Some Properties for Convolutions of Generalized Hypergeometric Functions

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Abstract

With the convolution products of generalized hypergeometric functions \( pF_q(z) \) and analytic functions \( f(z) \) in the open unit disk, the operator \( f^{a_1, a_2, \ldots, a_p}_{b_1, b_2, \ldots, b_q} (f) \) is introduced. The object of the present paper is to derive some interesting properties of operator \( f^{a_1, a_2, \ldots, a_p}_{b_1, b_2, \ldots, b_q} (f) \) associated with some classes of univalent functions.

1. Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( U = \{z : |z| < 1\} \). Denote by \( S \) the class of all functions in \( A \) which are univalent in \( U \).

A function \( f(z) \in A \) is said to be in the class \( R^t(A, B) \) if

\[
\left| \frac{f'(z) - 1}{t(A - B) - B(f'(z) - 1)} \right| < 1,
\]

where \( A \) and \( B \) are arbitrary fixed numbers with \(-1 \leq B < A \leq 1\) and \( t \in C \setminus \{0\} \) (\( C \) is the set of all complex numbers). Clearly, a function \( f(z) \) belongs to \( R^t(A, B) \) if and only if there exists a function \( w(z) \) regular in \( U \) satisfying \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)) such that

\[
1 + \frac{1}{t} (f'(z) - 1) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in U).
\]

The class \( R^t(A, B) \) was introduced by Dixit and Pal [4], recently.

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By giving specific values $t, A$ and $B$ in (1.2), we obtain the following subclasses studied by various researchers in earlier works:

(i) For $t = e^{-in}\cos \eta \ (|\eta| < \frac{\pi}{2})$, $A = 1 - 2\alpha \ (0 \leq \alpha < 1)$ and $B = -1$, we obtain the class of functions $f$ satisfying the condition

\begin{equation}
\left| \frac{e^{in}(f'(z) - 1)}{2(1 - \alpha)\cos \eta + e^{in}(f'(z) - 1)} \right| < 1 \quad (z \in U).
\end{equation}

In this case, the class $R(A, B)$ is equivalent to the class $R_\eta(\alpha)$ which is studied by Ponnusamy and Rohnning [11]. Here $R_\eta(\alpha)$ is the class of functions $f(z) \in A$ satisfying the condition

\[ \text{Re}(e^{in}(f'(z) - \alpha)) > 0 \quad (|\eta| < \frac{\pi}{2}, 0 \leq \alpha < 1, z \in U). \]

(ii) For $t = e^{-in}\cos \eta \ (|\eta| < \frac{\pi}{2})$, we obtain the class of functions $f(z) \in A$ satisfying the condition

\[ \left| \frac{e^{in}(f'(z) - 1)}{Be^{in}f'(z) - (A\cos \eta + iB\sin \eta)} \right| < 1 \quad (z \in U), \]

which was studied by Dashrath [3].

(iii) For $t = 1, A = \beta$ and $B = -\beta \ (0 < \beta \leq 1)$, we obtain the class of functions $f(z)$ satisfying the condition

\[ \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in U), \]

which was studied by Padmanabhan [10] and Caplinger and Cauchy [2].

Let $S^*(\alpha)$ and $C(\alpha)$ denote the subclasses of $S$ consisting of starlike and convex functions of order $\alpha \ (0 \leq \alpha < 1)$ in $U$, respectively. It is well-known that $S^*(\alpha) \subset S^*(0) \equiv S^*, C(\alpha) \subset C(0) \equiv C$ and $C(\alpha) \subset S^*(\alpha) \subset S$. For $\lambda > 0$, define the classes $S_\lambda^*$ and $C_\lambda$ by

\[ S_\lambda^* = \{f(z) \in A : \left| \frac{zf'(z)}{f(z)} \right| < \lambda, z \in U \} \]

and

\[ C_\lambda = \{f(z) \in A : zf'(z) \in S_\lambda^* \}, \]

respectively. It is a known fact that a sufficient condition for $f(z) \in A$ of the form (1.1) to belong to the class $S^*$ is that \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \). A simple extension of this result is the following [16]:

\begin{equation}
\sum_{n=2}^{\infty} (n + \lambda - 1)|a_n| \leq \lambda \implies f(z) \in S_\lambda^*.
\end{equation}
For $\lambda = \frac{1}{2}$, this was previously proved by Schild [18]. Since $f(z) \in C_\lambda$ if and only if $zf'(z) \in S_\lambda^*$, we have a corresponding result for $C_\lambda$,

\[(1.6) \sum_{n=2}^{\infty} n(n + \lambda - 1)|a_n| \leq \lambda \Rightarrow f(z) \in C_\lambda.\]

In this paper, we consider the generalized hypergeometric series $pF_q(z)$ defined by

\[(1.7) \quad pF_q(z) \equiv pF_q \left( \begin{array}{c} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{i=1}^{q}(b_i)_n} \frac{z^n}{(1)_n},\]

where $p$ and $q$ are positive integers and we assume that the variable $z$, the numerator parameters $a_1, a_2, \cdots, a_p$ and the denominator parameters $b_1, b_2, \cdots, b_q$ take on complex values, provided that $b_i \neq 0, -1, -2, \cdots; i = 1, 2, \cdots, q$. Here $(\lambda)_n$ is the Pochhammer symbol defined by

\[(\lambda)_n = \left\{ \begin{array}{ll} 1 & \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n \in \mathbb{N} = \{1, 2, \cdots\}. \end{array} \right.\]

