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Some Properties for Convolutions of Generalized Hypergeometric Functions

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Abstract

With the convolution products of generalized hypergeometric functions ${}_pF_q(z)$ and analytic functions $f(z)$ in the open unit disk, the operator $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(f)$ is introduced. The object of the present paper is to derive some interesting properties of operator $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(f)$ associated with some classes of univalent functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$. Denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in U.

A function $f(z)\in\mathcal{A}$ is said to be in the class $R^t(A,B)$ if

(1.2) $$\left|\frac{f'(z)-1}{t(A-B)-B(f'(z)-1)}\right|<1,$$

where A and B are arbitrary fixed numbers with $-1\leq B<A\leq 1$ and $t\in C\setminus\{0\}$ (C is the set of all complex numbers). Clearly, a function $f(z)$ belongs to $R^t(A,B)$ if and only if there exists a function $w(z)$ regular in U satisfying $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that

(1.3) $$1+\frac{1}{t}(f'(z)-1)=\frac{1+Aw(z)}{1+Bw(z)}\quad(z\in U).$$

The class $R^t(A,B)$ was introduced by Dixit and Pal [4], recently.

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By giving specific values t, A and B in (1.2), we obtain the following subclasses studied by various researchers in earlier works :

(i) For $t = e^{-i\eta}\cos\eta$ ($|\eta| < \frac{\pi}{2}$), $A = 1-2\alpha$ ($0 \le \alpha < 1$) and $B = -1$, we obtain the class of functions f satisfying the condition

$$\left|\frac{e^{i\eta}(f'(z)-1)}{2(1-\alpha)\cos\eta + e^{i\eta}(f'(z)-1)}\right| < 1 \quad (z \in U). \tag{1.4}$$

In this case, the class $R^t(A,B)$ is equivalent to the class $R_\eta(\alpha)$ which is studied by Ponnusamy and Rϕnning [11]. Here $R_\eta(\alpha)$ is the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$Re(e^{i\eta}(f'(z)-\alpha)) > 0 \quad (|\eta| < \frac{\pi}{2},\ 0 \le \alpha < 1,\ z \in U).$$

(ii) For $t = e^{-i\eta}\cos\eta$ ($|\eta| < \frac{\pi}{2}$), we obtain the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left|\frac{e^{i\eta}(f'(z)-1)}{Be^{i\eta}f'(z) - (A\cos\eta + iB\sin\eta)}\right| < 1 \quad (z \in U),$$

which was studied by Dashrath [3].

(iii) For $t = 1$, $A = \beta$ and $B = -\beta$ ($0 < \beta \le 1$), we obtain the class of functions $f(z)$ satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (z \in U),$$

which was studied by Padmanabhan [10] and Caplinger and Cauchy [2].

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{S}$ consisting of starlike and convex functions of order α ($0 \le \alpha < 1$) in U, respectively. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*, \mathcal{C}(\alpha) \subset \mathcal{C}(0) \equiv \mathcal{C}$ and $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$. For $\lambda > 0$, define the classes $\mathcal{S}^*_\lambda$ and $\mathcal{C}_\lambda$ by

$$\mathcal{S}^*_\lambda = \{f(z) \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)}\right| < \lambda, z \in U\}$$

and

$$\mathcal{C}_\lambda = \{f(z) \in \mathcal{A} : zf'(z) \in \mathcal{S}^*_\lambda\},$$

respectively. It is a known fact that a sufficient condition for $f(z) \in \mathcal{A}$ of the form (1.1) to belong to the class $\mathcal{S}^*$ is that $\sum_{n=2}^{\infty} n|a_n| \le 1$. A simple extension of this result is the following [16] :

$$\sum_{n=2}^{\infty}(n+\lambda-1)|a_n| \le \lambda \Longrightarrow f(z) \in \mathcal{S}^*_\lambda. \tag{1.5}$$

For $\lambda = \frac{1}{2}$, this was previously proved by Schild [18]. Since $f(z) \in \mathcal{C}_\lambda$ if and only if $zf'(z) \in \mathcal{S}^*_\lambda$, we have a corresponding result for $\mathcal{C}_\lambda$,

$$\sum_{n=2}^{\infty} n(n+\lambda-1)|a_n| \le \lambda \Longrightarrow f(z) \in \mathcal{C}_\lambda. \tag{1.6}$$

In this paper, we consider the generalized hypergeometric series ${}_pF_q(z)$ defined by

$$ {}_pF_q(z) \equiv {}_pF_q\left(\begin{matrix} a_1, a_2, \cdots, a_p \\ \\ b_1, b_2, \cdots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{i=1}^{q}(b_i)_n} \frac{z^n}{(1)_n} \tag{1.7}$$

where p and q are positive integers and we assume that the variable z, the numerator parameters $a_1, a_2, \cdots, a_p$ and the denominator parameters $b_1, b_2, \cdots, b_q$ take on complex values, provided that $b_i \neq 0, -1, -2, \cdots$; $i = 1, 2, \cdots, q$. Here $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n \in N = \{1, 2, \cdots\}. \end{cases}$$

For any complex number λ, we also use the assending fractorial notation

$$(\lambda)_n = \lambda(\lambda+1)_{n-1} \tag{1.8}$$

for $n \ge 1$ and $(\lambda)_0 = 1$ for $\lambda \neq 0$. If λ is neither zero nor a negative integer, then using the definition of the Gamma function, we can write

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}. \tag{1.9}$$

Furthermore, if we set

$$\omega = \sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j$$

it is known that the series ${}_pF_q(z)$, with $p = q + 1$, is
(i) absolutely convergent for $|z| = 1$ if Re $\omega > 0$,
(ii) conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 <$ Re $\omega \le 0$
and
(iii) divergent for $|z| = 1$ if Re $\omega \le -1$.

