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STARLIKE AND CONVEX FUNCTION OF COMPLEX ORDER INVOLVING A CERTAIN FRACTIONAL INTEGRAL OPERATOR

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Abstract

Let the classes $\mathcal{S}_0^*(b)$, $\mathcal{K}_0(b)$ and $\mathcal{C}_0(b)$ consist of functions which are starlike, convex and close-to-convex of complex order b introduced by Nasr and Aouf [2], [3]. The main object of the present paper is to investigate the starlike and convex functions of complex order involving a certain fractional integral operator. Futher relevant connections are also pointed out with various earlier results involving the Haramard product.

Key words : fractional integral, Hadamard product, starlike and convex functions of complex order
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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike of complex order b ($b \in \mathbb{C} \setminus \{0\}$) if and only if $z^{-1}f(z) \neq 0$ ($z \in \mathcal{U}$) and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.2}$$

We denote by $\mathcal{S}_0^*(b)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of complex order b. Futher, let $\mathcal{S}_1^*(b)$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < |b| \qquad (b \in \mathbb{C} \setminus \{0\}). \tag{1.3}$$

Here the inequality (1.2) is equivalent to

$$\operatorname{Re}\left\{\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > -1. \tag{1.4}$$

If $f(z) \in \mathcal{S}_1^*(b)$, then $f(z)$ satisfies (1.4) which implies that

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0.$$

Thus $\mathcal{S}_1^*(b)$ is a subclass of $\mathcal{S}_0^*(b)$.

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be convex of complex order b ($b \in \mathbb{C} \setminus \{0\}$) if and only if $f'(z) \neq 0$ ($z \in \mathcal{U}$) and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.5}$$

We denote by $\mathcal{K}_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of complex order b. Furthermore, let $\mathcal{K}_1(b)$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying

$$\left|\frac{zf''(z)}{f'(z)}\right| < |b| \qquad (b \in \mathbb{C} \setminus \{0\}). \tag{1.6}$$

We note that

$$f(z) \in \mathcal{K}_0(b) \Longleftrightarrow zf'(z) \in \mathcal{S}_0^*(b) \tag{1.7}$$

and

$$f(z) \in \mathcal{K}_1(b) \Longleftrightarrow zf'(z) \in \mathcal{S}_1^*(b) \tag{1.8}$$

for $b \in \mathbb{C} \setminus \{0\}$.

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be close-to-convex of complex order b ($b \in \mathbb{C} \setminus \{0\}$) if and only if there exists a function $g(z) \in \mathcal{K}_0(c)$ ($c \in \mathbb{C} \setminus \{0\}$) satisfying the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right)\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.9}$$

We denote by $\mathcal{C}_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of complex order b. Also let $\mathcal{C}_1(b)$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying

$$\left|\frac{f'(z)}{g'(z)} - 1\right| < |b| \tag{1.10}$$

for some $g \in \mathcal{K}_0(c)$ ($c \in \mathbb{C} \setminus \{0\}$).

We also have $\mathcal{K}_1(b) \subset \mathcal{K}_0(b)$ and $\mathcal{C}_1(b) \subset \mathcal{C}_0(b)$.

Remark. Setting $b = 1-\alpha$ ($0 \leq \alpha < 1$), we observe that $\mathcal{S}_0^*(1-\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{K}_0(1-\alpha) = \mathcal{K}(\alpha)$ and $\mathcal{C}_0(1-\alpha) = \mathcal{C}(\alpha)$, where $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{C}(\alpha)$ denote the usual classes of starlike, convex and close-to-convex of real order α, respectively. Indeed, letting $b = i\alpha$ ($\alpha \in \mathbb{R}$), we obtain that $f \in \mathcal{S}_0^*(i\alpha)$ implies that $\operatorname{Im}(zf'(z)/f(z)) > -\alpha$.

For the functions $f_j(z)$ ($j = 1, 2$) defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \qquad (a_{1,1} = a_{2,1} = 1), \tag{1.11}$$

let $(f_1 * f_2)(z)$ denote the Hadamard product or convolution of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{1,n+1}\, a_{2,n+1} z^{n+1}. \tag{1.12}$$

Let a, b, c be complex numbers with $c \neq 0, -1, -2, \cdots$. The Gaussian hypergeometric function ${}_2F_1(z)$ is defined by

$$\begin{aligned} {}_2F_1(z) &\equiv {}_2F_1(a, b; c; z) \\ &:= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.13}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of Γ-function, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\cdots\}). \end{cases}$$

Many essentially equivalent definitions of fractional calculus have been given in the literature (*cf.*, *e.g*, [9], [10,p.45]). For convenience, we recall here the following definitions due to Owa [4] and Saigo [8] which have been used rather frequently in the theory of analytic functions :

Definition 1. The fractional integral of order λ ($\lambda \in \mathbb{C}$) is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \qquad (\mathrm{Re}(\lambda) > 0), \tag{1.14}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real for $z-\zeta > 0$.

