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A CLASS OF FUNCTIONS DEFINED BY USING HADAMARD PRODUCT. II

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ABSTRACT. The object of the present paper is to obtain closure theorems, integral operators and several interesting results for the modified Hadamard products of functions belonging to the class $P_\alpha[\beta,\gamma]$ consisting of analytic functions with negative coefficients and defined by using Hadamard product $f * S_\alpha(z)$ of $f(z)$ and $S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$. Also we obtain distortion theorem for certain fractional integral operator of functions in the class $P_\alpha[\beta,\gamma]$.

KEY WORDS- Analytic, starlike, modified Hadamard product.
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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z: |z| < 1\}$. And let S denote the subclass of A consisting of analytic and univalent functions f(z) in the unit disc U. A function f(z) in S is said to be starlike of order α if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U) \tag{1.2}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Further, a function f(z) in S is said to be convex of order α if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U) \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$. And We donote by $K(\alpha)$ the class of all convex functions of order α. It is well-known that

$$f(z) \in K(\alpha) \text{ if and only if } zf'(z) \in S^*(\alpha). \tag{1.4}$$

These classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [8], and later were studied by Schild [9], MacGregor [3] and Pinchuk [7].

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.5}$$

is the well-known extremal function for the class $S^*(\alpha)$ (see [8,1]). Setting

$$C(\alpha,n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \qquad (n \geq 2). \tag{1.6}$$

$S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha,n)\, z^n. \tag{1.7}$$

Then we note that $C(\alpha,n)$ is decreasing in α and satisfies

$$\lim_{n\to\infty} C(\alpha,n) = \begin{cases} \infty & (\alpha < \frac{1}{2}) \\ 1 & (\alpha = \frac{1}{2}) \\ 0 & (\alpha > \frac{1}{2}). \end{cases} \tag{1.8}$$

Let $f * g(z)$ be the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.9}$$

then

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \qquad (1.10)$$

We say that the function f(z) defined by (1.1) belongs to the class $P_\alpha(\beta,\gamma)$ if f(z) satisfies the following condition

$$\left| \frac{(f * S_\alpha(z))' - 1}{(f * S_\alpha(z))' + (1-2\beta)} \right| < \gamma \qquad (z \in U) \qquad (1.11)$$

for $\beta(0 \le \beta < 1)$ and $\gamma(0 < \gamma \le 1)$.

Let T denote the subclass of A consisting of functions f(z) of the form

$$fz) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \ge 0). \qquad (1.12)$$

And we denote by $P_\alpha[\beta,\gamma]$ the class obtained by taking intersection of $P_\alpha(\beta,\gamma)$ with T, that is,

$$P_\alpha[\beta,\gamma] = P_\alpha(\beta,\gamma) \cap T. \qquad (1.13)$$

The class $P_\alpha[\beta,\gamma]$ was studied by Owa and Ahuja [6]. The class $P_\alpha[\beta,\gamma]$ is the generalization of the class $P^*(\beta,\gamma)$ which was introduced by Gupta and Jain [2]. In particular, $P_{1/2}[\beta,\gamma] = P^*(\beta,\gamma)$ when $\alpha = \frac{1}{2}$. Further we note that many classes defined by using the Hadamard product $f * S_\alpha(z)$ of f(z) and $S_\alpha(z)$ were introduced by Sheil-Small, Silverman and

Silvia [11], Silverman and Silvia ([12], [13]), and Ahuja and Silverman [1].

In order to prove our results for functions belonging to the class $P_\alpha[\beta,\gamma]$, we shall require the following lemma given by Owa and Ahuja [6].

LEMMA 1. Let the function f(z) be defined by (1.12). Then f(z) is in the class $P_\alpha[\beta,\gamma]$ if and only if

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\ a_n \le 2\gamma(1-\beta). \qquad (1.14)$$

The result is sharp.

2. Closure Theorems

Let the functions $f_i(z)$ be defined, for i = 1,2,..., m, by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i}\ z^n \qquad (a_{n,i} \ge 0) \qquad (2.1)$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $P_\alpha[\beta,\gamma]$.