As in the case of the function ${}_2F_1(z)$, we are led to the well-known Gauss summation theorem :

$${}_2F_1\left(\begin{matrix} a_1, a_2 \\ \\ b_1 \end{matrix} ; 1 \right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}, \quad \mathrm{Re}(b_1 - a_1 - a_2) > 0. \tag{1.10}$$

We recall thet the function ${}_2F_1(z)$ is bounded if $\mathrm{Re}(b_1 - a_1 - a_2) > 0$ and has a pole at $z = 1$ if $\mathrm{Re}(b_1 - a_1 - a_2) \leq 0$ (cf. [1]). Univalence, starlikeness and convexity properties of $z\,{}_2F_1(\binom{a_1,a_2}{b_1}; z)$ have been studied extensively in [12, 15].

For $f(z) \in \mathcal{A}$, we define the operator $I^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_q}(f)$ by

$$\left[I^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_q}(f)\right](z) = z\,{}_pF_q\left(\begin{matrix} a_1, a_2, \cdots, a_p \\ \\ b_1, b_2, \cdots, b_q \end{matrix}\; ; z\right) * f(z), \tag{1.11}$$

where the symbol " $*$" denotes the usual Hadamard product or convolution of power series.

2. Properties of the operators with $R^t(A, B)$

Now we introduce several lemmas which are needed for the proof of our main results.

Lemma 2.1 ([8]) *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_1 \in U$, then we can write*

$$z_1 w'(z_1) = m w(z_1),$$

where m is real and $m \geq 1$.

Lemma 2.2 ([4]) *Let a function $f(z)$ of the form* (1.1) *be in $R^t(A, B)$. Then*

$$|a_n| \leq \frac{(A-B)|t|}{n}.$$

Then result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A-B)tz^{n-1}}{1 + Bz^{n-1}}\right) dz \quad (n \geq 2, z \in U).$$

Lemma 2.3 ([4]) *Let a function $f(z)$ of the form* (1.1) *be in $\mathcal{A}$. If*

$$\sum_{n=2}^{\infty} (1 + |B|)n|a_n| \leq (A - B)|t| \quad (-1 \leq B < A \leq 1,\ t \in C \setminus \{0\})$$

then $f(z) \in R^t(A, B)$. The result is sharp for function

$$f(z) = z + \frac{(A-B)t}{(1+|B|)n} z^n \quad (n \geq 2, z \in U).$$

Our first result for the operators is contained in

Theorem 2. 1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{I^{a_1,a_2,\cdots,a_p,2}_{b_1,b_2,\cdots,b_q,1}(f)}{z}-1\right|^{1-\beta}\left|\frac{I^{a_1,a_2,\cdots,a_p,2,2}_{b_1,b_2,\cdots,b_q,1,1}(f)}{I^{a_1,a_2,\cdots,a_p,2}_{b_1,b_2,\cdots,b_q,1}(f)}-1\right|^{\beta}<\left(\frac{1}{2}\right)^{\beta} \tag{2.1}$$

for some fixed $\beta \geq 0$, then $I^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_q}(f)$ is univalent (close-to-convex) in U.

Proof. We note that

$$I^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_q}(f)=z+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n z^n$$

in $\mathcal{A}$. Define $w(z)$ by

$$w(z)=\frac{I^{a_1,a_2,\cdots,a_p,2}_{b_1,b_2,\cdots,b_q,1}(f)}{z}-1$$

for $z \in U$. Then it follows that $w(z)$ is analytic in U with $w(0)=0$. By (2.1), it is clear that

$$|w(z)|^{1-\beta}\left|\frac{zw'(z)}{1+w(z)}\right|^{\beta}=|w(z)|\left|\frac{zw'(z)}{w(z)(1+w(z))}\right|^{\beta}<\left(\frac{1}{2}\right)^{\beta}. \tag{2.2}$$

Suppose that there exists a point $z_1 \in U$ such that

$$\max_{|z|\leq|z_1|}|w(z)|=|w(z_1)|=1.$$

Then we can put

$$\frac{z_1w'(z_1)}{w(z_1)}=m\geq 1,$$

by Lemma 2.1. Therefore we obtain

$$|w(z_1)|\left|\frac{z_1w'(z_1)}{w(z_1)(1+w(z_1))}\right|^{\beta}\geq\left(\frac{m}{2}\right)^{\beta}\geq\left(\frac{1}{2}\right)^{\beta},$$

which contradicts the condition (2.2). This shows that

$$|w(z)|=\left|\frac{I^{a_1,a_2,\cdots,a_p,2}_{b_1,b_2,\cdots,b_q,1}(f)}{z}-1\right|<1,$$

which implies that $\mathrm{Re}\left[I^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_q}(f)\right]'(z)>0$ for $z \in U$. Therefore, by Noshiro-Warschawski Theroem [5], $I^{a_1,a_2,\cdots,a_p}_{b_1,b_2,\cdots,b_q}(f)$ is univalent (close-to-convex) in U.