For any complex number $\lambda$, we also use the ascending factorial notation

\[(\lambda)_n = \lambda(\lambda+1)n-1\]

for $n \geq 1$ and $(\lambda)_0 = 1$ for $\lambda \neq 0$. If $\lambda$ is neither zero nor a negative integer, then using the definition of the Gamma function, we can write

\[(1.9) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.\]

Furthermore, if we set

$$\omega = \sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j$$

it is known that the series $pF_q(z)$, with $p = q + 1$, is

(i) absolutely convergent for $|z| = 1$ if $\Re \omega > 0$,

(ii) conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 < \Re \omega \leq 0$

and

(iii) divergent for $|z| = 1$ if $\Re \omega \leq -1$.

As in the case of the function $2F_1(z)$, we are led to the well-known Gauss summation theorem:

\[(1.10) \quad 2F_1 \left( \begin{array}{c} a_1, a_2 \\ b_1 \end{array} ; 1 \right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}, \quad \Re(b_1 - a_1 - a_2) > 0.\]
We recall that the function $2F_1(z)$ is bounded if $\text{Re}(b_1 - a_1 - a_2) > 0$ and has a pole at $z = 1$ if $\text{Re}(b_1 - a_1 - a_2) \leq 0$ (cf. [1]). Univalence, starlikeness and convexity properties of $2F_1(a_1, a_2; z)$ have been studied extensively in [12, 15].

For $f(z) \in A$, we define the operator $I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f)$ by

$$I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f)(z) = z \frac{F_p(a_1, a_2 \ldots, a_p; b_1, b_2, \ldots, b_q; z) * f(z)}{f(z)},$$

where the symbol " * " denotes the usual Hadamard product or convolution of power series.

2. Properties of the operators with $R^t(A, B)$

Now we introduce several lemmas which are needed for the proof of our main results.

**Lemma 2.1 ([8])** Let $w(z)$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_1 \in U$, then we can write

$$z_1w'(z_1) = mw(z_1),$$

where $m$ is real and $m \geq 1$.

**Lemma 2.2 ([4])** Let a function $f(z)$ of the form (1.1) be in $R^t(A, B)$. Then

$$|a_n| \leq \frac{(A - B)|t|}{n}.$$ 

Then result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A - B)t z^{n-1}}{1 + B z^{n-1}}\right) dz \quad (n \geq 2, z \in U).$$

**Lemma 2.3 ([4])** Let a function $f(z)$ of the form (1.1) be in $A$. If

$$\sum_{n=2}^{\infty} (1 + |B|) n |a_n| \leq (A - B)|t| \quad (-1 \leq B < A \leq 1, \ t \in C \setminus \{0\})$$

then $f(z) \in R^t(A, B)$. The result is sharp for function

$$f(z) = z + \frac{(A - B)t}{(1 + |B|)n} z^n \quad (n \geq 2, z \in U).$$

Our first result for the operators is contained in
Theorem 2.1  If $f(z) \in \mathcal{A}$ satisfies

\[
I_{b_1,\ldots,b_p}^{a_1,\ldots,a_p,1}(f) - 1 < \left( \frac{1}{2} \right)^\beta
\]

for some fixed $\beta \geq 0$, then $I_{b_1,\ldots,b_p}(f)$ is univalent (close-to-convex) in $U$.

Proof.  We note that

\[
I_{b_1}^{a_1,b}(...,a_p,b_{2q},f) = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n-1}}{\prod_{i=1}^{q} (b_i)_{n-1}(1)_n} a_n z^n
\]

in $\mathcal{A}$. Define $w(z)$ by

\[
w(z) = \frac{I_{b_1,\ldots,b_p}^{a_1,\ldots,a_p,1}(f)}{z} - 1
\]

for $z \in U$. Then it follows that $w(z)$ is analytic in $U$ with $w(0) = 0$. By (2.1), it is clear that

\[
|w(z)|^{1-\beta} \left| \frac{zw'(z)}{1+w(z)} \right|^\beta = |w(z)| \left| \frac{zw'(z)}{w(z)(1+w(z))} \right|^\beta < \left( \frac{1}{2} \right)^\beta.
\]

Suppose that there exists a point $z_1 \in U$ such that

\[
\max_{|z| \leq |z_1|} |w(z)| = |w(z_1)| = 1.
\]

Then we can put

\[
\frac{z_1 w'(z_1)}{w(z_1)} = m \geq 1,
\]

by Lemma 2.1. Therefore we obtain

\[
|w(z_1)| \left| \frac{z_1 w'(z_1)}{w(z_1)(1+w(z_1))} \right|^\beta \geq \left( \frac{m}{2} \right)^\beta \geq \left( \frac{1}{2} \right)^\beta,
\]

which contradicts the condition (2.2). This shows that

\[
|w(z)| = \left| \frac{I_{b_1,\ldots,b_p}^{a_1,\ldots,a_p,1}(f)}{z} - 1 \right| < 1,
\]

which implies that Re $[I_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}(f)]'(z) > 0$ for $z \in U$. Therefore, by Noshiro-Warschawski Theroem [5], $I_{b_1,\ldots,b_p}(f)$ is univalent (close-to-convex) in $U$. 

