim} \\
X_0 \arrow[u, "A_{-1}"] \arrow[r, "T_0(t)"] & X_0 \arrow[d, "A_0^{-1}"] & \\
X_1 \arrow[u, "A_0"] \arrow[r, "T_1(t)"] & X_1 \arrow[d] & = D(A_0) \\
\vdots \arrow[u] & \vdots &
\end{tikzcd}$$

We point out that each space X_m is obtained as the (unique) completion of any of its subspaces X_n whenever $m \le n \in \mathbb{Z}$ (and for the appropriate norm). While this procedure yields a rather abstract object, it is possible to identify all Sobolev spaces with concrete function spaces in case of multiplication semigroups.

1.7 Example. We take X to be the function space $\mathrm{C}_0(\mathbb{R})$ and $q : \mathbb{R} \to \mathbb{C}$ a continuous function supposing, for simplicity, that $\sup_{s\in\mathbb{R}} \operatorname{Re} q(s) < 0$. We define $M_q f := q \cdot f$ with maximal domain and the corresponding multiplication semigroup by

$$T_q(t)f := \mathrm{e}^{tq} \cdot f$$

for $t \ge 0$, $f \in X$. The spaces X_n are then given by

$$X_n := \{q^{-n} \cdot f : f \in X\}.$$

An analogous result holds for multiplication semigroups on L^p-spaces. For more examples we refer to [NNR96].

In the next step we insert more spaces in a given Sobolev tower $(X_n)_{n\in\mathbb{Z}}$. Their definition is based on the following lemma.

1.8 Lemma. *Let $\big(T(t)\big)_{t\ge0}$ be a strongly continuous semigroup on a Banach space X with generator $\big(A, D(A)\big)$ and negative growth bound ω_0. For $x \in X$ the following assertions are equivalent.*

(a) $\sup_{t>0} \frac{1}{t} \|T(t)x - x\| < \infty$.
(b) $\sup_{\lambda>0} \lambda \|AR(\lambda, A)x\| < \infty$.
(c) There exists a sequence $(x_n) \subset D(A)$ such that $\lim_{n\to\infty} x_n = x$ *and* $\sup_{n\in\mathbb{N}} \|Ax_n\| < \infty$.

For the proof we refer to [vN92, Chapter 3.2] and note that, for reflexive Banach spaces, all properties are equivalent to

(d) $x \in D(A)$.

These equivalences are now applied to the semigroups $\big(T_n(t)\big)_{t\ge0}$ on the Banach spaces X_n, $n \in \mathbb{Z}$, in order to obtain the following intermediate spaces.

1.9 Definition. *For each $n \in \mathbb{Z}$, the space*

$$F_n := \left\{ x \in X_{n-1} : \sup_{t>0} \frac{1}{t} \|T_{n-1}(t)x - x\|_{n-1} < \infty \right\}$$

with norm

$$\|x\|_{F_n} := \sup_{t>0} \frac{1}{t} \|T_{n-1}(t)x - x\|_{n-1}$$

will be called the ***n*-th Favard class** *associated to $\big(T(t)\big)_{t\ge0}$.*

It is elementary to show that $(F_n, \|\cdot\|_{F_n})$ is a Banach space containing X_n as a closed subspace. Therefore, one has the following inclusions:

$$X_n \subset F_n \hookrightarrow X_{n-1} \subset F_{n-1}.$$

We now describe how the semigroups $\left(T_n(t)\right)_{t\geq 0}$ and their generators A_n behave on the Favard classes.
To that purpose we denote by A_{F_n} the part of A_{n-1} in F_n.

1.10 Proposition. *With the above definitions the following properties hold.*

(a) $T_{n-1}(t) \in \mathcal{L}(F_n)$ *and* $X_n = \{x \in F_n : \lim_{t\downarrow 0} \|T_{n-1}(t)x - x\|_{F_n} = 0\}$.
(b) $A_{F_n} F_{n+1} = A_{n-1} F_{n+1} = F_n$ *for all* $n \in \mathbb{Z}$.
(c) $\sigma(A_{F_n}) = \sigma(A_0)$ *for all* $n \in \mathbb{Z}$.

Proof. The assertion (a) and (b) have been shown in [NS93, Proposition 3.2] (for n=0). Since A_{F_n} is the part of A_{n-1} we obtain

$$\sigma(A_{F_n}) \subset \sigma(A_{n-1}).$$

Similarly, A_n is the part of A_{F_n} in X_n, hence

$$\sigma(A_n) \subset \sigma(A_{F_n}).$$

Since A_n and A_{n-1} are isomorphic, hence have equal spectrum, we obtain assertion (c). □

It is important to observe that the semigroup consisting of the restricted operators $T_{n-1}(t)_{|F_n}$ is, in general, *not* strongly continuous on F_n for $\|\cdot\|_{F_n}$. However, for each $x \in F_n$ the map

$$t \mapsto T_{n-1}(t)x \in F_n \hookrightarrow X_{n-1}$$

is continuous for $\|\cdot\|_{n-1}$, hence

$$t \mapsto \langle T_{n-1}(t)x, x'\rangle$$

is continuous for each $x \in F_n$, $x' \in X'_{n-1}$. This dual space can be identified with the domain $D(A'_n) \subseteq X'_n$ of the adjoint A'_n of A_n.