Theorem 2. 2 *Let* a_j $(j = 1,2,\cdots,p)$ $\in C\setminus\{0\}$, b_i $(i = 1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_i > 0$ $(i = 1,2,\cdots,q)$, *and* $\sum_{i=1}^{q}\mathrm{Re}b_i > \sum_{j=1}^{p}|a_j|$. *If* $f(z) \in R^t(A,B)$ *satisfies*

$$(2.3)\qquad {}_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p| \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right) \le \frac{1}{1+|B|}+1,$$

then

$$z\ {}_pF_q\left(\begin{matrix}a_1,a_2,\cdots,a_p \\ \\ b_1,b_2,\cdots,b_q\end{matrix};z^k\right) * f(z) \in R^t(A,B),$$

where $k \in N$.

Proof. By Lemma 2.3, it suffices to show that

$$(2.4)\qquad T_1 := \sum_{n=2}^{\infty}(1+|B|)(k(n-1)+1)\left|\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_{k(n-1)+1}\right| \le (A-B)|t|.$$

From Lemma 2.2 and the fact that $|(a)_n| \le (|a|)_n$ and $(\mathrm{Re}b)_n \le |(b)_n|$, $\mathrm{Re}b > 0$, we have

$$\begin{aligned}
T_1 &\le \sum_{n=2}^{\infty}(A-B)(1+|B|)|t|\left\{\frac{\prod_{j=1}^{p}(|a_j|)_{n-1}}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_{n-1}(1)_{n-1}}\right\} \\
&= (A-B)(1+|B|)|t|\left\{\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(|a_j|)_n}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_n(1)_n}-1\right\} \\
&= (A-B)(1+|B|)|t|\left\{{}_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p| \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right)-1\right\} \\
&\le (A-B)|t|
\end{aligned}$$

by (2.3). This completes the proof of Theroem 2.2.

Corollary 2. 1 *Let* a_j $(j = 1,2,\cdots,q+1)$ $\in C\setminus\{0\}$, b_i $(i = 1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_m > |a_m|+1$ $(m = 1,2,\cdots,q-1)$, *and* $\mathrm{Re}b_q > |a_q|+|a_{q+1}|$. *If* $f(z) \in R^t(A,B)$ *satisfies*

$$(2.5)\qquad \frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|)}{\Gamma(\mathrm{Re}b_q-|a_q|)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}\left(\prod_{m=1}^{q-1}\frac{\mathrm{Re}b_m-1}{\mathrm{Re}b_m-|a_m|-1}\right) \le \frac{1}{1+|B|}+1,$$

then

$$z\ {}_{q+1}F_q\left(\begin{matrix}a_1,a_2,\cdots,a_{q+1} \\ \\ b_1,b_2,\cdots,b_q\end{matrix};z^k\right) * f(z) \in R^t(A,B),$$

where $k \in N$.

Proof. We note that

$$(2.6)\quad {}_{q+1}F_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_{q+1}| \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right)=\sum_{n=0}^{\infty}\frac{(|a_1|)_n\cdots(|a_{q+1}|)_n}{(\mathrm{Re}b_1)_n\cdots(\mathrm{Re}b_q)_n(1)_n}$$

$$=\left(\prod_{m=1}^{q-1}\frac{\Gamma(\mathrm{Re}b_m)\Gamma(\mathrm{Re}b_m-|a_m|-1)}{\Gamma(\mathrm{Re}b_m-|a_m|)\Gamma(\mathrm{Re}b_m-1)}\right)\frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|)}{\Gamma(\mathrm{Re}b_q-|a_q|)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}$$

$$=\left(\prod_{m=1}^{q-1}\frac{\mathrm{Re}b_m-1}{\mathrm{Re}b_m-|a_m|-1}\right)\frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|)}{\Gamma(\mathrm{Re}b_q-|a_q|)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}.$$

Hence we have

$$\sum_{n=2}^{\infty}(1+|B|)(k(n-1)+1)\left|\frac{\prod_{j=1}^{q+1}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_{k(n-1)+1}\right|$$

$$\leq (A-B)(1+|B|)|t|\left\{\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q+1}(|a_j|)_n}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_n(1)_n}-1\right\}$$

$$=(A-B)(1+|B|)|t|\left\{{}_{q+1}F_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_{q+1}| \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right)-1\right\}$$

$$\leq (A-B)|t|,$$

by assumption. This completes our proof.

