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Some Duality Properties of Combinatorial Optimization Games *

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Abstract

We introduce an integer programming formulation for a class of combinatorial optimization games, which includes many interesting problems on graphs. Based on the theorem, proved in our previous paper [6], that the core is nonempty if and only if the associated linear program has an integer optimal solution, we prove some duality properties between related prime/dual linear programs for the corresponding games to be totally balanced.

1 Introduction

Game theory has a profound influence on methodologies of many different branches of sciences, especially those of economics, operations research and management sciences. The concept of core [22] lays down an important principle for a collective decision: Every subgroup of the players would not be able to do better if they break away from the decision of all players and form their own coalition. In addition, the thesis of bounded rationality is introduced as a crucial concept for game theoretical solutions to have practically meanful implementations in real life situations [25, 18, 21]. Informally, this states that players would not spend an unbounded amount of computational resources to gain a small amount of improvements in the outcome. There have been more and more studies on the computational aspects of game theory problems, though early works may even be traced back to two decades ago [15, 11, 20, 5, 21, 9, 17]. An extensive discussion can be found in a review by Kalai on interplays of operations research, game theories, and theoretical computer science [12].

Games associated with combinatorial optimization have long attracted the attention of researchers. An important feature of these games is that the value of each subset of players can be presented succinctly as the optimal solution to a combinatorial optimalization sub-problem for these players. Shapley and Shubik studied a market in which players start with a vector for the amount of commodities they own and wish to redistribute the commodities so as to maximize their utility functions [23]. Shapley and Shubik also studied an assignment game for which whether an imputation is in the core can be tested efficiently [24]. Claus and Kleitman initiated the discussion of the cost allocation problem of communication networks shared by many users

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and introduced several cost allocation criteria [3]. Bird [1], and independently, Claus and Granot [2] formulated it as a minimum cost spanning tree game (a terminology coined by Granot and Huberman [11]). Megiddo introduced an alternative formulation with Steiner trees [16]. Tamir studied a traveling salesman cost allocation game [26], and network synthesis games [27]. Deng and Papadimitriou discussed a game for which the game value for any subset of players is the total weight of the edges in the subgraph induced by them [5]. Faigle, et al., studied an Euclidean TSP game and a matching game [8, 9]. Nagamochi, et al., studied a minimum base game on matroid [17].

In another direction, Owen introduced a linear production game in which each player j controls a certain resource vector b^j [19]. Jointly, they maximize a linear objective function cx, subject to the resource constraints $Ax \leq \sum_{all\ j} b^j$. The value a subset S of players can achieve on their own is the maximum they can achieve with resources collectively owned by this subset: $\max\{cx : Ax \leq \sum_{j \in S} b^j\}$. Dubey and Shapley studied games related to some nonlinear programs which result in totally balanced games, that is, games for which cores of subgames are all nonempty [7]. Kalai and Zemel considered a class of combinatorial optimization game associated with the maximum flow from a source to a sink on a network, where each player controls one arc in the network [13, 14]. The maximum flow game is totally balanced, and on the other hand, every non-negative totally balanced game is a maximum flow game [13, 14]. Curiel proved that the class of linear programming games is equivalent to the class of *totally balanced games* [4]. These reductions for the equivalence proof, however, involve in exponential time and space in the number of players [4]. Therefore, complexity for computational problems for the cores of these models are not necessarily the same.

The motivation of our study is to design a general model which allows for general mathematical/computational methods to deal with computational issues for combinatorial optimization games. In this paper, we focus on a class of combinatorial optimization games with their game values defined by the following integer programs of packing type, $\max\{y^t 1 : y^t A \leq 1^t, y \geq 0, y \text{ integral}\}$, and of covering type $\min\{1^t x : Ax \geq 1, x \geq 0, x \text{ integral}\}$, where matrix A is of 0-1 values and vector 1 is of all ones. We showed in [6] that the core for such a game is nonempty if and only if the corresponding linear programming relaxation has an integer optimal solution. This result opens the door for techniques central to combinatorial optimization problems to be applied to our cooperative game problems. Based on this, we show in this paper tight results in terms of totally balanced games, which reveals an asymmetry between the games of packing type and the games of covering type: The total balancedness of a packing game implies that the corresponding covering game has nonempty core, while the total balancedness of a covering game implies that the corresponding packing game is also totally balanced.