1.11 Lemma. *For each* $y \in X_n$ *one has*

$$\|y\|_n := \sup\{|\langle y, y'\rangle| : y' \in D(A'_n), \|y'\| \leq 1\}.$$

Proof. Take $x' \in X'_n$ and $y \in X_n$. Then

$$\begin{aligned}\langle y, x'\rangle &= \lim_{\mu\to\infty} \langle \mu R(\mu, A_n)y, x'\rangle \\ &= \lim_{\mu\to\infty} \langle y, \mu R(\mu, A_n)' x'\rangle \\ &= \lim_{\mu\to\infty} \langle y, \mu R(\mu, A'_n) x'\rangle\end{aligned}$$

with $\mu R(\mu, A'_n)x' \in D(A'_n)$ and $\|\mu R(\mu, A'_n)x'\| \leq \|x'\|$. This proves the assertion. □

These considerations allow to obtain $\langle R(\lambda, A_{F_n})x, x'\rangle$ for $x \in F_n$, $x' \in X'_{n-1} = D(A'_n)$ and $\lambda > 0$ as the resolvent integral

$$\int_0^\infty \mathrm{e}^{-\lambda s} \langle T_{n-1}(s)x, x'\rangle \, ds$$

and to estimate the norm of $R(\lambda, A_{F_n})$ in F_n. We conclude, using the normalizing assumption made at the beginning, that

$$\|\lambda R(\lambda, A_{F_n})\|_{F_n} \leq 1 \quad \text{for } \lambda > 0,$$

i.e., A_{F_n} is a *Hille–Yosida operator* on F_n (see [NS93] for the terminology). The same estimate holds for the part A_Y of A_{F_n} in any closed subspace Y satisfying $X_n \subset Y \subset F_n$. This proves one implication in the following theorem while the other has been shown in [NS93, Theorem 1.7]. See also Theorem 4.3.6 in [vN92].

1.12 Theorem. *Let $(B; D(B))$ be a closed operator on a Banach space Y. Then B is a Hille–Yosida operator if and only if there exists a Sobolev tower $(X_n)_{n\in\mathbb{Z}}$ and corresponding semigroup generators A_n such that*

$$X_0 \subset Y \subset F_0$$

as closed subspaces and the given operator B is the part of A_{-1} in Y.

As a typical example we mention the first derivative

$$Bf := f'$$

on the space $Y := \mathrm{C}_b(\mathbb{R})$ with maximal domain. Then one obtains $X_0 = \mathrm{C}_{\mathrm{ub}}(\mathbb{R})$ and $F_0 = \mathrm{L}^\infty(\mathbb{R})$. See [NNR96] for more details.

1.13 Comment. (i) Extrapolation spaces have been introduced in many places, e.g., [PG82], [Nag83], [PG84], [Har86], [Ama87], [Ver97]. See [Sin96] for a recent review.

(ii) In [vN92, Chapter 4.3], there is a "duality " approach to the extrapolated Favard class F_0.

(iii) Recent applications of these extrapolation spaces can be found, e.g., in [Ama95], [NS93], [NR], [Rha95a], [Rha95b].

2. Extrapolation spaces and boundary perturbation

In this section, we use the construction of Sobolev towers to study so called "boundary perturbations". To that purpose we use the abstract setting proposed by Greiner [Gre87].

2.1 Assumptions. Let $(A_m, D(A_m))$ be a closed, linear operator on a Banach space X_0 and consider a linear operator, called *boundary operator*,

$$L : D(A_m) \to Y$$

which is surjective and bounded for the graph norm on $D(A_m)$. Finally, we assume that the restriction $A_0 := A_{m|\ker L}$ is the generator of a strongly continuous semigroup $\big(T_0(t)\big)_{t\geq 0}$ on X_0 having growth bound $\omega_0 < 0$.

With these assumptions we can construct the Sobolev tower $(X_n)_{n\in\mathbb{Z}}$ corresponding to the semigroup $\big(T_0(t)\big)_{t\geq 0}$. Therefore, the generator A_0 extends to a bijection $A_{-1} : X_0 \to X_{-1}$ and maps $D(A_m)$ onto a subspace Z_0 satisfying

$$X_0 \hookrightarrow Z_0 \hookrightarrow X_{-1}.$$

We now try to describe the action of A_{-1} on $D(A_m)$.