Theorem 1. Let the functions $f_i(z)$ $(i = 1, 2, \ldots, m)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$. Then the function h(z) defined by

$$h(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{2.2}$$

also belongs to the class $P_\alpha[\beta,\gamma]$, where

$$b_n = \frac{1}{m} \sum_{i=1}^{m} a_{n,i}. \tag{2.3}$$

Proof. Since $f_i(z) \in P_\alpha[\beta,\gamma]$, it follows from Lemma 1 that

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i} \le 2\gamma(1-\beta) \quad (i=1,2,\ldots,m). \tag{2.4}$$

Therefore

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)b_n = \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\left[\frac{1}{m}\sum_{i=1}^{m} a_{n,i}\right]$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i}\right\} \le 2\gamma(1-\beta). \tag{2.5}$$

Hence, by Lemma 1, $h(z) \in P_\alpha[\beta,\gamma]$. Thus we have the theorem.

Theorem 2. Let the functions $f_i(z)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ for each $i = 1, 2, \ldots, m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{m} d_i f_i(z) \qquad (d_i \geq 0) \tag{2.6}$$

is also in the same class $P_\alpha[\beta,\gamma]$, where

$$\sum_{i=1}^{m} d_i = 1. \tag{2.7}$$

Proof. According to the definition of $h(z)$, we can write that

$$h(z) = z - \sum_{n=2}^{\infty} \left[\sum_{i=1}^{m} d_i a_{n,i} \right] z^n. \tag{2.8}$$

By means of Lemma 1, we have

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i} \leq 2\gamma(1-\beta) \tag{2.9}$$

for every $i = 1, 2, \ldots, m$. Hence we can observe that

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n) \left[\sum_{i=1}^{m} d_i a_{n,i} \right]$$

$$= \sum_{i=1}^{m} d_i \left\{ \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_{n,i} \right\}$$

$$\leq \left[\sum_{i=1}^{m} d_i \right] 2\gamma(1-\beta) = 2\gamma(1-\beta) \tag{2.10}$$

which implies that $h(z) \in P_\alpha[\beta,\gamma]$. Thus we have the theorem.

Theorem 3. The class $P_\alpha[\beta,\gamma]$ is closed under convex linear combination.

Proof. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$. Then it is sufficient to prove that the function

$$h(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad (0 \leq \lambda \leq 1) \tag{2.11}$$

is in the calss $P_\alpha[\beta,\gamma]$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{n=2}^{\infty} \left\{\lambda\, a_{n,1} + (1-\lambda)a_{n,2}\right\} z^n, \tag{2.12}$$

with the aid of Lemma 1, we have

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\left\{\lambda a_{n,1}+(1-\lambda)a_{n,2}\right\} \leq 2\gamma(1-\beta) \tag{2.13}$$

which implies $h(z) \in P_\alpha[\beta,\gamma]$.

3. Integral Operators

Theorem 4. Let the function $f(z)$ defined by (1.12) be in the class $P_\alpha[\beta,\gamma]$, and let d be a real number such that $d > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{d+1}{z^d} \int_0^z t^{d-1} f(t)\, dt \tag{3.1}$$

also belongs to the class $P_\alpha[\beta,\gamma]$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n, \tag{3.2}$$

where

$$b_n = \left(\frac{d+1}{d+n}\right) a_n. \tag{3.3}$$

Therefore,

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)b_n = \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)\left(\frac{d+1}{d+n}\right) a_n$$

$$\leq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_n \leq 2\gamma(1-\beta), \tag{3.4}$$

since $f(z) \in P_\alpha[\beta,\gamma]$. Hence, by Lemma 1, $F(z) \in P_\alpha[\beta,\gamma]$.

THEOREM 5. Let the function $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ be in the class $P_\alpha[\beta,\gamma]$, and let d be a real number such that $d > -1$. Then the function $f(z)$ defined by (3.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_n \left[\frac{(1+\gamma)C(\alpha,n)(d+1)}{2\gamma(1-\beta)(d+n)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{3.5}$$

The result is sharp.

PROOF. From (3.1), we have

$$f(z) = \frac{z^{1-d}\,(z^d F'(z))'}{(d+1)} \quad (d > -1)$$

$$= z - \sum_{n=2}^{\infty} \left(\frac{d+n}{d+1}\right) a_n z^n. \tag{3.6}$$

In order to obtain the required result it suffices to show that $|f'(z) - 1| < 1$ in $|z| < R^*$. Now

$$|f'(z) - 1| = \left|-\sum_{n=2}^{\infty} n\left(\frac{d+n}{d+1}\right)a_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} n\left(\frac{d+n}{d+1}\right)a_n |z|^{n-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{n=2}^{\infty} n\left[\frac{d+n}{d+1}\right]a_n \; |z|^{n-1} < 1. \tag{3.7}$$