Theorem 2.2 Let $a_j (j = 1, 2, \cdots, p) \in C \setminus \{0\}$, $b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}$, $\operatorname{Re} b_i > 0 (i = 1, 2, \cdots, q)$, and $\sum_{i=1}^{q} \operatorname{Re} b_i > \sum_{j=1}^{p} |a_j|$. If $f(z) \in R^t(A, B)$ satisfies

\begin{equation}
\begin{aligned}
_{p}F_{q}\left(\begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p| \\
\operatorname{Re} b_1, \operatorname{Re} b_2, \cdots, \operatorname{Re} b_q
\end{array} ; 1 \right) \leq \frac{1}{1 + |B|} + 1,
\end{aligned}
\end{equation}

then

\begin{equation}
z_{p}F_{q}\left(\begin{array}{c}
a_1, a_2, \cdots, a_p \\
b_1, b_2, \cdots, b_q
\end{array} ; z^k \right) \ast f(z) \in R^t(A, B),
\end{equation}

where $k \in N$.

Proof. By Lemma 2.3, it suffices to show that

\begin{equation}
T_1 := \sum_{n=2}^{\infty} (1 + |B|)(k(n-1) + 1) \left| \frac{\prod_{j=1}^{p} |a_j|^{n-1}}{\prod_{i=1}^{q} \operatorname{Re} b_i^{n-1}} a_{k(n-1)+1} \right| \leq (A-B)|t|.
\end{equation}

From Lemma 2.2 and the fact that $|a_n| \leq |a_{n}|$ and $(\operatorname{Re} b)_n \leq |b_{n}|$, $\operatorname{Re} b > 0$, we have

\begin{align}
T_1 &\leq \sum_{n=2}^{\infty} (A-B)(1 + |B|)|t| \left\{ \frac{\prod_{j=1}^{p} |a_j|^{n-1}}{\prod_{i=1}^{q} \operatorname{Re} b_i^{n-1}} \right\} \\
&= (A-B)(1 + |B|)|t| \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} |a_j|^{n}}{\prod_{i=1}^{q} \operatorname{Re} b_i^{n}} - 1 \right\} \\
&= (A-B)(1 + |B|)|t| \left\{ \frac{\prod_{j=1}^{p} |a_j|}{\prod_{i=1}^{q} \operatorname{Re} b_i} - 1 \right\} \\
&\leq (A-B)|t|
\end{align}

by (2.3). This completes the proof of Theorem 2.2.

Corollary 2.1 Let $a_j (j = 1, 2, \cdots, q+1) \in C \setminus \{0\}$, $b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}$, $\operatorname{Re} b_m > |a_m| + 1 (m = 1, 2, \cdots, q-1)$, and $\operatorname{Re} b_q > |a_q| + |a_{q+1}|$. If $f(z) \in R^t(A, B)$ satisfies

\begin{equation}
\begin{aligned}
\frac{\Gamma(\operatorname{Re} b_q) \Gamma(\operatorname{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\operatorname{Re} b_q - |a_q|) \Gamma(\operatorname{Re} b_q - |a_{q+1}|)} \left( \prod_{m=1}^{q-1} \frac{\operatorname{Re} b_m - 1}{\operatorname{Re} b_m - |a_m| - 1} \right) \leq \frac{1}{1 + |B|} + 1,
\end{aligned}
\end{equation}

then

\begin{equation}
z_{q+1}F_{q}\left(\begin{array}{c}
a_1, a_2, \cdots, a_{q+1} \\
b_1, b_2, \cdots, b_q
\end{array} ; z^k \right) \ast f(z) \in R^t(A, B),
\end{equation}

where $k \in N$. 
Proof. We note that

\[
\sum_{n=0}^{\infty} \frac{(|a_1|)_n \cdots (|a_{q+1}|)_n}{(\text{Re} b_1)_n \cdots (\text{Re} b_q)_n (1)_n} = q+1F_{q+1} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_{q+1}|
\end{array} ; 1 \right) \frac{1}{(\text{Re} b_1)_n \cdots (\text{Re} b_q)_n (1)_n}
\]

(2.6)

\[
= \left( \prod_{m=1}^{q} \frac{\Gamma(\text{Re} b_m) \Gamma(\text{Re} b_m - |a_m| - 1)}{\Gamma(\text{Re} b_m - |a_m|) \Gamma(\text{Re} b_m - 1)} \right) \frac{\Gamma(\text{Re} b_q) \Gamma(\text{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re} b_q - |a_q|) \Gamma(\text{Re} b_q - |a_{q+1}|)}.
\]

Hence we have

\[
\sum_{n=2}^{\infty} (1 + |B|)(k(n - 1) + 1) \left| \frac{\prod_{j=1}^{p} |a_j|^{n-1}}{\prod_{i=1}^{q} (\text{Re} b_i)^n (1)_n} \right| \leq (A - B)(1 + |B|)|t|
\]

by assumption. This completes our proof.