Definition 2. For $\alpha, \beta, \eta \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$, the fractional integral operator $\mathcal{I}_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$\mathcal{I}_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}) f(\zeta) d\zeta, \tag{1.15}$$

where the function ${}_2F_1$ is Gauss's hypergeometric function defined by (1.13).

The definition (1.15) is an interesting extension of both the Riemann-Liou- ville and Erdélyi-Kober fractional operators in terms of Gauss's hypergeometric functions. Indeed, in its special case, it is treated alike the definition (1.14).

It is easy to observe that

$$\mathcal{I}_{0,z}^{\alpha,-\alpha,\eta} f(z) = \mathcal{D}_z^{-\alpha} f(z) \qquad (\mathrm{Re}(\alpha) > 0). \tag{1.16}$$

By using the fractional integral, we now introduce the linear operator Ω^λ given by

$$\Omega^\lambda f(z) = \Gamma(2-\lambda) z^\lambda \mathcal{D}_z^\lambda f(z) \qquad (\mathrm{Re}(\lambda) < 0) \tag{1.17}$$

for $f(z) \in \mathcal{A}$.

The operator $\mathcal{I}_{0,z}^{\alpha,\beta,\eta}$ is also modified by defining $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ in the form

$$\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} \mathcal{I}_{0,z}^{\alpha,\beta,\eta} f(z) \tag{1.18}$$

for $f(z) \in \mathcal{A}$ and $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$.

2. Main results

In order to prove our main results, we shall require the following lemmas to be used in the sequel.

Lemma 1. (Jack [1]) *Let $\omega(z)$ be analytic in $\mathcal{U}$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ $(r < 1)$ at a point z_0, we can write*

$$z_0\omega'(z_0) = k\,\omega(z_0), \tag{2.1}$$

where k is real and $k \geq 1$.

Lemma 2. (Ruscheweyh and Sheil-Small [7]) *Let $\phi(z)$ and $g(z)$ be analytic in $\mathcal{U}$ and satisfy*

$$\phi(0) = g(0) = 0, \qquad \phi'(0) \neq 0, \qquad \text{and} \qquad g'(0) \neq 0.$$

Suppose that for each σ $(|\sigma| = 1)$ and ρ $(|\rho| = 1)$

$$\phi(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) g(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}).$$

Then, for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying the inequality $\mathrm{Re}\{F(z)\} > 0$ $(z \in \mathcal{U})$, we have

$$\mathrm{Re}\left(\frac{(\phi * G)(z)}{(\phi * g)(z)}\right) > 0 \qquad (z \in \mathcal{U}), \tag{2.2}$$

where $G(z) = F(z)g(z)$.

Lemma 3. ([7]) *Let $\phi(z)$ be convex and $g(z)$ starlike in $\mathcal{U}$. Then, for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying $\mathrm{Re}\{F(z)\} > 0$ $(z \in \mathcal{U})$, we have*

$$\mathrm{Re}\left(\frac{(\phi * Fg)(z)}{(\phi * g)(z)}\right) > 0 \qquad (z \in \mathcal{U}), \tag{2.3}$$

Lemma 4. (*cf.*, Owa, Saigo and Srivastava [5]) *Let* $\alpha, \beta, \eta \in \mathbb{C}$ *and* $\mathrm{Re}(\alpha) > 0$, *and let* $k > \mathrm{Re}(\beta - \eta) - 1$. *Then*

$$\mathcal{I}_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+\eta)\Gamma(k+\alpha+\eta+1)} z^{k-\beta} \,. \tag{2.4}$$

Applying the above lemmas, we derive

Theorem 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}_0^*(b)$ *and satisfy*

$$h(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) b f(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}) \tag{2.5}$$

for each ρ ($|\rho| = 1$) *and* σ ($|\sigma| = 1$), *where*

$$h(z) = z + \sum_{n=2}^{\infty} \frac{(2-\beta+\eta)_{n-1}(1)_n}{(2-\beta)_{n-1}(2+\alpha+\eta)_{n-1}} z^n \,, \tag{2.6}$$

and for $\alpha, \beta, \eta \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$ *and* $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. *Then* $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z)$ *belongs to the class* $\mathcal{S}_0^*(b)$.