But Lemma 1 confirms that

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} a_n \le 1. \tag{3.8}$$

Hence (3.7) will be satisfied if

$$\frac{n(d+n)}{(d+1)}|z|^{n-1} \le \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} \qquad (n \ge 2)$$

or if

$$|z| \le \left[\frac{(1+\gamma)C(\alpha,n)(d+1)}{2\gamma(1-\beta)(d+n)}\right]^{\frac{1}{n-1}} \qquad (n \ge 2). \tag{3.9}$$

The required result follows now from (3.9). The result is sharp for the function

$$f(z) = z - \frac{2\gamma(1-\beta)(d+n)}{n(1+\gamma)C(\alpha,n)(d+1)} z^n \qquad (n \ge 2). \tag{3.10}$$

4. Modified Hadamard Products

Let the functions $f_i(z)$ $(i = 1,2)$ be defined by (2.1).

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1}\, a_{n,2}\, z^n. \tag{4.1}$$

Theorem 6. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, and $0 < \gamma \le 1$. Then $f_1 * f_2(z) \in P_\alpha[\delta(\alpha,\beta,\gamma),\gamma]$, where

$$\delta(\alpha,\beta,\gamma) = 1 - \frac{\gamma(1-\beta)^2}{2(1+\gamma)(1-\alpha)}. \tag{4.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest $\delta = \delta(\alpha,\beta,\gamma)$ such that

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\delta)}\, a_{n,1}\, a_{n,2} \le 1. \tag{4.3}$$

Since

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\, a_{n,1} \le 1 \tag{4.4}$$

and

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} a_{n,2} \leq 1, \tag{4.5}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{n=2}^{\infty} \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} \sqrt{a_{n,1}\, a_{n,2}} \leq 1. \tag{4.6}$$

Thus it is sufficient to show that

$$\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\delta)} a_{n,1}\, a_{n,2} \leq \frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)} \sqrt{a_{n,1}\, a_{n,2}} \quad (n \geq 2), \tag{4.7}$$

that is, that

$$\sqrt{a_{n,1}\, a_{n,2}} \leq \frac{(1-\delta)}{(1-\beta)}. \tag{4.8}$$

Note that

$$\sqrt{a_{n,1}\, a_{n,2}} \leq \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha,n)} \qquad (n \geq 2). \tag{4.9}$$

Consequently, we need only to prove that

$$\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha,n)} \leq \frac{(1-\delta)}{(1-\beta)} \qquad (n \geq 2), \tag{4.10}$$

or, equivalently, that

$$\delta \leq 1 - \frac{2\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \qquad (n \geq 2). \tag{4.11}$$

Since

$$A(n) = 1 - \frac{2\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \tag{4.12}$$

is an increasing function of n $(n \geq 2)$, for $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, letting $n = 2$ in (4.12), we obtain

$$\delta \leq A(2) = 1 - \frac{\gamma(1-\beta)^2}{2(1+\gamma)(1-\alpha)}, \tag{4.13}$$

which completes the proof of Theorem 6.

Finally, by taking the functions given by

$$f_i(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (i = 1,2) \tag{4.14}$$

we can see that the result is sharp.

Theorem 7. Let the function $f_1(z)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$, and the function $f_2(z)$ defined by (2.1) be in the class $P_\alpha[\tau,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \tau < 1$, and $0 < \gamma \leq 1$, then f_1 *

$f_2(z) \in P_\alpha[\zeta(\alpha,\beta,\tau,\gamma),\gamma]$, where

$$\zeta(\alpha,\beta,\tau,\gamma) = 1 - \frac{\gamma(1-\beta)(1-\tau)}{2(1+\gamma)(1-\alpha)}. \tag{4.15}$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 6, we get

$$\zeta \le B(n) = 1 - \frac{2\gamma(1-\beta)(1-\tau)}{n(1+\gamma)C(\alpha,n)} \quad (n \ge 2). \tag{4.16}$$

Since the function B(n) is an increasing function of n ($n \ge 2$), for $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, $0 \le \tau < 1$, and $0 < \gamma \le 1$, letting n = 2 in (4.16) we obtain

$$\zeta \le B(2) = 1 - \frac{\gamma(1-\beta)(1-\tau)}{2(1+\gamma)(1-\alpha)}, \tag{4.17}$$

which evidently proves Theorem 7.

