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On angular estimate of analytic functions

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Abstract

In this paper, we obtained the following result

$$|arg(zp'(z)-p(z))| > \frac{\pi}{2}(\beta+\frac{2}{\pi}Tan^{-1}(-\beta))$$

$$\Rightarrow \quad |arg p(z)| < \frac{\pi}{2}\beta$$

where β and $p(z)$ satisfy the conditions of Theorem 1.

Introduction and Results.

If $f(z)$ and $g(z)$ are analytic in the unit disk $E=\{z:|z|<1\}$, then $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if $g(z)$ is univalent in E, $f(0)=g(0)$ and $f(E)\subset g(E)$.

In this paper we need the following lemma.

Lemma 1. Let $p(z)$ be analytic in E, $p(0)=1$, $p(z)\neq 0$ in E and suppose that there exists a point $z_0\in E$ such that

$$|arg p(z)| < \frac{\pi}{2}\beta \quad for \quad |z|<|z_0|$$

and

$$|arg p(z_0)| = \frac{\pi}{2}\beta$$

where $0<\beta$.

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\beta$$

where

$$k\geq 1 \quad when \quad arg p(z_0) = \frac{\pi}{2}\beta$$

and

$$k\leq -1 \quad when \quad arg p(z_0) = -\frac{\pi}{2}\beta$$

We owe this lemma to [2].

In [1], Miller and Mocanu proved the following theorem.

Theorem A. Let $\beta_0 = 1.21872\cdots$ be the solution of

$$\beta\pi = \frac{3}{2}\pi - Tan^{-1}\beta$$

and let

$$\alpha = \alpha(\beta) = \beta + \frac{2}{\pi}Tan^{-1}\beta$$

for $0 < \beta < \beta_0$.

If $p(z)$ is analytic in E, with $p(0) = 1$, then

$$p(z) + zp'(z) \prec (\frac{1+z}{1-z})^{\alpha}$$

$$\Rightarrow \quad p(z) \prec (\frac{1+z}{1-z})^{\beta}$$

or

$$|arg(p(z) + zp'(z))| < \frac{\pi}{2}\alpha \qquad E$$

$$\Rightarrow \quad |argp(z)| < \frac{\pi}{2}\beta \qquad E.$$

Applying Theorem A, we can obtain many interesting results.

Corresponding Theorem A, we will obtain a result which is probably useful to obtain some results for meromorphic functions.

Theorem 1. Let $p(z)$ be analytic in E, $p(0) = 1$, $p(z) \neq 0$ in E and suppose that

$$|arg(zp'(z) - p(z))| > \frac{\pi}{2}(\beta + \frac{2}{\pi}Tan^{-1}(-\beta)) \quad in \quad E$$

where $0 < \beta$ and the branche of $Tan^{-1}(-\beta)$ is restricted in $\frac{\pi}{2} < Tan^{-1}(-\beta) < \pi$.

Then we have

$$|argp(z)| < \frac{\pi}{2}\beta \quad for \quad E.$$

Proof. If there exists a point $z_0 \in E$ such that

$$|argp(z)| < \frac{\pi}{2}\beta \quad for \quad |z| < |z_0|$$

and

$$|argp(z_0)| = \frac{\pi}{2}\beta,$$

then from Lemma 1, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = i\beta k$$

where

(1) $$k \geq 1 \quad when \quad arg p(z_0) = \frac{\pi}{2}\beta$$

and

(2) $$k \leq -1 \quad when \quad arg p(z_0) = -\frac{\pi}{2}\beta.$$

When $arg p(z_0) = \frac{\pi\beta}{2}$, then from (1), we have

$$\begin{aligned} \arg(z_0p'(z_0) - p(z_0)) &= \arg p(z_0)(\frac{z_0p'(z_0)}{p(z_0)} - 1) \\ &= \frac{\pi}{2}\beta + \arg(i\beta k - 1) \\ &\leq \frac{\pi}{2}\beta + Tan^{-1}(-\beta) \\ &= \frac{\pi}{2}(\beta + \frac{2}{\pi}Tan^{-1}(\beta)) \end{aligned}$$

and if $\arg p(z_0) = -\frac{\pi\beta}{2}$, then from (2), we also have

$$\begin{aligned} \arg(z_0p'(z_0) - p(z_0)) &= \arg p(z_0)(\frac{z_0p'(z_0)}{p(z_0)} - 1) \\ &= -\frac{\pi}{2}\beta + \arg(i\beta k - 1) \\ &\geq -\frac{\pi}{2}\beta - Tan^{-1}(-\beta) \\ &= -\frac{\pi}{2}(\beta + \frac{2}{\pi}Tan^{-1}(\beta)). \end{aligned}$$

These contradict the assumption of the theorem and therefore this completes the proof.

Remark. It is trivial that $\frac{\pi}{2}(\beta + \frac{2}{\pi}Tan^{-1}(-\beta)) > \pi$ for $0 < \beta$.

Applying the same method as the proof of Theorem 1 and from Lemma 1, we can generalize Theorem A as the following.

Theorem A′. Let $p(z)$ be analytic in E, $p(0) = 1$, $p(z) \neq 0$ in E and suppose that

$$|arg(p(z) + zp'(z))| < \frac{\pi}{2}(\beta + \frac{2}{\pi}Tan^{-1}\beta) \quad in \quad E$$

where $0 < \beta$.

Then we have

$$|arg p(z)| < \frac{\pi}{2}\beta \quad in \quad E.$$

References

[1] S.S Miller and P.T.Mocanu , Marx-Strohhäcker differential subordination systems, Proc. Amer. Math. Soc., 99(3), 527-534(1987).

[2] M.Nunokawa, On the order of strongly convex functions, Proc. Japan Acad., 69. A, No.7, 234-237(1993).

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