In Section 5, we study the above relation between packing and covering games for the the maximum matching game and the vertex-cover games on graphs. In this case, we see that one of them is totally balanced if and only if the other is totally balanced, and this occurs if and only if the underlying graph is bipartite.

2 Packing and Covering Games

For a cooperative game (N, v), we have a set N of players, and a value function $v : 2^S \rightarrow R$: for each subset $S \subseteq N$ of players, v(S) is the revenue the subset can obtain by forming a coalition of the players in S only. The question is how to distribute the total value $v(N)$ to the players, i.e., to find an imputation $x : N \rightarrow R_+$ such that $\sum_{i \in N} x(i) = v(N)$. Usually we denote $\sum_{i \in S} x(i)$ by $x(S)$. Then, the above condition can be written as $x(N) = v(N)$. The concept of core introduces a principle to resolve this problem. An imputation x is in the core if and only if $\forall S \subseteq N : x(S) \geq v(S)$. That is, no subset of players can gain advantage by breaking away from the collective decision and forming their own coalition. The above formulation works only for the revenue distribution problem. For cost allocation, the definitions is similar with the above inequalities in the reversed direction.

In this paper, we are interested in the following special subclass of combinatorial optimization games: i.e., packing and covering games. We restrict A to be an $m \times n$ $\{0,1\}$-matrix. Let 1_k and 0_k denote the column vectors with all ones and all zeros, respectively, of dimension k. We may denote these vectors by 1 and 0 for simplicity. Let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be the corresponding index sets, and let t denote the transposition operation. Consider the following linear program,

$$\begin{aligned} LP(c, A, \max): \quad & \max \quad y^t c \\ & s.t. \quad y^t A \leq 1_n^t, \quad y \geq 0_m, \end{aligned}$$

and its dual,

$$\begin{aligned} DLP(c, A, \max): \quad & \min \quad 1_n^t x \\ & s.t. \quad Ax \geq c, \quad x \geq 0_n, \end{aligned}$$

where c is an m-dimensional column vector $\in R^m$, y is an m-dimensional column vector of variables and x is an n-dimensional column vector of variables.

We denote the corresponding integer programming version of $LP(c, A, \max)$ by $ILP(c, A, \max)$. Since A is a $\{0,1\}$-matrix, the integrality constraints are equivalent to require y to have $\{0,1\}$ values. We define the *packing game* $Game(c, A, \max)$ as follows, where $\overline{S} = N - S$:

1. The player set is N.

2. For each subset $S \subseteq N$, $v(S)$ is defined as the value of the following integer program:

$$\begin{aligned} \max \quad & y^t c \\ s.t. \quad & y^t A_{M,S} \leq 1_{|S|}^t, \quad y^t A_{M,\overline{S}} \leq 0_{n-|S|}^t, \\ & y \in \{0,1\}^m, \end{aligned}$$

where $A_{T,S}$ is the submatrix of A with row set T and column set S, and $v(\emptyset)$ is defined to be 0.

Since this is a maximization problem, we may as well assume that $c_j > 0$ for j with $A_{j\cdot} \neq 0$. Otherwise, we can always choose $y_j = 0$.

We then introduce a *covering game* $Game(d, A, \min)$ for cost minimization problems in the similar manner:

1. The player set is M.

2. For each subset $T \subseteq M$, $v(T)$ is defined as the value of the following integer program:

$$\begin{array}{ll} \min & d^t x \\ s.t. & A_{T,N} x \geq 1_{|T|}, \quad x \in \{0,1\}^n, \end{array}$$

where $v(\emptyset)$ is defined to be 0.