Proof. Note from (1.18), (2.4) and (2.6) that

$$\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) = z + \sum_{n=2}^{\infty} \frac{(2-\beta+\eta)_{n-1}(1)_n}{(2-\beta)_{n-1}(2+\alpha+\eta)_{n-1}} a_n z^n = (h * f)(z),$$

which readily yields

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z)} - 1\right) &= \frac{h(z) * \left(\sum_{n=0}^{\infty} (n+b) a_{n+1} z^{n+1}\right)}{b\,(h*f)\,(z)} \\ &= \frac{(h * [(b-1)f + zf'])(z)}{(h * bf)(z)}. \end{aligned} \tag{2.7}$$

as $a_1 = 1$.
Therefore, putting $\phi(z) = h(z)$, $g(z) = bf(z)$ and $F(z) = 1 + 1/b[(zf'(z))/f(z) - 1]$ in Lemma 2, we conclude from (2.7) that

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z)} - 1\right)\right\} > 0 \,,$$

which completes the proof of Theorem 1.

Corollary 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}_0^*(b)$ *and satisfy*

$$u(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) bf(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\})$$

for each ρ $(|\rho| = 1)$ *and* σ $(|\sigma| = 1)$, *where*

$$u(z) = z + \sum_{n=2}^{\infty} \frac{(1)_n}{(2-\lambda)_{n-1}} z^n \qquad (\mathrm{Re}(\lambda) < 0). \tag{2.8}$$

Then $\Omega^\lambda f(z)$ *belongs to the class* $\mathcal{S}_0^*(b)$.

Proof. Setting $\alpha = -\beta = -\lambda$ in Theorem 1 and taking Remark 2 into account, we have Corollary 1.

Corollary 2. *Let* $h(z)$ *be convex and let* $f(z) \in \mathcal{S}_1^*(b)$ $(|b| \leq 1)$, *where* $h(z)$ *is given by* (2.6) *with the same assumptions of* α, β *and* η *in Theorem 1. Then* $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) = (h \prec f)(z)$ *belongs to the class* $\mathcal{S}_0^*(b)$.

Proof. From the hypothesis, we obtain

$$f(z) \in \mathcal{S}_1^*(b) \subset \mathcal{S}^*(0) = \mathcal{S}^* \qquad (|b| \leq 1).$$

By applying Lemma 3 in view of Theorem 1, we have the desirous result immediately.

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{K}_0(b)$ *and satisfy*

$$h(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) bzf'(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}) \tag{2.9}$$

for each ρ $(|\rho| = 1)$ *and* σ $(|\sigma| = 1)$, *where* $h(z)$ *is given by* (2.6) *and for* $\alpha, \beta, \eta \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$ *and* $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. *Then* $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z)$ *belongs to the class* $\mathcal{K}_0(b)$.

Proof. Applying (1.7) and Theorem 1, we observe that

$$f(z) \in \mathcal{K}_0(b) \Longleftrightarrow zf'(z) \in \mathcal{S}_0^*(b) \Longrightarrow \mathcal{J}_{0,z}^{\alpha,\beta,\eta} zf'(z) \in \mathcal{S}_0^*(b)$$

$$\Longleftrightarrow (h * zf')(z) \in \mathcal{S}_0^*(b) \Longleftrightarrow z(h*f)'(z) \in \mathcal{S}_0^*(b)$$

$$\Longleftrightarrow (h * f)(z) \in \mathcal{K}_0(b) \Longleftrightarrow \mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{K}_0(b),$$

which evidently proves Theorem 2.

Taking $\alpha = -\beta = -\lambda$ in Theorem 2, we get

Corollary 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}_0(b)$ and satisfy*

$$u(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) bzf'(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}) \tag{2.10}$$

for each ρ ($|\rho| = 1$) and σ ($|\sigma| = 1$), where $u(z)$ is given by (2.8). Then $\Omega^\lambda f(z)$ belongs to the class $\mathcal{K}_0(b)$.