Theorem 2.3 Let \(a_j (j = 1, 2, \cdots, p) \in C \setminus \{0\}, b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \text{Re} b_i > 0 (i = 1, 2, \cdots, q), \) and \(\sum_{i=1}^{q} \text{Re} b_i > \sum_{j=1}^{p} |a_j|\). If \(f(z) \in R^t(A, B)\) satisfies

\[
p+2F_{q+2} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p|, \lambda + 1, 1
\end{array} ; 1 \right) \leq \frac{1}{(A - B)|t|} + 1,
\]

then \(I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_p} (f) \in S_\lambda^*\) where \(\lambda > 0\).

Proof. Suppose that \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R^t(A, B)\). Then, by (1.5) it suffices to show that

\[
T_2 := \sum_{n=2}^{\infty} (n + \lambda - 1) \left| \frac{\prod_{j=1}^{p} (a_j)^{n-1}}{\prod_{i=1}^{q} (\text{Re} b_i)^n (1)^n} a_n \right| \leq \lambda.
\]

From Lemma 2.3, we observe that

\[
T_2 \leq \sum_{n=2}^{\infty} (n + \lambda - 1) \frac{(A - B)|t|}{n} \left| \frac{\prod_{j=1}^{p} (|a_j|)_n}{\prod_{i=1}^{q} (\text{Re} b_i)_n (1)_n} \right|
\]
\[= (A - B)|t| \sum_{n=1}^{\infty} \frac{\lambda(\lambda + 1)_{n}(1)_{n}}{(\lambda)_{n}} \left\{ \frac{\prod_{j=1}^{\infty}(|a_{j}|)_{n}}{\prod_{i=1}^{\infty}(\text{Re } b_{i})_{n}(1)_{n}} \right\} \]

\[\leq \lambda \]

by (2.7), which completes the proof of Theorem 2.3.

**Corollary 2.2** Let \(a_{j} \ (j = 1, 2, \cdots, q + 1) \in C \setminus \{0\}, \ b_{i} \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \)
\(\text{Re } b_{m} > |a_{m}| + 1 \ (m = 1, 2, \cdots, q - 1), \ |a_{q}| < 1 \) and \(\text{Re } b_{q} - 2 > \lambda > |a_{q+1}| + 1. \) If \(f(z) \in R^{t}(A, B)\) satisfies

\[\frac{(\lambda - 1)(\text{Re } b_{q} - 1)}{(1 - |a_{q}|)(\lambda - |a_{q+1}| - 1)(\text{Re } b_{q} - \lambda - 2)} \times \left( \prod_{m=1}^{q-1} \frac{\text{Re } b_{m} - 1}{\text{Re } b_{m} - |a_{m}| - 1} \right) \leq \frac{1}{(A - B)|t| + 1}, \]

then \(f_{b_{1}, b_{2}, \cdots, b_{q}}^{a_{1}, a_{2}, \cdots, a_{q+1}}(f) \in S_{\lambda}^{*}\) where \(\lambda > 0.\)

**Proof.** We note that

\[q+3F_{q+2} \left( \begin{array}{c}
|a_{1}|, \ |a_{2}|, \cdots, |a_{q+1}|, \lambda + 1, 1 \\
\text{Re } b_{1}, \text{Re } b_{2}, \cdots, \text{Re } b_{q}, \lambda, 2
\end{array} ; 1 \right) \]

\[= \left( \prod_{m=1}^{q-1} \frac{\Gamma(\text{Re } b_{m}) \Gamma(\text{Re } b_{m} - |a_{m}| - 1)}{\Gamma(\text{Re } b_{m}) \Gamma(\text{Re } b_{m} - |a_{m}| - 1)} \right) \frac{\Gamma(2) \Gamma(1 - |a_{q}|) \Gamma(\lambda) \Gamma(\lambda - |a_{q+1}| - 1)}{\Gamma(2) \Gamma(1 - |a_{q}|) \Gamma(\lambda) \Gamma(\lambda - |a_{q+1}| - 1)} \times \frac{\Gamma(\text{Re } b_{q}) \Gamma(\text{Re } b_{q} - 1) \Gamma(\text{Re } b_{q} - \lambda - 2)}{\Gamma(\text{Re } b_{q}) \Gamma(\text{Re } b_{q} - 1) \Gamma(\text{Re } b_{q} - \lambda - 1)} \]

\[= \left( \prod_{m=1}^{q-1} \frac{\text{Re } b_{m} - 1}{\text{Re } b_{m} - |a_{m}| - 1} \right) \frac{1}{1 - |a_{q}|} \frac{\lambda - 1}{\lambda - |a_{q+1}| - 1} \frac{\text{Re } b_{q} - 1}{\text{Re } b_{q} - \lambda - 2}. \]

