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Some Problems in Algebraic Analysis of Singular Perturbations

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As we have recently written up a monograph titled "Algebraic Analysis of Singular Perturbations" (to be published by Iwanami; [KT 3]), here we like to list up our next targets; some of them are mentioned also at the end of [KT 3].

[I] As we expounded in [KT 3, Chap. 3], we can describe the monodromic structure of a second order Fuchsian equation in terms of period integrals of its WKB solution. The method used there strongly indicates that global structure of solutions of equations with irregular singularities should be also analyzed by the exact WKB analysis. We surmise that the notion of Stokes graphs ([KT 3,

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Chap. 3, §2]) should play an important role then and that clarifying the relation between confluence of regular singularities and structure of Stokes graphs might be important and useful. (Cf. [SAKT])

[II] Thanks to the efforts of the Nice school conducted by F. Pham, the geometric structure of the Borel transform of a WKB solution has been considerably clarified. However, reflecting the complexity of the problem, it still requires, we think, some more clarification. For example, it seems to us that the description of the sheet structure of such a multi-valued function has not yet been satisfactory, particularly when the number of relevant periods is equal to or bigger than 3. As a somewhat more analytic issue we also propose to try to interpret the resurgent function theory of Ecalle by our approach based upon the transformation of an equation to some canonical equation. We hope that analyzing connection automorphisms (cf. e.g. [P]) in terms of the transformation of an equation with two simple turning points into its canonical form ([AKT 1, §3]) is probably the first step toward this problem.

[III] The exact WKB analysis so far done deals with second order equations only. It is theoretically quite unpleasant; furthermore several equations in physics do require exact WKB analysis for higher order equations, which is probably the most reliable tool in handling exponentially small terms. The main trouble in developing such a theory is, as Berk, Nevins and Roberts ([BNR]) (probably first) observed, due to the crossing of Stokes curves, which does not occur for second order equations. As Aoki and we ([AKT 2]) detected, each troublesome crossing point (the so-called ordered crossing point) is connected with a **new turning point** by a (new) Stokes curve [on the condition that we restrict our consideration to operators with simple discriminant]. Let us note that a new turning point is, by definition, the x-component of a self-intersection point of a bicharacteristic curve associated with the Borel transformed operator defined on (x, y)-space with y being the dual variable of the large parameter. We also note that a bicharacteristic strip is a non-singular curve because of the simple discriminant assumption but that a self-intersection point easily appears in a bicharacteristic curve as the Borel transformed operator is considered on a two-dimensional space. Although our recipe for describing the Stokes geometry making use of new turning points has not yet been logically completed, we hope it is practically complete. At the current stage we plan to investigate concrete examples encountered in physics, assuming that our proposal be complete. In the course of the concrete computation, we also plan to think over whether the simple discriminant assumption is reasonably generic or not in application; one possibility is that we might be obliged to generalize our framework to incorporate into the consideration the Hermitian requirement of the operator in question. In parallel with higher order ordinary differential equations with a large parameter, we propose to study holonomic systems with a large parameter; its Borel transform is a subholonomic system. We hope that our study of the Painlevé transcendents ([KT 1], [AKT 3], [KT 3], [KT 4], ...) will be a guidepost, although the equations treated there, namely the Schrödinger equation (SL_J) together with the deformation equation (D_J), might be too restricted. We also note that dealing with $n \times n$ first order systems sometimes makes the discussion more transparent, particularly when we consider higher (i.e., $n > 2$) order equations. Thus rewriting the exact WKB analysis in the matrix form seems to be worth doing.

[IV] Although in a series of articles ([KT 1], [AKT 3] and [KT 4]) we have clarified the formal aspect of asymptotic analysis of Painlevé transcendents, we are not yet completely satisfied with the current situation:

(i) To endow a reasonable analytic meaning with the 2-parameter solution of (P_J) is certainly an important issue. See [KT3, p.118-p.120] and [T] for some discussion of this point. See also the interesting report of T. Aoki in this proceedings, where some toy-model for this problem is analyzed concretely.
(ii) The result obtained in [KT4] claims that for each 2-parameter solution $\lambda_J(\tilde{t}; \alpha, \beta)$ we can find parameters α' and β' and formal transformations $x(\tilde{x}, \tilde{t})$ and $t(\tilde{t})$ so that

$$x(\lambda_J(\tilde{t}; \alpha, \beta), \tilde{t}) = \lambda_{\mathrm{I}}(t(\tilde{t}); \alpha', \beta')$$

holds. We have not, however, given an explicit relation between (α, β) and (α', β') except for the top terms. For the practical application this is not very satisfactory. We believe giving the complete correspondence between (α, β) and (α', β') should be important for explicitly writing down the connection formula for general Painlevé transcendents. One related interesting problem is to confirm that the constant E that appears in the canonical form of (SL_J) near its double turning points is independent of the parameters that (P_J) contains. See [KT 2] for this problem.

[V] How much can we do the asymptotic analysis of non-linear differential equations beyond Painlevé equations? We might dare say that the results obtained so far suggest the possibility of establishing a general theory for non-linear equations: One evidence is that all the formal solutions we have used (WKB solutions in the Schrödinger case and 2-parameter multiple-scale solutions in the Painlevé case) are, in a sense, constructed in such a way that the equation in question should be transformed to $(-d^2/dx^2 + \eta^2)\psi(x, \eta) = 0$, i.e., the equation whose solutions are given by exponential functions, by using these formal solutions. This problem of generalization is certainly a very important issue, but it is somewhat too vague as stated in this form. Probably one trial of worth doing might be the analysis of (partial) differential equations with conservation laws. For example, non-linear WKB analysis of Miura and Kruskal ([MK]) seems to us not to be very widely known in spite of its interesting and illuminating contents. In view of the importance in the exact WKB analysis of the Riemann surface determined by the potential, Novikov's works ([N], [GN]) dealing with the modulation of the Riemann surface seem to be also worth attention. (See the report of S. Tajima in this proceedings concerning these topics.)

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