Again we can assume $d_j > 0$ for all j. Otherwise we may always choose $x_j = 1$ to simplify the problem. Since the value of the game is defined by a solution to the minimization problem, this is in fact a problem of sharing the cost of the game. Thus, we would revise the definitions of imputation and core. A vector $w : M \rightarrow R_+$ is an *imputation* if $w(M) = v(M)$, and an imputation is in the *core* if $w(T) \leq v(T)$ holds for all $T \subseteq M$.

We note at this point that two games $Game(c, A, \max)$ and $Game(d, A, \min)$ with $c = d$ are *not* dual in the sense of the underlying linear programs since the rôles of objective function and the right hand side of the constraint are not interchanged. In the case of $c = d = 1$, however, the corresponding linear relaxations become dual to each other.

3 Properties of the Core

In this section, we describe several mathematical theorems for the core of packing/covering games, which were given in [6].

Lemma 1 *A vector $z : N \rightarrow R_+$ is in the core of $Game(c, A, \max)$ if and only if*

1. *$z(N) = v(N)$ (i.e., z is an imputation),*
2. *$z(S_i) \geq c_i$ for all $i \in M$, where $S_i = \{j \in N \mid A_{ij} = 1\}$ (i.e., z is feasible to the dual $DLP(c, A, \max)$ of $LP(c, A, \max)$).*

Theorem 1 *The core for $Game(c, A, \max)$ is nonempty if and only if $LP(c, A, \max)$ has an integer optimal solution. In such case, a vector $z : N \rightarrow R_+$ is in the core if and only if it is an optimal solution to $DLP(c, A, \max)$.*

Similarly, we have the following lemma and theorem for the minimization game.

Lemma 2 *A vector $w : M \rightarrow R_+$ is in the core of $Game(d, A, \min)$ if and only if*

1. *$w(M) = v(M)$ (i.e., w is an imputation),*
2. *$w(T_j) \leq d_j$ for all $j \in N$, where $T_j = \{i \in M \mid A_{ij} = 1\}$ (i.e., z is feasible to the dual $DLP(d, A, \min)$ of $LP(d, A, \min)$).*

Theorem 2 *The core for $Game(d, A, \min)$ is nonempty if and only if $LP(d, A, \min)$ has an integer optimal solution. In such case, a vector $w : M \rightarrow R_+$ is in the core if and only if it is an optimal solution to $DLP(d, A, \min)$.*

4 Duality Properties for Totally Balanced Games

For a game with set of players N and game value $v : 2^N \to R_+$, the game with a subset S of players with $\emptyset \neq S \subseteq N$ and the game value $v_S(S') = v(S')$ for all $S' \subseteq S$ is called the *subgame* induced by S. A game is called *totally balanced* if any of its subgames has nonempty core [13]. In this section, we discuss the relationship of the total balancedness between $Game(1_m, A, \max)$ and $Game(1_n, A, \min)$.

Given a packing game $Game(c, A, \max)$, a subset $S \subseteq N$ of players induces the following subgame:

1. The player set is S.
2. For each subset $S' \subseteq S$, the game value $v_S(S')$ is defined as the value of the following integer program:

$$\begin{aligned} \max \quad & y^t c \\ s.t. \quad & y^t A_{M,S'} \leq 1^t_{|S'|}, \quad y^t A_{M,N-S'} \leq 0^t_{n-|S'|}, \\ & y \in \{0,1\}^m. \end{aligned}$$

By noting that constraint $y^t A_{M,N-S} \leq 0^t_{n-|S|}$ is always implied for any $S' \subseteq S$, we see that the subgame is equivalent to the packing game $Game(c_U, A_{U,S}, \max)$, where $U = \{i \in M \mid A_{ij} = 0, \text{ for all } j \in N - S\}$.