Hence we have

\[\sum_{n=2}^{\infty} (n + \lambda - 1) \left| \frac{\prod_{j=1}^{q+1}(a_{j})_{n-1}}{\prod_{i=1}^{q}(b_{i})_{n-1}(1)_{n-1}} a_{n} \right| \]

\[= (A - B)|t| \sum_{n=1}^{\infty} \frac{\lambda(\lambda + 1)_{n}(1)_{n}}{(\lambda)_{n}} \left\{ \frac{\prod_{j=1}^{\infty}(|a_{j}|)_{n}}{\prod_{i=1}^{\infty}(\text{Re } b_{i})_{n}(1)_{n}} \right\} \]
\[
= \lambda(A - B)|t| \left\{ q+3F_{q+2} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_{q+1}|, \lambda + 1, 1 \\
\Re b_1, \Re b_2, \cdots, \Re b_q, \lambda, 2 \end{array} \right) - 1 \right\}
\leq \lambda,
\]
by assumption, which completes the proof of Corollary 2.2.

**Theorem 2.4** Let \( a_j (j = 1, 2, \cdots, p) \in C \setminus \{0\}, \) \( b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \Re b_i > 0 (i = 1, 2, \cdots, q), \) and \( \sum_{i=1}^{q} \Re b_i > \sum_{j=1}^{p} |a_j| + 1. \) If \( f(z) \in R^t(A, B) \) satisfies

\[
(2.10) \quad \begin{array}{c}
p+1F_{q+1} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p|, \lambda + 1 \\
\Re b_1, \Re b_2, \cdots, \Re b_q, \lambda \end{array} \right) \leq \frac{1}{(A - B)|t|} + 1,
\end{array}
\]

then \( I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_p} (f) \in C_\lambda \) where \( \lambda > 0. \)

**Proof.** Since the proof follows from Lemma 2.3 and by using the method of the proof of Theorem 2.3, we omit the details.

**Corollary 2.3** Let \( a_j (j = 1, 2, \cdots, q + 1) \in C \setminus \{0\}, \) \( b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \Re b_m > |a_m| + 1 (m = 1, 2, \cdots, q - 1), \) and \( \Re b_q - 2 > \lambda > |a_q| + |a_{q+1}|. \) If \( f(z) \in R^t(A, B) \) satisfies

\[
(2.11) \quad \frac{(\Re b_q - 1)\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{(\Re b_q - \lambda - 2)\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \times \left( \prod_{m=1}^{q-1} \frac{\Re b_m - 1}{\Re b_m - |a_m| - 1} \right) \leq \frac{1}{(A - B)|t|} + 1,
\]

then \( I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_{q+1}} (f) \in C_\lambda \) where \( \lambda > 0. \)

**Proof.** We note that

\[
q+2F_{q+1} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_{q+1}|, \lambda + 1 \\
\Re b_1, \Re b_2, \cdots, \Re b_q, \lambda \end{array} \right) = \left( \prod_{m=1}^{q-1} \frac{\Gamma(\Re b_m)\Gamma(\Re b_m - |a_m| - 1)}{\Gamma(\Re b_m - |a_m|)\Gamma(\Re b_m - 1)} \right) \frac{\Gamma(\Re b_q)\Gamma(\Re b_q - \lambda - 2)}{\Gamma(\Re b_q - 1)\Gamma(\Re b_q - \lambda - 1)} \times \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \times \left( \prod_{m=1}^{q-1} \frac{\Re b_m - 1}{\Re b_m - |a_m| - 1} \right) \frac{\Re b_q - 1}{\Re b_q - \lambda - 2} \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)}.
\]
Hence we observe that
\[
\sum_{n=2}^{\infty} n(n + \lambda - 1) \left| \frac{\prod_{j=1}^{q}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}} a_n \right| = (A - B)|t| \sum_{n=1}^{\infty} \frac{\lambda(\lambda + 1)_{n}}{(\lambda)_{n}} \left\{ \frac{\prod_{j=1}^{p}(|a_j|)_{n}}{\prod_{i=1}^{q}(\Re b_i)_{n}(1)_{n}} \right\} = \lambda(A - B)|t| \{q + 2F_{q+1} - 1\} \leq \lambda,
\]
by assumption. This completes our proof.

**Theorem 2.5** Let \( a_j (j = 1, 2, \cdots, p) \in C \setminus \{0\}, \) \( b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \Re b_i > 0 (i = 1, 2, \cdots, q), \) and \( \Sigma_{i=1}^{q} \Re b_i > \Sigma_{j=1}^{p} |a_j| + 1. \) If

\[
(2.12) \quad k \frac{\prod_{j=1}^{p}|a_j|}{\prod_{i=1}^{q} \Re b_i} pF_q \left( \begin{array}{c}
|a_1| + 1, |a_2| + 1, \cdots, |a_p| + 1 \\
\Re b_1 + 1, \Re b_2 + 1, \cdots, \Re b_q + 1 
\end{array} ; 1 \right) + pF_q \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p| \\
\Re b_1, \Re b_2, \cdots, \Re b_q 
\end{array} ; 1 \right) \leq (A - B)|t| + (1 + \Re b_q + 1),
\]

then
\[
z pF_q \left( \begin{array}{c}
a_1, a_2, \cdots, a_p \\
b_1, b_2, \cdots, b_q 
\end{array} ; z^k \right) \in R^t(A, B),
\]
where \( k \in N. \)