Finally the result is best possible for the functions

$$f_1(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \tag{4.18}$$

and

$$f_2(z) = z - \frac{\gamma(1-\tau)}{2(1+\gamma)(1-\alpha)} z^2. \tag{4.19}$$

Corollary 1. Let the functions $f_i(z)$ $(i = 1,2,3)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, and $0 < \gamma \le 1$, then $f_1 * f_2 * f_3(z) \in P_\alpha[\eta(\alpha,\beta,\gamma),\gamma]$, where

$$\eta(\alpha,\beta,\gamma) = 1 - \frac{\gamma^2(1-\beta)^3}{4(1+\gamma)^2(1-\alpha)^2} . \tag{4.20}$$

The result is best possible for the functions

$$f_i(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2 \quad (i = 1,2,3). \tag{4.21}$$

Proof. From Theorem 6, we have $f_1 * f_2(z) \in P_\alpha[\delta(\alpha,\beta,\gamma),\gamma]$, where δ is given by (4.2). We use now Theorem 7, we get $f_1 * f_2 * f_3(z) \in P_\alpha[\eta(\alpha,\beta,\gamma),\gamma]$, where

$$\eta(\alpha,\beta,\gamma) = 1 - \frac{\gamma(1-\beta)(1-\delta)}{2(1+\gamma)(1-\alpha)} = 1 - \frac{\gamma^2(1-\beta)^3}{4(1+\gamma)^2(1-\alpha)^2} .$$

This completes the proof of Corollary 1.

Theorem 8. Let the functions $f_i(z)$ $(i = 1,2)$ defined by (2.1) be in the class $P_\alpha[\beta,\gamma]$ with $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, and $0 < \gamma \le 1$. Then the function

$$h(z) = z - \sum_{n=2}^{\infty} \left(a_{n,1}^2 + a_{n,2}^2\right) z^n \tag{4.21}$$

belongs to the class $P_\alpha[\varphi(\alpha,\beta,\gamma),\gamma]$, where

$$\varphi(\alpha,\beta,\gamma) = 1 - \frac{\gamma(1-\beta)^2}{(1+\gamma)(1-\alpha)}. \tag{4.22}$$

The result is sharp for the functions $f_i(z)$ $(i = 1,2)$ defined by (4.14).

Proof. By virtue of Lemma 1, we obtain

$$\sum_{n=2}^{\infty}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2 a_{n,1}^2 \leq \left[\sum_{n=2}^{\infty}\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\, a_{n,1}\right]^2 \leq 1 \tag{4.23}$$

and

$$\sum_{n=2}^{\infty}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2 a_{n,2}^2 \leq \left[\sum_{n=2}^{\infty}\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\, a_{n,2}\right]^2 \leq 1. \tag{4.24}$$

It follows from (4.23) and (4.24) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2\left(a_{n,1}^2 + a_{n,2}^2\right) \leq 1. \tag{4.25}$$

Therefore, we need to find the largest $\varphi = \varphi(\alpha,\beta,\gamma)$ such that

$$\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\varphi)} \leq \frac{1}{2}\left[\frac{n(1+\gamma)C(\alpha,n)}{2\gamma(1-\beta)}\right]^2 \qquad (n \geq 2) \tag{4.26}$$

that is,

$$\varphi \le 1 - \frac{4\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \qquad (n \ge 2). \tag{4.27}$$

Since

$$D(n) = 1 - \frac{4\gamma(1-\beta)^2}{n(1+\gamma)C(\alpha,n)} \tag{4.28}$$

is an increasing function of n $(n \ge 2)$, for $0 \le \alpha \le \frac{1}{2}$, $0 \le \beta < 1$, and $0 < \gamma \le 1$, we readily have

$$\varphi \le D(2) = 1 - \frac{\gamma(1-\beta)^2}{(1+\gamma)(1-\alpha)}, \tag{4.29}$$

and Theorem 8 follows at once.

5. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [14].

Definition 1. For real numbers $\eta > 0$, ρ and δ, the fractional integral operator $I_{0,z}^{\eta,\rho,\delta}$ is defined by

$$I_{0,z}^{\eta,\rho,\delta}f(z) = \frac{z^{-\eta-\rho}}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}F(\eta+\rho,-\delta;\eta;1-\frac{t}{z})f(t)dt \tag{5.1}$$

where f(z) is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}), \qquad z \longrightarrow 0,$$

where

$$\varepsilon > \mathrm{Max}\,(0, \rho-\delta) - 1,$$

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{5.2}$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)\ldots(\nu+n-1) & (n \in \mathbb{N} = \{1,2,\ldots\}), \end{cases} \tag{5.3}$$

and the multiplicity of $(z - t)^{\eta-1}$ is removed by requiring $\log (z-t)$ to be real when $z - t > 0$.