Theorem 2. 3 *Let* a_j $(j=1,2,\cdots,p)$ $\in C\setminus\{0\}$, b_i $(i=1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_i>0$ $(i=1,2,\cdots,q)$, *and* $\sum_{i=1}^{q}\mathrm{Re}b_i>\sum_{j=1}^{p}|a_j|$. *If* $f(z)\in R^t(A,B)$ *satisfies*

$$(2.7)\quad {}_{p+2}F_{q+2}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|,\lambda+1,1 \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,\lambda,2\end{matrix};1\right)\leq\frac{1}{(A-B)|t|}+1,$$

then $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(f)\in S_\lambda^*$ *where* $\lambda>0$.

Proof. Suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in R^t(A,B)$. Then, by (1.5) it suffices to show that

$$(2.8)\quad T_2:=\sum_{n=2}^{\infty}(n+\lambda-1)\left|\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n\right|\leq\lambda.$$

From Lemma 2.3, we observe that

$$T_2\leq\sum_{n=2}^{\infty}(n+\lambda-1)\frac{(A-B)|t|}{n}\left\{\frac{\prod_{j=1}^{p}(|a_j|)_{n-1}}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_{n-1}(1)_{n-1}}\right\}$$

$$= (A-B)|t|\sum_{n=1}^{\infty}\frac{\lambda(\lambda+1)_n}{(\lambda)_n}\frac{(1)_n}{(2)_n}\left\{\frac{\prod_{j=1}^{p}(|a_j|)_n}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_n(1)_n}\right\}$$

$$= \lambda(A-B)|t|\left\{{}_{p+2}F_{q+2}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|,\lambda+1,1\\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,\lambda,2\end{matrix};1\right)-1\right\}$$

$$\leq \lambda$$

by (2.7), which completes the proof of Theroem 2.3.

Corollary 2. 2 *Let* a_j $(j=1,2,\cdots,q+1)$ $\in C\setminus\{0\}$, b_i $(i=1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_m > |a_m|+1$ $(m=1,2,\cdots,q-1)$, $|a_q|<1$ *and* $\mathrm{Re}b_q-2>\lambda>|a_{q+1}|+1$. *If* $f(z)\in R^t(A,B)$ *satisfies*

$$\text{(2.9)}\qquad \frac{(\lambda-1)(\mathrm{Re}b_q-1)}{(1-|a_q|)(\lambda-|a_{q+1}|-1)(\mathrm{Re}b_q-\lambda-2)} \times\left(\prod_{m=1}^{q-1}\frac{\mathrm{Re}b_m-1}{\mathrm{Re}b_m-|a_m|-1}\right)\leq\frac{1}{(A-B)|t|}+1,$$

then $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_{q+1}}(f)\in S_\lambda^*$ *where* $\lambda>0$.

Proof. We note that

$${}_{q+3}F_{q+2}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_{q+1}|,\lambda+1,1\\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,\lambda,2\end{matrix};1\right)$$

$$=\left(\prod_{m=1}^{q-1}\frac{\Gamma(\mathrm{Re}b_m)\Gamma(\mathrm{Re}b_m-|a_m|-1)}{\Gamma(\mathrm{Re}b_m-|a_m|)\Gamma(\mathrm{Re}b_m-1)}\right)\frac{\Gamma(2)\Gamma(1-|a_q|)}{\Gamma(2-|a_q|)\Gamma(1)}\frac{\Gamma(\lambda)\Gamma(\lambda-|a_{q+1}|-1)}{\Gamma(\lambda-|a_{q+1}|)\Gamma(\lambda-1)}$$
$$\times\frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q-\lambda-2)}{\Gamma(\mathrm{Re}b_q-1)\Gamma(\mathrm{Re}b_q-\lambda-1)}$$
$$=\left(\prod_{m=1}^{q-1}\frac{\mathrm{Re}b_m-1}{\mathrm{Re}b_m-|a_m|-1}\right)\frac{1}{1-|a_q|}\frac{\lambda-1}{\lambda-|a_{q+1}|-1}\frac{\mathrm{Re}b_q-1}{\mathrm{Re}b_q-\lambda-2}.$$

Hence we have

$$\sum_{n=2}^{\infty}(n+\lambda-1)\left|\frac{\prod_{j=1}^{q+1}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n\right|$$
$$=(A-B)|t|\sum_{n=1}^{\infty}\frac{\lambda(\lambda+1)_n}{(\lambda)_n}\frac{(1)_n}{(2)_n}\left\{\frac{\prod_{j=1}^{q+1}(|a_j|)_n}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_n(1)_n}\right\}$$

$$= \lambda(A-B)|t| \left\{ {}_{q+3}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_{q+1}|, \lambda+1, 1 \\ \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q, \lambda, 2 \end{matrix} ; 1 \right) - 1 \right\}$$

$$\leq \lambda,$$

by assumption, which completes the proof of Corollary 2.2.