Similarly for a covering game $Game(d, A, \min)$, a subset $T \subseteq M$ of players induces the following subgame:

1. The player set is T.
2. For each subset $T' \subseteq T$, $v_T(T')$ is defined as the value of the following integer program:

$$\begin{aligned} \min \quad & d^t x \\ s.t. \quad & A_{T',N} x \geq 1_{|T'|}, \quad x \in \{0,1\}^n. \end{aligned}$$

Clearly, this subgame is equivalent to the covering game $Game(d, A_{T,N}, \min)$. In this case, if $Game(d, A, \min)$ is totally balanced, so is $Game(d, A_{T,N}, \min)$.

Theorem 3 *If $Game(1_m, A, \max)$ is totally balanced, then the core for $Game(1_n, A, \min)$ is nonempty.*

Proof: Since $Game(1_m, A, \max)$ is totally balanced, it has a nonempty core. Therefore, by Theorem 1, the following linear program has an integer optimal solution y^*.

$$\begin{aligned} LP(1_m, A, \max) : \quad \max \quad & y^t 1_m \\ s.t. \quad & y^t A \leq 1^t_n, \quad y \geq 0_m. \end{aligned}$$

Without loss of generality, assume $y^*_1 = y^*_2 = \cdots = y^*_r = 1$ and $y^*_{r+1} = y^*_{r+2} = \cdots = y^*_m = 0$. We can rearrange the columns of A so that $A_{11} = A_{12} = \cdots = A_{1p} = 1$ and $A_{1(p+1)} = A_{1(p+2)} =$

$\cdots = A_{1n} = 0$. Then, by the feasibility of y^*, all entries in the submatrix $A_{\{2,\ldots,r\},\{1,\ldots,p\}}$ are zeros.

Consider the following linear programs for $1 \leq k \leq p$:

$$\begin{aligned} LP_k: \quad & \max \quad y^t 1_m \\ & s.t. \quad y^t A \leq 1_n^t, \quad y^t A_{\cdot k} \leq 0, \quad y \geq 0_m, \end{aligned}$$

which correspond to the subgames $Game(1_m, A_{M,N-\{k\}}, \max)$, and let y^{k*} be their optimal solutions. By the total balancedness, $1_m^t y^{k*}$ are integers which are at least $1_m^t y^* - 1$, for all k (since a vector $y \in \{0,1\}^n$ such that $y_1 = 0$ and $y_i = y_i^*$ for all $i \neq 1$ is a feasible solution to LP_k).

We now claim that $1_m^t y^{k*} = 1_m^t y^* - 1$ holds for at least one k. Assume not, and we will have $1_m^t y^{k*} = 1_m^t y^*$, $k = 1, 2, \cdots, p$. Let $y' = \frac{1}{p}[(1,0,\cdots,0)^t + \sum_{k=1}^p y^{k*}]$. Then for $p+1 \leq j \leq n$,

$$(y')^t A_{\cdot j} = \frac{1}{p}(\sum_{k=1}^{p} y^{k*} A_{\cdot j}) \leq 1.$$

For $1 \leq j \leq p$, we have $(y^{j*})^t A_{\cdot j} \leq 0$ and $(y^{k*})^t A_{\cdot j} \leq 1$ for all $k \in \{1,2,\ldots,p\} - \{j\}$. Since $(1,0,\cdots,0)A_{\cdot j} = 1$, this implies $(y')^t A_{\cdot j} \leq 1$ for all $j = 1, 2, \cdots, n$. Therefore, y' is a feasible solution to $LP(1_m, A, \max)$, but $1_m^t y' = 1_m^t y^* + \frac{1}{p}$, a contradiction to the optimality of y^*.

Based on this claim, we prove that $LP(1_m, A, \min)$ also has an integer optimal solution by induction on the number n of columns of A (which proves by Theorem 2 that $Game(1_n, A \min)$ has nonempty core). For the base case of $n = 1$, the matrix A must be a vector of all ones, since otherwise, $LP(1_m, A, \max)$ is unbounded. Then $x_1 = 1$ is the optimal solution for $LP(1_1, A, \min)$, which is an integer solution.