**Proof.** By Lemma 2.3, it suffices to show that

\[
(2.13) \quad T_3 := \sum_{n=2}^{\infty} (1 + \Re b_q)(k(n - 1) + 1) \left| \frac{\prod_{j=1}^{p}|a_j|_{n-1}}{\prod_{i=1}^{q}(\Re b_i)_{n-1}(1)_{n-1}} \right| \leq (A - B)|t|.
\]

Then we have,
\[
T_3 \leq (1 + \Re b_q) \sum_{n=2}^{\infty} (kn - (k - 1)) \frac{\prod_{j=1}^{p}|a_j|_{n-1}}{\prod_{i=1}^{q}(\Re b_i)_{n-1}(1)_{n-1}}
= (1 + \Re b_q) k p+1 F_{q+1} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p|, 2 \\
\Re b_1, \Re b_2, \cdots, \Re b_q, 1 
\end{array} \right)
\]
$-(1+|B|)(k-1)_{pq}F-(1+|B|) = (1+|B|)k \frac{\prod_{j=1}^{p}|a_{j}|}{\prod_{i=1}^{q}\text{Re} b_{i}}pqF + (1+|B|)_{pq}F-(1+|B|)
\leq (A-B)|t|$

by (2.12), which completes the proof of Theorem 2.5.

Corollary 2.4 Let $a_{j} (j = 1, 2, \cdots, q + 1) \in C \setminus \{0\}$, $b_{i} (i = 1, 2, \cdots, q) \in C \setminus \{0\}$, $\text{Re} b_{m} > |a_{m}|+1 (m = 1, 2, \cdots, q-1)$, and $\text{Re} b_{q} > |a_{q}|+1$. If

(2.14) \[ \frac{\Gamma(\text{Re} b_{q})\Gamma(\text{Re} b_{q} - |a_{q}| - |a_{q+1}| - 1)}{\Gamma(\text{Re} b_{q} - |a_{q}|)\Gamma(\text{Re} b_{q} - |a_{q+1}| - 1)} \left( \prod_{m=1}^{q-1} \frac{1}{\text{Re} b_{m} - |a_{m}| - 1} \right)
\leq \frac{(A-B)|t|}{1+|B|} + 1,
\]

then

$z_{q+1}F_{q} \left( a_{1}, a_{2}, \cdots, a_{q+1} \right) \in R^{t}(A, B)$,

where $k \in N$.

Proof. We note that

$q+1F_{q} \left( |a_{1}|+1, |a_{2}|+1, \cdots, |a_{q+1}|+1 \right)
\text{Re} b_{1}+1, \text{Re} b_{2}+1, \cdots, \text{Re} b_{q}+1
\leq \left( \prod_{m=1}^{q-1} \frac{\Gamma(\text{Re} b_{m} + 1)\Gamma(\text{Re} b_{m} - |a_{m}| - 1)}{\Gamma(\text{Re} b_{m})\Gamma(\text{Re} b_{m} - |a_{m}|)} \right) \frac{\Gamma(\text{Re} b_{q} + 1)\Gamma(\text{Re} b_{q} - |a_{q}| - |a_{q+1}| - 1)}{\Gamma(\text{Re} b_{q} - |a_{q}|)\Gamma(\text{Re} b_{q} - |a_{q+1}|)}
\leq \left( \prod_{m=1}^{q-1} \frac{\text{Re} b_{m}}{\text{Re} b_{m} - |a_{m}| - 1} \right) \frac{\Gamma(\text{Re} b_{q} + 1)\Gamma(\text{Re} b_{q} - |a_{q}| - |a_{q+1}| - 1)}{\Gamma(\text{Re} b_{q} - |a_{q}|)\Gamma(\text{Re} b_{q} - |a_{q+1}|)}.

From above equality and (2.6), we have the result of Corollary 2.4.
3. Uniformly starlikeness and convexity

A function $f(z) \in A$ is said to be uniformly starlike in $U$ if it satisfies

\begin{equation}
\Re \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq 0
\end{equation}

for all $(z, \zeta) \in U \times U$. We denote by $UST$ the subclass of $A$ consisting of all uniformly starlike functions in $U$. Further, a function $f(z) \in A$ is said to be uniformly convex in $U$ if and only if

\begin{equation}
\Re \left\{ 1 + (z - \zeta)\frac{f''(z)}{f'(z)} \right\} \geq 0
\end{equation}

for all $(z, \zeta) \in U \times U$. We also denote by $UCV$ the class of all such functions.

The classes $UST$ and $UCV$ were defined by Goodman [6,7] and studied recently by Rønning [13]. By the result of Rønning [13], we see that $f(z) \in UCV$ if and only if

\begin{equation}
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| (z \in U).
\end{equation}

In view of definitions of $UST$ and $UCV$, we define the following classes:

**Definition 3.1** A function $f(z)$ in $A$ is said to be a member of the class $UST(\alpha)$ if it satisfies

\begin{equation}
\Re \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq \alpha \quad ((z, \zeta) \in U \times U)
\end{equation}

for some real $\alpha (0 \leq \alpha < 1)$.

