数理解析研究所講究録
1012巻 1997年 140-142

ON SCHWARZ LEMMA FOR THE HALF PLANE

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ABSTRACT. The object of the present paper is to derive the properties for functions which are analytic in the upper half plane.

1. INTRODUCTION. Let $\mathbb{D}$ denote the upper half plane which is given as

$$\mathbb{D} = \{z: z \in \mathbb{C}, \operatorname{Im}(z) > 0\}.$$

For $\phi(z)$ which are analytic in $\mathbb{D}$, Stankiewicz and Stankiewicz [2] (also Raducanu and Pascu [1]) have shown that

THEOREM A. Let $\phi: \mathbb{D} \longrightarrow \mathbb{D}$ be analytic in $\mathbb{D}$. If

(1.1) $$\lim_{\mathbb{D} \ni z \to \infty} (\phi(z) - z) = 0,$$

then

(1.2) $$\operatorname{Im}(\phi(z)) \geqq \operatorname{Im}(z) \qquad (z \in \mathbb{D}).$$

If there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Im}(\phi(z_0)) = \operatorname{Im}(z_0)$, then $\phi(z) = z + \alpha$, where $\alpha \in \mathbb{R}$ (the set of all real numbers).

In this paper, we obtain similar properties of functions $\phi(z)$.

2. MAIN RESULTS. Our main theorem of this paper is contained in

THEOREM 1. Let $\phi: \mathbb{D} \longrightarrow \mathbb{D}$ be analytic in $\mathbb{D}$. If, for all $\varepsilon > 0$, there exists $r = r(\varepsilon)$ such that

(2.1) $$\operatorname{Im}(\phi(z+i\varepsilon)) \geqq \operatorname{Im}(z) \qquad (z \in \mathbb{D}_r^-),$$

with

1991 Mathematics Subject Classification. 30C45.

$$\mathbb{D}_r^- = \{z\colon \operatorname{Im}(z) \geqq 0,\ |z| \geqq r\},$$

then

(2.2) $$\operatorname{Im}(\phi(z)) \geqq \operatorname{Im}(z) \qquad (z \in \mathbb{D}).$$

If there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Im}(\phi(z_0)) = \operatorname{Im}(z_0)$, then $\phi(z) = z + \alpha$, where $\alpha \in \mathbb{R}$.

PROOF. Let ε be a positive real number, $g\colon \mathbb{D} \longrightarrow \mathbb{D}$ be the function defined by $g(z) = \phi(z+i\varepsilon)$, and let

$$\mathbb{D}_r^+ = \{z\colon \operatorname{Im}(z) \geqq 0,\ |z| \leqq r\},$$

where $r = r(\varepsilon)$. If $\operatorname{Im}(g(z_0) - z_0) = \min_{z\varepsilon\mathbb{D}_r^+} \operatorname{Im}(g(z) - z)$, then $z_0 \in \partial\mathbb{D}_r^+$ and

(2.3) $$\operatorname{Im}(g(z) - z) \geqq \operatorname{Im}(g(z_0) - z_0) \qquad (z \in \mathbb{D}_r^+).$$

There are two possible cases:

(i) $\operatorname{Im}(z_0) = 0$ and hence $\operatorname{Im}(g(z) - z) \geqq \operatorname{Im}(g(z_0)) > 0$,

(ii) $\operatorname{Im}(z_0) > 0$. In this case, because $|z_0| = r$ and from the hypothesis we have $\operatorname{Im}(g(z_0) - z_0) \geqq 0$.

Therefore, in both cases, we obtain

$$\operatorname{Im}(g(z)) \geqq \operatorname{Im}(z) \qquad (z \in \mathbb{D}_r^+).$$

If $\varepsilon \to 0$ (and hence $r = r(\varepsilon) \to \infty$), this shows that $\operatorname{Im}(\phi(z)) \geqq \operatorname{Im}(z)$ in $\mathbb{D}$. If there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Im}(\phi(z_0) - z_0) = 0$, then the function $\operatorname{Im}(\phi(z) - z)$ is constant 0 in $\mathbb{D}$. Hence $\phi(z) - z = \alpha$, $\alpha \in \mathbb{R}$. This completes the proof of Theorem 1.

Next we derive

THEOREM 2. If the function $\phi\colon \mathbb{D} \longrightarrow \mathbb{D}$ is analytic in $\mathbb{D}$, and

(2.4) $$\lim_{z\to\infty} \mathrm{Im}(\phi(z) - z) \geqq 0 \qquad (z \in \mathbb{D}),$$

then $\mathrm{Im}(\phi(z)) \geqq \mathrm{Im}(z)$ for all $z \in \mathbb{D}$. If there exists a point $z_0 \in \mathbb{D}$ such that $\mathrm{Im}(\phi(z_0)) = \mathrm{Im}(z_0)$, then $\phi(z) = z + \alpha$, where $\alpha \in \mathbb{R}$.

PROOF. Replacing z by $z + i\varepsilon$ in (2.4), where $\varepsilon > 0$, we have

$$\lim_{z\to\infty} \mathrm{Im}(g(z) - z) \geqq \varepsilon > 0 \qquad (z \in \mathbb{D}).$$

Applying the definition of $g(z)$ in Theorem 1, we see the condition in Theorem 2 is safisfied.

REMARKS. (i) If $\lim_{z\to\infty}(\phi(z) - z) = 0 \quad (z \in \mathbb{D})$, then $\lim_{z\to\infty} \mathrm{Im}(\phi(z) - z) \geqq 0 \quad (z \in \mathbb{D})$. Therefore, Theorem A is a consequence of Theorem 2.

(ii) Functions $\phi_1(z) = z + \log z$ (we choose the principal branch for $\log z$) and $\phi_2(z) = z + e^{\varepsilon z} + i$ satisfy the conditions in Theorem 1, but not satisfy the conditions in Theorem A.

ACKNOWLEDGMENT. This research of the second author was supported, in part, by the Japanese Ministry of Education, Science and Culture under a Grant-in-Aid for General Scientific Research.

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