Theorem 2. 4 *Let* a_j $(j = 1, 2, \cdots, p)$ $\in C \setminus \{0\}$, b_i $(i = 1, 2, \cdots, q)$ $\in C \setminus \{0\}$, $\mathrm{Re}b_i > 0$ $(i = 1, 2, \cdots, q)$, *and* $\sum_{i=1}^{q} \mathrm{Re}b_i > \sum_{j=1}^{p} |a_j| + 1$. *If* $f(z) \in R^t(A, B)$ *satisfies*

$$(2.10) \qquad {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, \lambda+1 \\ \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q, \lambda \end{matrix} ; 1 \right) \leq \frac{1}{(A-B)|t|} + 1,$$

then $I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_p}(f) \in \mathcal{C}_\lambda$ *where* $\lambda > 0$.

Proof. Since the proof follows from Lemma 2.3 and by using the method of the proof of Theorem 2.3, we omit the details.

Corollary 2. 3 *Let* a_j $(j = 1, 2, \cdots, q+1)$ $\in C \setminus \{0\}$, b_i $(i = 1, 2, \cdots, q)$ $\in C \setminus \{0\}$, $\mathrm{Re}b_m > |a_m| + 1$ $(m = 1, 2, \cdots, q-1)$, *and* $\mathrm{Re}b_q - 2 > \lambda > |a_q| + |a_{q+1}|$. *If* $f(z) \in R^t(A, B)$ *satisfies*

$$(2.11) \qquad \frac{(\mathrm{Re}b_q - 1)\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{(\mathrm{Re}b_q - \lambda - 2)\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \times \left(\prod_{m=1}^{q-1} \frac{\mathrm{Re}b_m - 1}{\mathrm{Re}b_m - |a_m| - 1} \right) \leq \frac{1}{(A-B)|t|} + 1,$$

then $I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_{q+1}}(f) \in \mathcal{C}_\lambda$ *where* $\lambda > 0$.

Proof. We note that

$$\begin{aligned} & {}_{q+2}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_{q+1}|, \lambda+1 \\ \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q, \lambda \end{matrix} ; 1 \right) \\ = & \left(\prod_{m=1}^{q-1} \frac{\Gamma(\mathrm{Re}b_m)\Gamma(\mathrm{Re}b_m - |a_m| - 1)}{\Gamma(\mathrm{Re}b_m - |a_m|)\Gamma(\mathrm{Re}b_m - 1)} \right) \frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q - \lambda - 2)}{\Gamma(\mathrm{Re}b_q - 1)\Gamma(\mathrm{Re}b_q - \lambda - 1)} \\ & \times \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \\ = & \left(\prod_{m=1}^{q-1} \frac{\mathrm{Re}b_m - 1}{\mathrm{Re}b_m - |a_m| - 1} \right) \frac{\mathrm{Re}b_q - 1}{\mathrm{Re}b_q - \lambda - 2} \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)}. \end{aligned}$$

Hence we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n+\lambda-1)\left|\frac{\prod_{j=1}^{q+1}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n\right| \\
&= (A-B)|t|\sum_{n=1}^{\infty}\frac{\lambda(\lambda+1)_n}{(\lambda)_n}\left\{\frac{\prod_{j=1}^{q+1}(|a_j|)_n}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_n(1)_n}\right\} \\
&= \lambda(A-B)|t|\left\{ {}_{q+2}F_{q+1}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_{q+1}|,\lambda+1 \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,\lambda\end{matrix};1\right)-1\right\} \\
&\leq \lambda,
\end{aligned}$$

by assumption. This completes our proof.

Theorem 2. 5 *Let* a_j $(j = 1,2,\cdots,p)$ $\in C\setminus\{0\}$, b_i $(i = 1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_i > 0$ $(i = 1,2,\cdots,q)$, *and* $\sum_{i=1}^{q}\mathrm{Re}b_i > \sum_{j=1}^{p}|a_j|+1$. *If*

$$k\frac{\prod_{j=1}^{p}|a_j|}{\prod_{i=1}^{q}\mathrm{Re}b_i}\,{}_pF_q\left(\begin{matrix}|a_1|+1,|a_2|+1,\cdots,|a_p|+1 \\ \mathrm{Re}b_1+1,\mathrm{Re}b_2+1,\cdots,\mathrm{Re}b_q+1\end{matrix};1\right) + {}_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p| \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right) \leq \frac{(A-B)|t|}{1+|B|}+1, \tag{2.12}$$

then

$$z\,{}_pF_q\left(\begin{matrix}a_1,a_2,\cdots,a_p \\ b_1,b_2,\cdots,b_q\end{matrix};z^k\right)\in R^t(A,B),$$

where $k\in N$.