**Definition 3.2** A function $f(z)$ belonging to $A$ is called as a member of the class $UCV(\alpha)$ if and only if

\begin{equation}
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha \left| \frac{zf''(z)}{f'(z)} \right| (z \in U)
\end{equation}

for some real $\alpha (\alpha \geq 0)$.

Note that $UST(\alpha) \subset UST (0 \leq \alpha < 1)$, $UCV(\alpha) \subset UCV (\alpha \geq 1)$ and $UCV \subset UCV(\alpha) (0 \leq \alpha < 1)$. Now, we derive the following lemmas for functions $f(z) \in A$ to be in the classes $UST(\alpha)$ and $UCV(\alpha)$.

**Lemma 3.1** If $f(z) \in A$ satisfies $\sum_{n=2}^{\infty} n(n(\alpha+1) - \alpha)|a_n| \leq 1$, then $f(z)$ is in $UCV(\alpha)$. 
Proof. It suffices to show that

$$\alpha \left| \frac{zf''(z)}{f'(z)} \right| - \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \leq 1.$$  \hfill (3.6)

We have

$$\alpha \left| \frac{zf''(z)}{f'(z)} \right| - \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \leq (\alpha + 1) \left| \frac{zf''(z)}{f'(z)} \right|$$

$$= \frac{\left| \alpha + 1 \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} \right|}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty} (\alpha + 1)n(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}.$$  

Now this last expression is bounded above by 1 provided that \(\sum_{n=2}^{\infty} n(n+\alpha-1)|a_n| \leq 1\).

Lemma 3.2 If \(f(z) \in A\) satisfies \(\sum_{n=2}^{\infty} ((3-\alpha)n-2)|a_n| \leq 1-\alpha\), then \(f(z)\) is in \(UST(\alpha)\).

Proof. It suffices to show that

$$\left| \frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} - 1 \right| \leq 1 - \alpha.$$  \hfill (3.7)

We have

$$\left| \frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} - 1 \right| = \frac{\sum_{n=2}^{\infty} a_n (z^{n-1} + z^{n-2}\zeta + \cdots + \zeta^{n-1}) - \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty} 2(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|},$$

which is bounded above by \(1 - \alpha\) if \(\sum_{n=2}^{\infty} ((3-\alpha)n-2)|a_n| \leq 1 - \alpha\).

Theorem 3.1 Let \(a_j (j = 1, 2, \ldots, p) \in C \setminus \{0\}\), \(b_i (i = 1, 2, \ldots, q) \in C \setminus \{0\}\), \(\text{Re} b_i > 0 (i = 1, 2, \ldots, q)\), and \(\sum_{i=1}^{q} \text{Re} b_i > \sum_{j=1}^{p} |a_j| + 1\). If \(f(z) \in R^t(A, B)\) satisfies

$$\frac{(1 + \alpha) \prod_{j=1}^{p} |a_j|}{\prod_{i=1}^{q} \text{Re} b_i} F_q \left( \begin{array}{c} |a_1| + 1, |a_2| + 1, \ldots, |a_p| + 1 \\ \text{Re} b_1 + 1, \text{Re} b_2 + 1, \ldots, \text{Re} b_q + 1 \end{array} ; 1 \right)$$

$$+ \sum_{p} F_q \left( \begin{array}{c} |a_1|, |a_2|, \ldots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \ldots, \text{Re} b_q \end{array} ; 1 \right) \leq \frac{1}{(A-B)|t|} + 1,$$

then \(f_{b_1,b_2,\ldots,b_q}(f) \in UCV(\alpha)\).
Proof. By Lemma 3.1, we need only to show that

\begin{equation}
S_1 := \sum_{n=2}^{\infty} n(n(\alpha + 1) - \alpha) \left| \frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}} a_n \right| \leq 1.
\end{equation}

From Lemma 2.2, we have,

\begin{align*}
S_1 & \leq (A - B) |t| \sum_{n=2}^{\infty} (n(\alpha + 1) - \alpha) \frac{\prod_{j=1}^{p}|a_j|_{n-1}}{\prod_{i=1}^{q}(\text{Re} b_i)_{n-1}} \\
& = (A - B) |t|(\alpha + 1)_{p+1}F_{q+1} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p| \\
\text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q, 1
\end{array} ; 1 \right) \\
& - (A - B) |t| \alpha F_q \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p| \\
\text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q
\end{array} ; 1 \right) - (A - B) |t|
\end{align*}

\begin{align*}
& = (A - B) |t|(\alpha + 1) \frac{\prod_{j=1}^{p}|a_j|}{\prod_{i=1}^{q}\text{Re} b_i} \left( \begin{array}{c}
|a_1| + 1, |a_2| + 1, \cdots, |a_p| + 1 \\
\text{Re} b_1 + 1, \text{Re} b_2 + 1, \cdots, \text{Re} b_q + 1
\end{array} ; 1 \right) \\
& + (A - B) |t| \alpha F_q \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p| \\
\text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q
\end{array} ; 1 \right) - (A - B) |t|
\end{align*}

\begin{align*}
& \leq 1
\end{align*}

by (3.8), which completes the proof of Theorem 3.1.