Remark. For $\rho = -\eta$, we note that

$$I_{0,z}^{\eta,-\eta,\delta} f(z) = D_z^{-\eta} f(z),$$

where $D_z^{-\eta} f(z)$ is the fractional integral of order η which was introduced by Owa ([4],[5]).

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [14].

Lemma 2. If $\eta > 0$ and $n > \rho - \delta - 1$, then

$$I_{o,z}^{\eta,\rho,\delta} z^n = \frac{\Gamma(n+1)\ \Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1)\ \Gamma(n+\eta+\delta+1)} z^{n-\rho}. \tag{5.4}$$

With the aid of the Lemma 2, we prove

Theorem 9. Let $\eta > 0$, $\rho < 2$, $\eta+\delta > -2$, $\rho-\delta < 2$, $\rho(\eta+\delta) \leq 3\eta$. If the function $f(z)$ defined by (1.12) is in the class $P_\alpha[\beta,\gamma]$ with $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta < 1$, and $0 < \gamma \leq 1$. Then

$$|I_{o,z}^{\eta,\rho,\delta} f(z)| \geq \frac{\Gamma(2-\rho+\delta)\,|z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} \left\{1 - \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|\right\} \tag{5.5}$$

and

$$|I_{o,z}^{\eta,\rho,\delta} f(z)| \leq \frac{\Gamma(2-\rho+\delta)\,|z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} \left\{1 + \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)} |z|\right\} \tag{5.6}$$

for $z \in U_o$, where

$$U_o = \begin{cases} U & (\rho \leq 1) \\ U - \{0\} & (\rho > 1). \end{cases}$$

The equalities in (5.5) and (5.6) are attained for the function f(z) given by

$$f(z) = z - \frac{\gamma(1-\beta)}{2(1+\gamma)(1-\alpha)} z^2. \tag{5.7}$$

PROOF. By using Lemma 2, we have

$$I_{0,z}^{\eta,\rho,\delta} f(z) = \frac{\Gamma(2-\rho+\delta)}{\Gamma(2-\rho)\ \Gamma(2+\eta+\delta)} z^{1-\rho}$$

$$- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\ \Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1)\ \Gamma(n+\eta+\delta+1)} a_n z^{n-\rho}. \tag{5.8}$$

Letting

$$H(z) = \frac{\Gamma(2-\rho)\ \Gamma(2+\eta+\delta)}{\Gamma(2-\rho+\delta)} z^{\rho} I_{0,z}^{\eta,\rho,\delta} f(z)$$

$$= z - \sum_{n=2}^{\infty} h(n)\ a_n z^n, \tag{5.9}$$

where

$$h(n) = \frac{(2-\rho+\delta)_{n-1} (1)_n}{(2-\rho)_{n-1} (2+\eta+\delta)_{n-1}} \qquad (n \geq 2), \tag{5.10}$$

we can see that h(n) is non-increasing for integers $n \geq 2$, and we have

$$0 < h(n) \leq h(2) = \frac{2(2-\rho+\delta)}{(2-\rho)(2+\eta+\delta)}. \tag{5.11}$$

We note that $C(\alpha,n+1) \geq C(\alpha,n)$ for $0 \leq \alpha \leq \frac{1}{2}$ and $n \geq 2$. Hence by using Lemma 1, we obtain that

$$2(1+\gamma)C(\alpha,2)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_n \leq 2\gamma(1-\alpha) \tag{5.12}$$

which implies that

$$\sum_{n=2}^{\infty} a_n \leq \frac{\gamma(1-\alpha)}{2(1+\gamma)(1-\alpha)}. \tag{5.13}$$

Therefore, by using (5.11) and (5.13), we have

$$|H(z)| \geq |z| - h(2)\ |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\geq |z| - \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)}\ |z|^2 \tag{5.14}$$

and

$$|H(z)| \leq |z| + h(2)\ |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |z| + \frac{\gamma(1-\beta)(2-\rho+\delta)}{(1+\gamma)(1-\alpha)(2-\rho)(2+\eta+\delta)}|z|^2. \qquad (5.15)$$

This completes the proof of Theorem 9.

REMARK 1. Taking $\rho = -\eta = -\lambda$ in Theorem 9, we get the result obtained by Owa and Ahuja [6, Theorem 10].

REMARK 2. Taking $\alpha = 1/2$ in the above results we obtain corresponding results for the class $P^*[\alpha,\gamma]$ defined by Gupta and Jain [2].

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