Proof. By Lemma 2.3, it suffices to show that

$$T_3 := \sum_{n=2}^{\infty}(1+|B|)(k(n-1)+1)\left|\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}\right| \leq (A-B)|t|. \tag{2.13}$$

Then we have,

$$\begin{aligned}
T_3 &\leq (1+|B|)\sum_{n=2}^{\infty}(kn-(k-1))\frac{\prod_{j=1}^{p}(|a_j|)_{n-1}}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_{n-1}(1)_{n-1}} \\
&= (1+|B|)k\,{}_{p+1}F_{q+1}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|,2 \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,1\end{matrix};1\right)
\end{aligned}$$

$$-(1+|B|)(k-1)_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|\\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right)-(1+|B|)$$

$$= (1+|B|)k\frac{\prod_{j=1}^p|a_j|}{\prod_{i=1}^q \mathrm{Re}b_i}\,{}_pF_q\left(\begin{matrix}|a_1|+1,|a_2|+1,\cdots,|a_p|+1\\ \\ \mathrm{Re}b_1+1,\mathrm{Re}b_2+1,\cdots,\mathrm{Re}b_q+1\end{matrix};1\right)$$

$$+(1+|B|)_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|\\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right)-(1+|B|)$$

$$\leq (A-B)|t|$$

by (2.12), which completes the proof of Theroem 2.5.

Corollary 2.4 *Let* a_j $(j=1,2,\cdots,q+1)$ $\in C\setminus\{0\}$, b_i $(i=1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_m>|a_m|+1$ $(m=1,2,\cdots,q-1)$, *and* $\mathrm{Re}b_q>|a_q|+|a_{q+1}|+1$. *If*

$$\frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|-1)}{\Gamma(\mathrm{Re}b_q-|a_q|)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}\left(\prod_{m=1}^{q-1}\frac{1}{\mathrm{Re}b_m-|a_m|-1}\right) \tag{2.14}$$

$$\times\left\{k\prod_{j=1}^{q+1}|a_j|+\left(\prod_{m=1}^{q-1}\mathrm{Re}b_m-1\right)(\mathrm{Re}b_q-|a_q|-|a_{q+1}|-1)\right\}$$

$$\leq \frac{(A-B)|t|}{1+|B|}+1,$$

then

$$z\ {}_{q+1}F_q\left(\begin{matrix}a_1,a_2,\cdots,a_{q+1}\\ \\ b_1,b_2,\cdots,b_q\end{matrix};z^k\right)\in R^t(A,B),$$

where $k\in N$.

Proof. We note that

$${}_{q+1}F_q\left(\begin{matrix}|a_1|+1,|a_2|+1,\cdots,|a_{q+1}|+1\\ \\ \mathrm{Re}b_1+1,\mathrm{Re}b_2+1,\cdots,\mathrm{Re}b_q+1\end{matrix};1\right)$$

$$=\left(\prod_{m=1}^{q-1}\frac{\Gamma(\mathrm{Re}b_m+1)\Gamma(\mathrm{Re}b_m-|a_m|-1)}{\Gamma(\mathrm{Re}b_m)\Gamma(\mathrm{Re}b_m-|a_m|)}\right)\frac{\Gamma(\mathrm{Re}b_q+1)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|-1)}{\Gamma(\mathrm{Re}b_q-|a_q|)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}$$

$$=\left(\prod_{m=1}^{q-1}\frac{\mathrm{Re}b_m}{\mathrm{Re}b_m-|a_m|-1}\right)\frac{\Gamma(\mathrm{Re}b_q+1)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|-1)}{\Gamma(\mathrm{Re}b_q-|a_q|)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}.$$

From above equality and (2.6), we have the result of Corollary 2.4.

3. Uniformly starlikeness and convexity

A function $f(z) \in \mathcal{A}$ is said to be uniformly starlike in U if it satisfies

$$\operatorname{Re}\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\} \geq 0 \tag{3.1}$$

for all $(z,\zeta) \in U \times U$. We denote by UST the subclass of $\mathcal{A}$ consisting of all uniformly starlike functions in U. Further, a functions $f(z) \in \mathcal{A}$ is said to be uniformly convex in U if and only if

$$\operatorname{Re}\left\{1+(z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0 \tag{3.2}$$

for all $(z,\zeta) \in U \times U$. We also denote by UCV the class of all such functions.

The classes UST and UCV were defined by Goodman [6,7] and studied recently by Rϕnning [13]. By the result of Rϕnning [13], we see that $f(z) \in UCV$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U). \tag{3.3}$$

In view of definitions of UST and UCV, we define the following classes :

Definition 3. 1 *A function $f(z)$ in $\mathcal{A}$ is said to be a member of the class $UST(\alpha)$ if it satisfies*

$$\operatorname{Re}\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\} \geq \alpha \quad ((z,\zeta) \in U \times U) \tag{3.4}$$

for some real α $(0 \leq \alpha < 1)$.

Definition 3. 2 *A function $f(z)$ belonging to $\mathcal{A}$ is called as a member of the class $UCV(\alpha)$ if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \alpha\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U) \tag{3.5}$$

for some real α $(\alpha \geq 0)$.

Note that $UST(\alpha) \subset UST$ $(0 \leq \alpha < 1)$, $UCV(\alpha) \subset UCV$ $(\alpha \geq 1)$ and $UCV \subset UCV(\alpha)$ $(0 \leq \alpha < 1)$. Now, we derive the following lemmas for functions $f(z) \in \mathcal{A}$ to be in the classes $UST(\alpha)$ and $UCV(\alpha)$.