2.2 Lemma. *The operator A_{-1} restricted to $D(A_m)$ can be represented as*

$$\mathcal{A} := \begin{pmatrix} 0 & L \\ 0 & A_m \end{pmatrix} : Z_1 \to Z_0,$$

where we take $Z_1 := \{0\} \times D(A_m)$ and $Z_0 := Y \times X_0$.

Proof. From [Gre87, Lemma 1.2] we know that $D(A_m) = \ker A_m \times D(A_0)$ and that $\ker A_m$ is isomorphic to Y. Therefore, A_{-1} induces a bijection from $D(A_0)$ onto X_0 and from $\ker A_m$ onto (an isomorphic copy of) Y. The operator matrix $\left(\begin{smallmatrix} 0 & L \\ 0 & A_m \end{smallmatrix}\right)$ from $\{0\} \times D(A_m)$ onto $Y \times X_0$ does just that. □

We again visualize the situation by a diagram.

2.3 Diagram.

$$
\begin{tikzcd}
\vdots & \vdots & \\
X_{-1} \arrow[u] \arrow[r, "T_{-1}(t)"] & X_{-1} \arrow[u] & \\
 & & Z_0 := Y \times X_0 \arrow[ul] \\
X_0 \arrow[uu, "A_{-1}"] \arrow[r, "T_0(t)"] & X_0 \arrow[uu, "A_{-1}"'] \arrow[ur] & \\
 & & Z_1 = \{0\} \times D(A_m) \arrow[ul] \arrow[uu, "\mathcal{A} := \begin{pmatrix} 0 & L \\ 0 & A_m \end{pmatrix}"'] \\
X_1 \arrow[uu, "A_0"] \arrow[r, "T_1(t)"] & X_1 = D(A_0) = \ker L \arrow[uu, "A_0"'] \arrow[ur] & \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

The operator A_0 will now be perturbed in the following way.

2.4 Definition. *For a bounded, linear operator $B : X_0 \to Y$ we consider $\mathcal{B} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : Z_0 \to Z_0$ and define*

$$A_{-1} + \mathcal{B} : X_0 \to X_{-1}.$$

We observe that $\mathcal{B}$, while being bounded from Z_0 to Z_0, is only relatively A_{-1}-bounded if considered as an operator in X_{-1}.
In the next step we make assumptions on B and A_0 guaranteeing that the additive perturbation $A_{-1} + \mathcal{B}$ with domain X_0 remains a generator of a strongly continuous semigroup on X_{-1}.

2.5 Theorem. *If A_0 and B satisfy one of the following conditions, then $A_{-1} + \mathcal{B}$ with domain $D(A_{-1} + \mathcal{B}) = X_0$ generates a strongly continuous semigroup on X_{-1}.*

(i) The space Z_0 is contained in the extrapolated Favard class F_0.

(ii) The semigroup $\big(T_0(t)\big)_{t \geq 0}$ is analytic and the A_{-1}-bound of $\mathcal{B}$ is small enough.

Proof. (i) is the Desch–Schappacher perturbation theorem from [DS89], while (ii) is Kato's perturbation theorem for analytic semigroups. □

If the assertion of Theorem 2.5 holds we also obtain a strongly continuous semigroup on X_0 (use Proposition 1.2.(ii)). Its generator is the part of $A_{-1} + \mathfrak{B}$ in X_0. In order to identify this operator we use our knowledge on how $A_{-1} + \mathfrak{B}$ maps Z_1 into Z_0. In fact, it follows from Lemma 2.2 that

$$A_{-1} + \mathfrak{B} = \begin{pmatrix} 0 & L+B \\ 0 & A_m \end{pmatrix} : Z_1 \to Z_0.$$

Taking the part of this operator in X_0 we obtain the following result.

2.6 Corollary. *Let $A_{-1} + \mathfrak{B}$ with domain $D(A_{-1} + \mathfrak{B}) = X_0$ be the generator of a strongly continuous semigroup on X_{-1}. Then the operator*

$$A_{L,B}x := A_m x$$

for all $$x \in D(A_{L,B}) := \{x \in D(A_m) : Lx + Bx = 0\}$$

is the generator of a strongly continuous semigroup on X_0.

In this way we obtained the operator $A_{L,B}$ with perturbed domain (or, boundary perturbation) as the "lower level" of an additively perturbed Sobolev tower. In particular, case (i) in Theorem 2.5 corresponds to Theorem 2.1 in [Gre87], while case (ii) is a variant of Greiner's Theorem 2.4. Clearly, other properties of the perturbed semigroup, like spectral or compactness properties, can be reduced in the same way to an additive perturbation. We refer to [NR], [Rha95b] or [Rha95a] where this idea has been applied to concrete situations.

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