Corollary 4. *Let $h(z)$ be convex and let $f(z) \in \mathcal{K}_1(b)$ ($|b| \leq 1$), where $h(z)$ is given by (2.6) with the same assumption of α, β and η there. Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) = (h \prec f)(z)$ belongs to the class $\mathcal{K}_0(b)$.*

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\left|\frac{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))'}{g'(z)} - 1\right|^\sigma \left|\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))''}{g'(z)} - \frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))' g''(z)}{\{g'(z)\}^2}\right|^\delta < |b|^{\sigma+\delta} \qquad (z \in \mathcal{U}) \tag{2.11}$$

for some $\sigma \geq 0$, $\delta \geq 0$ and $g(z) \in \mathcal{K}_0(c)$. Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ *and* $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. *Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z)$ belongs to the class $\mathcal{C}_1(b)$.*

Proof. If we define

$$\omega(z) = \frac{1}{b}\left(\frac{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))'}{g'(z)} - 1\right) \tag{2.12}$$

for $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{K}_0(c)$, then it is an elementary matter to show that $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$. Noting that

$$bz\omega'(z) = \frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))''}{g'(z)} - \frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))' g''(z)}{\{g'(z)\}^2},$$

we know that the condition (2.11) leads us to

$$|b\omega(z)|^\sigma |bz\omega'(z)|^\delta < |b|^{\sigma+\delta}.$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \qquad (\omega(z_0) \neq 1). \tag{2.13}$$

Then, using Lemma 1, we see

$$|b\omega(z_0)|^\sigma |bz_0\omega'(z_0)|^\delta = |b|^{\sigma+\delta} k^\delta \geq |b|^{\sigma+\delta},$$

which contradicts (2.11). Therefore we conclude $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. This implies that

$$\left|\frac{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'}{g'(z)} - 1\right| < |b| \qquad (z \in \mathcal{U}),$$

which completes the proof of Theorem 3.

Letting $\alpha = -\beta = -\lambda$ in Theorem 3, we have

Corollary 5. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\left|\frac{(\Omega^{\lambda}f(z))'}{g'(z)}-1\right|^{\sigma}\left|\frac{z(\Omega^{\lambda}f(z))''}{g'(z)}-\frac{z(\Omega^{\lambda}f(z))'g''(z)}{\{g'(z)\}^{2}}\right|^{\delta}<|b|^{\sigma+\delta}\qquad(z\in\mathcal{U})\tag{2.14}$$

for some $\sigma \geq 0$, $\delta \geq 0$, and $g(z) \in \mathcal{K}_0(c)$. Then $\Omega^{\lambda}f(z)$ belongs to the class $\mathcal{C}_1(b)$.

Putting $g(z) = z \in \mathcal{K}_0(1)$, Theorem 3 gives

Corollary 6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\left|(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'-1\right|^{\sigma}\left|z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))''\right|^{\delta}<|b|^{\sigma+\delta}\qquad(z\in\mathcal{U})\tag{2.15}$$

for some $\sigma \geq 0$ and $\delta \geq 0$. Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)$ belongs to the class $\mathcal{C}_1(b)$.

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\begin{aligned}\left|a\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}-1\right)+(1-a)\frac{z^{2}(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))''}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}\right|&\\ <|b|\,[1+(1-a)(1-|b|)]&\qquad(z\in\mathcal{U})\end{aligned}\tag{2.16}$$

for some $a \leq 1$ and $|b| \leq 1$. Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)$ belongs to the class $\mathcal{S}_1^*(b)$.

Proof. If we set

$$\omega(z)=\frac{1}{b}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}-1\right)\qquad(f\in\mathcal{A}),\tag{2.17}$$

then the function $\omega(z)$ is regular in $\mathcal{U}$ and $\omega(0) = 0$. By using the logarithmic differentiation on both sides of (2.17), we have

$$\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}=b\omega(z)+\frac{bz\omega'(z)}{1+b\omega(z)}\,.$$

This yields

$$\begin{aligned}a\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}-1\right)+(1-a)\frac{z^{2}(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))''}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}&\\ =b\omega(z)\left\{1+(1-a)\left(b\omega(z)+\frac{z\omega'(z)}{\omega(z)}\right)\right\}&\,.\end{aligned}$$

Assume that there exists $z_0 \in \mathcal{U}$ such that (2.13) holds true for the function $\omega(z)$ in (2.17). Then, writing $\omega(z_0) = e^{i\theta}$, and using Lemma 1, we deduce

$$\left| b\omega(z_0)\left\{1+(1-a)\left(b\omega(z_0)+\frac{z_0\omega'(z_0)}{\omega(z_0)}\right)\right\}\right| = |b||1+(1-a)(k+be^{i\theta})|$$
$$\geq |b||1+(1-a)(1-|b|)| \, ,$$

which contradicts (2.16). Thus we obtain

$$|\omega(z)| = \left|\frac{1}{b}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}-1\right)\right| < 1 \qquad (z \in \mathcal{U}),$$

which completes the proof of Theorem 4.