Lemma 3. 1 *If $f(z) \in \mathcal{A}$ satisfies $\sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)|a_n| \leq 1$, then $f(z)$ is in $UCV(\alpha)$.*

Proof. It suffices to show that

$$\alpha\left|\frac{zf''(z)}{f'(z)}\right| - \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \leq 1. \tag{3.6}$$

We have

$$\begin{aligned}\alpha\left|\frac{zf''(z)}{f'(z)}\right| - \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) &\leq (\alpha+1)\left|\frac{zf''(z)}{f'(z)}\right| \\ &= \left|\frac{(\alpha+1)\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1+\sum_{n=2}^{\infty} n a_n z^{n-1}}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}(\alpha+1)n(n-1)|a_n|}{1-\sum_{n=2}^{\infty} n|a_n|}.\end{aligned}$$

Now this last expression is bounded above by 1 provided that $\sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)|a_n| \leq 1$.

Lemma 3. 2 *If $f(z) \in \mathcal{A}$ satisfies $\sum_{n=2}^{\infty}((3-\alpha)n-2)|a_n| \leq 1-\alpha$, then $f(z)$ is in $UST(\alpha)$.*

Proof. It suffices to show that

$$\left|\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} - 1\right| \leq 1-\alpha. \tag{3.7}$$

We have

$$\begin{aligned}\left|\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} - 1\right| &= \left|\frac{\sum_{n=2}^{\infty} a_n(z^{n-1}+z^{n-2}\zeta+\cdots+\zeta^{n-1}) - \sum_{n=2}^{\infty} n a_n z^{n-1}}{1+\sum_{n=2}^{\infty} n a_n z^{n-1}}\right| \\ &\leq \frac{\sum_{n=2}^{\infty} 2(n-1)|a_n|}{1-\sum_{n=2}^{\infty} n|a_n|},\end{aligned}$$

which is bounded above by $1-\alpha$ if $\sum_{n=2}^{\infty}((3-\alpha)n-2)|a_n| \leq 1-\alpha$.

Theorem 3. 1 *Let a_j $(j = 1,2,\cdots,p)$ $\in C\setminus\{0\}$, b_i $(i = 1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\operatorname{Re}b_i > 0$ $(i = 1,2,\cdots,q)$, and $\sum_{i=1}^{q}\operatorname{Re}b_i > \sum_{j=1}^{p}|a_j|+1$. If $f(z) \in R^t(A,B)$ satisfies*

$$\begin{aligned}&(1+\alpha)\frac{\prod_{j=1}^{p}|a_j|}{\prod_{i=1}^{q}\operatorname{Re}b_i}\,{}_pF_q\left(\begin{matrix}|a_1|+1,|a_2|+1,\cdots,|a_p|+1 \\ \operatorname{Re}b_1+1,\operatorname{Re}b_2+1,\cdots,\operatorname{Re}b_q+1\end{matrix};1\right) \\ &\quad+{}_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p| \\ \operatorname{Re}b_1,\operatorname{Re}b_2,\cdots,\operatorname{Re}b_q\end{matrix};1\right) \leq \frac{1}{(A-B)|t|}+1,\end{aligned} \tag{3.8}$$

then $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(f) \in UCV(\alpha)$.

Proof. By Lemma 3.1, we need only to show that

$$S_1 := \sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha) \left| \frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}} a_n \right| \leq 1. \tag{3.9}$$

From Lemma 2.2, we have,

$$\begin{aligned} S_1 &\leq (A-B)|t| \sum_{n=2}^{\infty} (n(\alpha+1)-\alpha) \frac{\prod_{j=1}^{p}(|a_j|)_{n-1}}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_{n-1}(1)_{n-1}} \\ &= (A-B)|t|(\alpha+1)\, {}_{p+1}F_{q+1}\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p|, 2 \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q, 1 \end{matrix} ;1 \right) \\ &\quad -(A-B)|t|\alpha\, {}_pF_q\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p| \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q \end{matrix} ;1 \right) - (A-B)|t| \\ &= (A-B)|t|(\alpha+1) \frac{\prod_{j=1}^{p}|a_j|}{\prod_{i=1}^{q}\mathrm{Re}b_i}\, {}_pF_q\left(\begin{matrix} |a_1|+1, |a_2|+1, \cdots, |a_p|+1 \\ \mathrm{Re}b_1+1, \mathrm{Re}b_2+1, \cdots, \mathrm{Re}b_q+1 \end{matrix} ;1 \right) \\ &\quad +(A-B)|t|\, {}_pF_q\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_p| \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q \end{matrix} ;1 \right) - (A-B)|t| \\ &\leq 1 \end{aligned}$$

by (3.8), which completes the proof of Theroem 3.1.