**Corollary 3.1** Let \( a_j (j = 1, 2, \cdots, q+1) \in C \setminus \{0\} \), \( b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\} \), \( \text{Re} b_m > |a_m| + 1 \) \((m = 1, 2, \cdots, q-1)\), and \( \text{Re} b_q > |a_q| + |a_{q+1}| + 1 \). If \( f(z) \in R^t(A, B) \) satisfies

\begin{equation}
\frac{\Gamma(\text{Re} b_q)\Gamma(\text{Re} b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\text{Re} b_q - |a_q|)\Gamma(\text{Re} b_q - |a_{q+1}|)} \left( \begin{array}{c}
1 \\
\prod_{m=1}^{q-1} \text{Re} b_m - |a_m| - 1
\end{array} \right) \\
\times \left\{ (\alpha + 1) \prod_{j=1}^{q+1} |a_j| + \left( \prod_{m=1}^{q-1} \text{Re} b_m - 1 \right) (\text{Re} b_q - |a_q| - |a_{q+1}| - 1) \right\}
\end{equation}

\begin{align*}
& \leq \frac{1}{(A - B)|t|} + 1,
\end{align*}

then \( f_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_{q+1}}(f) \in UCV(\alpha) \).

Proof. Since the proof is similarly the proof of Corollary 2.4, we omit the details.
Theorem 3.2 Let \( a_j \ (j = 1, 2, \cdots, p) \in C \setminus \{0\}, \) \( b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \text{Re} b_i > 0 \ (i = 1, 2, \cdots, q), \) and \( \sum_{i=1}^{q} \text{Re} b_i > \sum_{j=1}^{p} |a_j|. \) If \( f(z) \in R^t(A, B) \) satisfies

\[
(3 - \alpha) p F_q \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p| \\
\text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q \\
1
\end{array} ; 1 \right) - 2 p+1 F_{q+1} \left( \begin{array}{c}
|a_1|, |a_2|, \cdots, |a_p|, 1 \\
\text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q, 2 \\
1
\end{array} ; 1 \right) \leq (1 - \alpha) \left( \frac{1}{(A - B)|t|} + 1 \right),
\]

then \( I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_p} (f) \in UST(\alpha), \) for some \( \alpha \ (0 \leq \alpha < 1). \)

Proof. By Lemma 3.2, we need only to show that

\[
S_2 := \sum_{n=2}^{\infty} \left( (3 - \alpha)n - 2 - \frac{\prod_{j=1}^{p} (|a_j|)_{n-1}}{\prod_{i=1}^{q} (\text{Re} b_i)_{n-1}(1)_{n-1}} \right) \leq 1 - \alpha.
\]

From Lemma 2.2, we have,

\[
S_2 \leq (A - B)|t|(3 - \alpha) \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} (|a_j|)_{n-1}}{\prod_{i=1}^{q} (\text{Re} b_i)_{n-1}(1)_{n-1}} - 2(A - B)|t| \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} (|a_j|)_{n-1}(1)_{n-1}}{\prod_{i=1}^{q} (\text{Re} b_i)_{n-1}(1)_{n-1}} \leq 1 - \alpha
\]

by (3.11), which completes the proof of Theorem 3.2.

Corollary 3.2 Let \( a_j \ (j = 1, 2, \cdots, q + 1) \in C \setminus \{0\}, \) \( b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \text{Re} b_m \geq |a_m| + 1 \ (m = 1, 2, \cdots, q - 1), \) and \( \text{Re} b_q > |a_q| + 1, \) \( |a_{q+1}| < 1. \) If \( f(z) \in R^t(A, B) \) satisfies

\[
(3 - \alpha) \frac{\Gamma(\text{Re} b_q - 1)\Gamma(\text{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re} b_q - |a_q| - 1)\Gamma(\text{Re} b_q - |a_{q+1}|)} \leq \frac{2}{1 - |a_{q+1}|}
\]

\[
\times \left( \sum_{m=1}^{q} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \leq (1 - \alpha) \left( \frac{1}{(A - B)|t|} + 1 \right),
\]

then \( I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_{q+1}} (f) \in UST(\alpha), \) for some \( \alpha \ (0 \leq \alpha < 1). \)
Proof. We note that
\[
\begin{split}
q+2F_{q+1}\left(\begin{array}{c}
|a_{1}|, |a_{2}|, \ldots, |a_{q+1}|, 1 \\
\text{Re}b_{1}, \text{Re}b_{2}, \ldots, \text{Re}b_{q}, 2
\end{array}
; 1\right) \\
= \left(\prod_{m=1}^{q-1} \frac{\text{Re}b_{m} - 1}{\text{Re}b_{m} - |a_{m}| - 1}\right) \frac{\text{Re}b_{q} - 1}{(\text{Re}b_{q} - |a_{q}| - 1)(1 - |a_{q+1}|)}.
\end{split}
\]
From above equality and (2.6), we have the result of Corollary 3.2.

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