Taking $\alpha = -\beta = -\lambda$ in Theorem 4, we have

Corollary 7. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\text{(2.18)} \quad \left|a\left(\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}-1\right)+(1-a)\frac{z^2(\Omega^\lambda f(z))''}{\Omega^\lambda f(z)}\right| < |b|[1+(1-a)(1-|b|)] \quad (z \in \mathcal{U})$$

for some $a \leq 1$ and $|b| \leq 1$. Then $\Omega^\lambda f(z)$ belongs to the class $\mathcal{S}_1^*(b)$.

Theorem 5. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*
(2.19)

$$\left|a\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)}-1\right)+(1-a)\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}\right| < |b|\left(1+\frac{1-a}{1+|b|}\right) \quad (z \in \mathcal{U})$$

for some $a \leq 1$. Suppose also that $\alpha,\beta,\eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)$ belongs to the class $\mathcal{S}_1^*(b)$.

The proof of Theorem 5 is much akin to that of Theorem 4, and we omit the details involved.

Theorem 6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\text{(2.20)} \quad \left|(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'-1\right|^\sigma \left|1+\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'}\right|^\delta < |b|^\sigma\left(\frac{1+2|b|}{1+|b|}\right)^\delta \quad (z \in \mathcal{U})$$

for some $\sigma \geq 0$ and $\delta \geq 0$. Suppose also that $\alpha,\beta,\eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)$ belongs to the class $\mathcal{C}_1(b)$.

Proof. Define the function $\omega(z)$ by

$$\text{(2.21)} \qquad \omega(z) = \frac{1}{b}\{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z))'-1\} \, .$$

Then it follows that $\omega(z)$ is analytic in $\mathcal{U}$ with $\omega(0) = 0$. Substituting for $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z)$ into the left-hand side of (2.20) from (2.21), we get

$$\left|(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))' - 1\right|^{\sigma} \left|1 + \frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z))'}\right|^{\delta} = |b\omega(z)|^{\sigma} \left|\frac{1 + b(\omega(z) + z\omega'(z))}{1 + b\omega(z)}\right|^{\delta}.$$

Assume that there exist a point $z_0 \in \mathcal{U}$ satisfying (2.13) for the function $\omega(z)$ in (2.21). Then, applying Lemma 1, we obtain

$$\begin{aligned} |b\omega(z_0)|^{\sigma} \left|\frac{1 + b(\omega(z_0) + z_0\omega'(z_0))}{1 + b\omega(z_0)}\right|^{\delta} &= |b|^{\sigma} \left|(k+1) - \frac{k}{1 + b\omega(z_0)}\right|^{\delta} \\ &\geq |b|^{\sigma} \left(\frac{1 + 2|b|}{1 + |b|}\right)^{\delta}, \end{aligned}$$

which contradicts the condition (2.20). Hence we have $\mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{C}_1(b)$.

Theorem 7. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy*

$$\text{(2.22)} \qquad \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right) > \frac{|2b-1| - 1}{2(|2b-1| + 1)} \quad \text{if } \left|b - \frac{1}{2}\right| < \frac{1}{2}$$

or

$$\text{(2.23)} \qquad \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right) < \frac{|2b-1| - 1}{2(|2b-1| + 1)} \quad \text{if } \left|b - \frac{1}{2}\right| > \frac{1}{2}$$

for some $g(z) \in \mathcal{K}_0(c)$. Then $f(z)$ belongs to the class $\mathcal{C}_0(b)$.