Corollary 3. 1 *Let* a_j $(j = 1, 2, \cdots, q+1)$ $\in C \setminus \{0\}$, b_i $(i = 1, 2, \cdots, q)$ $\in C \setminus \{0\}$, $\mathrm{Re}b_m > |a_m| + 1$ $(m = 1, 2, \cdots, q-1)$, *and* $\mathrm{Re}b_q > |a_q| + |a_{q+1}| + 1$. *If* $f(z) \in R^t(A, B)$ *satisfies*

$$\begin{aligned} &\frac{\Gamma(\mathrm{Re}b_q)\Gamma(\mathrm{Re}b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\mathrm{Re}b_q - |a_q|)\Gamma(\mathrm{Re}b_q - |a_{q+1}|)} \left(\prod_{m=1}^{q-1} \frac{1}{\mathrm{Re}b_m - |a_m| - 1} \right) \\ &\quad \times \left\{ (\alpha+1) \prod_{j=1}^{q+1} |a_j| + \left(\prod_{m=1}^{q-1} \mathrm{Re}b_m - 1 \right) (\mathrm{Re}b_q - |a_q| - |a_{q+1}| - 1) \right\} \\ &\leq \frac{1}{(A-B)|t|} + 1, \end{aligned} \tag{3.10}$$

then $I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_{q+1}}(f) \in UCV(\alpha)$.

Proof. Since the proof is similarly the proof of Corollary 2.4, we omit the details.

Theorem 3. 2 *Let* a_j $(j=1,2,\cdots,p)$ $\in C\setminus\{0\}$, b_i $(i=1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_i>0$ $(i=1,2,\cdots,q)$, *and* $\sum_{i=1}^{q}\mathrm{Re}b_i>\sum_{j=1}^{p}|a_j|$. *If* $f(z)\in R^t(A,B)$ *satisfies*

$$(3.11)\qquad (3-\alpha)\ {}_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p| \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right) -2\ {}_{p+1}F_{q+1}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|,1 \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,2\end{matrix};1\right)\leq(1-\alpha)\left(\frac{1}{(A-B)|t|}+1\right),$$

then $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(f)\in UST(\alpha)$, *for some* α $(0\leq\alpha<1)$.

Proof. By Lemma 3.2, we need only to show that

$$(3.12)\qquad S_2:=\sum_{n=2}^{\infty}((3-\alpha)n-2)\left|\frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n\right|\leq 1-\alpha.$$

From Lemma 2.2, we have,

$$\begin{aligned} S_2 &\leq (A-B)|t|(3-\alpha)\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}(|a_j|)_{n-1}}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_{n-1}(1)_{n-1}} \\ &\quad -2(A-B)|t|\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}(|a_j|)_{n-1}}{\prod_{i=1}^{q}(\mathrm{Re}b_i)_{n-1}(1)_{n-1}}\frac{(1)_{n-1}}{(2)_{n-1}} \\ &= (A-B)|t|(3-\alpha)\ {}_pF_q\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p| \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q\end{matrix};1\right) \\ &\quad -2(A-B)|t|\ {}_{p+1}F_{q+1}\left(\begin{matrix}|a_1|,|a_2|,\cdots,|a_p|,1 \\ \\ \mathrm{Re}b_1,\mathrm{Re}b_2,\cdots,\mathrm{Re}b_q,2\end{matrix};1\right)-(1-\alpha)(A-B)|t| \\ &\leq 1-\alpha \end{aligned}$$

by (3.11), which completes the proof of Theroem 3.2.

Corollary 3. 2 *Let* a_j $(j=1,2,\cdots,q+1)$ $\in C\setminus\{0\}$, b_i $(i=1,2,\cdots,q)$ $\in C\setminus\{0\}$, $\mathrm{Re}b_m>|a_m|+1$ $(m=1,2,\cdots,q-1)$, *and* $\mathrm{Re}b_q>|a_q|+1$, $|a_{q+1}|<1$. *If* $f(z)\in R^t(A,B)$ *satisfies*

$$(3.13)\qquad \left\{(3-\alpha)\frac{\Gamma(\mathrm{Re}b_q-1)\Gamma(\mathrm{Re}b_q-|a_q|-|a_{q+1}|)}{\Gamma(\mathrm{Re}b_q-|a_q|-1)\Gamma(\mathrm{Re}b_q-|a_{q+1}|)}-\frac{2}{1-|a_{q+1}|}\right\} \times\left(\prod_{m=1}^{q}\frac{\mathrm{Re}b_m-1}{\mathrm{Re}b_m-|a_m|-1}\right)\leq(1-\alpha)\left(\frac{1}{(A-B)|t|}+1\right),$$

then $I_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_{q+1}}(f)\in UST(\alpha)$, *for some* α $(0\leq\alpha<1)$.

Proof. We note that

$$
{}_{q+2}F_{q+1}\left(\begin{matrix} |a_1|, |a_2|, \cdots, |a_{q+1}|, 1 \\ \\ \mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_q, 2 \end{matrix} ; 1\right)
$$

$$
= \left(\prod_{m=1}^{q-1} \frac{\mathrm{Re}b_m - 1}{\mathrm{Re}b_m - |a_m| - 1}\right) \frac{\mathrm{Re}b_q - 1}{(\mathrm{Re}b_q - |a_q| - 1)(1 - |a_{q+1}|)}.
$$

From above equality and (2.6), we have the result of Corollary 3.2.

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