Proof. Let us introduce the function $\omega(z)$ by

$$\text{(2.24)} \qquad 1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right) = \frac{1 + \omega(z)}{1 - \omega(z)}$$

for some $g(z) \in \mathcal{K}_0(c)$ and $f(z) \in \mathcal{A}$. Differentiating both side of (2.24) logarithmically, we obtain

$$\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} = \frac{(2b-1)z\omega'(z)}{1 + (2b-1)\omega(z)} + \frac{z\omega'(z)}{1 - \omega(z)}.$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that (2.13) holds true for the function $\omega(z)$ in (2.24). Then, letting $\omega(z_0) = e^{i\theta}$ and $2b - 1 = |2b-1|e^{i\phi}$, and using Lemma 1, we have

$$\begin{aligned} \operatorname{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g''(z_0)}{g'(z_0)}\right) &= \operatorname{Re}\left(\frac{(2b-1)k\omega(z_0)}{1 + (2b-1)\omega(z_0)}\right) + \operatorname{Re}\left(\frac{k\omega(z_0)}{1 - \omega(z_0)}\right) \\ &= \frac{k|2b-1|(|2b-1| + \cos(\theta + \phi))}{1 + |2b-1|^2 + 2|2b-1|\cos(\theta + \phi)} - \frac{k}{2} \end{aligned}$$

for $k \geq 1$ and $z_0 \in \mathcal{U}$. Hence, let

$$h(t) = \frac{|2b-1|+t}{1+|2b-1|^2+2|2b-1|t} \qquad (-1 \leq t \leq 1).$$

If $|b-1/2| \leq 1/2$, then $h(t)$ is monotone increasing and

$$\begin{aligned} \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g''(z_0)}{g'(z_0)}\right) &\leq \frac{|2b-1|k}{|2b-1|+1} - \frac{k}{2} \\ &\leq \frac{|2b-1|-1}{2(|2b-1|+1)}. \end{aligned}$$

If, on the other hand, $|b-1/2| \geq 1/2$, then $h(t)$ is monotone decreasing and

$$\begin{aligned} \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g''(z_0)}{g'(z_0)}\right) &\geq \frac{|2b-1|k}{|2b-1|+1} - \frac{k}{2} \\ &\geq \frac{|2b-1|-1}{2(|2b-1|+1)}. \end{aligned}$$

These contradict (2.22) and (2.23), which evidently completes the proof of Theorem 6.

Corollary 8. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{A}$ and satisfy*

$$\text{(2.25)} \qquad \mathrm{Re}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'} - \frac{zg''(z)}{g'(z)}\right) > \frac{|2b-1|-1}{2(|2b-1|+1)} \quad \text{if } \left|b-\frac{1}{2}\right| < \frac{1}{2}$$

or

$$\text{(2.26)} \qquad \mathrm{Re}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'} - \frac{zg''(z)}{g'(z)}\right) < \frac{|2b-1|-1}{2(|2b-1|+1)} \quad \text{if } \left|b-\frac{1}{2}\right| > \frac{1}{2}$$

for some $g(z) \in \mathcal{K}_0(c)$. Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\min\{\mathrm{Re}(\alpha+\eta), \mathrm{Re}(-\beta+\eta), \mathrm{Re}(-\beta)\} > -2$. Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z)$ belongs to the class $\mathcal{C}_0(b)$.

Theorem 8. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{A}$ and satisfy*

$$\text{(2.27)} \qquad \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) > \frac{-|2b-1|-3}{2(|2b-1|+1)} \quad \text{if } \left|b-\frac{1}{2}\right| \leq \frac{1}{2}$$

or

$$\text{(2.28)} \qquad \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{-|2b-1|-3}{2(|2b-1|+1)} \quad \text{if } \left|b-\frac{1}{2}\right| > \frac{1}{2}.$$

Then $f(z)$ belongs to the class $\mathcal{S}_0^*(b)$.

Proof. The proof of Theorem 8 runs parallel to that of Theorem 7 with

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right) = \frac{1+\omega(z)}{1-\omega(z)},$$

and we omit the details involved.

Corollary 9. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{A}$ and satisfy*

$$\text{(2.29)} \qquad \operatorname{Re}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'} - \frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z)}\right) > \frac{-|2b-1|-3}{2(|2b-1|+1)} \quad if \quad \left|b-\frac{1}{2}\right| \leq \frac{1}{2}$$

or

$$\text{(2.30)} \qquad \operatorname{Re}\left(\frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))''}{(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'} - \frac{z(\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z))'}{\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z)}\right) < \frac{-|2b-1|-3}{2(|2b-1|+1)} \quad if \quad \left|b-\frac{1}{2}\right| > \frac{1}{2}.$$

Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$ *and* $\min\{\operatorname{Re}(\alpha+\eta), \operatorname{Re}(-\beta+\eta), \operatorname{Re}(-\beta)\} > -2$. *Then $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}(z)$ belongs to the class $\mathcal{S}_0^*(b)$.*

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