For general n, let x^* be the optimal solution of $LP(1_n, A, \min)$. By the above claim, we may assume without loss of generality that $1_m^t y^{1*} = 1_m^t y^* - 1$. Let $S = N - \{1\}$ and $T = \{i \in M \mid A_{i1} = 0\}$. It is easy to see that the linear program LP_1 and its dual DLP_1 can be written as follows.

$$\begin{aligned} LP_1: \quad & \max \quad y_T^t 1_{|T|} \\ & s.t. \quad y_T^t A_{T,S} \leq 1_{n-1}^t, \quad y_T \geq 0_{|T|}. \end{aligned}$$

$$\begin{aligned} DLP_1: \quad & \min \quad 1_{|S|}^t x_S \\ & s.t. \quad A_{T,S} x_S \geq 1_{|T|}, \quad x_S \geq 0_{|S|}. \end{aligned}$$

Since $Game(1_m, A, \max)$ is totally balanced, $Game(1_{|S|}, A_{T,S}, \max)$ is totally balanced and has a nonempty core. Thus, by Theorem 1, we have an integer optimal solution y_T^0 to LP_1. By induction hypothesis, we also have an integer optimal solution x_S^0 to DLP_1. Define $w \in \{0,1\}^n$ by $w_1 = 1$ and $w_j = (x_S^0)_j$ for $j \in S$. Then $A_{T,N} w = A_{T,S} x_S^0 \geq 1_{|T|}$ and $A_{M-T,N} w = 1_{|M-T|} + A_{M-T,S} x_T^0 \geq 1_{|M-T|}$. Therefore w is a feasible integer solution to $LP(1_n, A, \min)$, and furthermore,

$$1_n^t w = 1 + 1_{|S|}^t x_S^0 = 1 + 1_{|T|}^t y_T^0 = 1 + 1_m^t y^{1*} = 1_m^t y^* (\text{=optimum value of } LP(1_m, A, \max))$$

implies that it is an optimal solution to $LP(1_n, A, \min)$ by the duality theory of linear programming. By Theorem 2, therefore, the core for $Game(1_n, A, \min)$ is nonempty. □

We remark in passing that a weaker condition such as only the nonempty core for $Game(1_m, A, \max)$ would not give the same result: The following matrix A has a nonempty core for $Game(1_6, A, \max)$ but the core for $Game(1_4, A, \min)$ is empty.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

To see this, first note that $y^* = (1, 0, 0, 0, 0, 1)^t$ and $x^* = (1/2, 1/2, 1/2, 1/2)^t$ are optimal solutions to $LP(1_6, A, \max)$ and $LP(1_4, A, \min)$, respectively, because they satisfy the complementary slackness condition of linear programming. Since y^* is integer, $Game(1_6, A, \max)$ has a nonempty core by Theorem 1. However, $LP(1_4, A, \min)$ cannot have an integer optimal solution x whose entries consist of two ones and two zeros, because, for any choice of two entries, the corresponding entries in some row of A are zeros. Then, by Theorem 2, the core for $Game(1_4, A, \min)$ is empty.

In addition, a stronger result that $Game(1_n, A, \min)$ is totally balanced would not hold either: The following matrix A gives a totally balanced game $Game(1_6, A, \max)$ but $Game(1_3, A, \min)$ is not totally balanced.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

This can be seen as follows. First note that A contains 3×3 identity matrix. For any choice of a subset $S \subseteq N = \{1, 2, 3\}$, it is easy to see that y^* with $y_i^* = 1$ for $i \in S$ and $y_i^* = 0$ for other i is an optimal solution to $LP(1_6, A, \max)$. Hence, by Theorem 1, $Game(1_6, A, \max)$ is totally balanced. However, for $T = \{4, 5, 6\} \subseteq M = \{1, \ldots, 6\}$, $LP(1_3, A_{T,N}, \min)$ has the optimal value $3/2$, and hence no integer optimal solution.

Surprisingly, for the opposite direction, a stronger result holds, as will be stated in Theorem 4. First we prove the next lemma, which is similar to Theorem 3 but requires a somewhat different proof.

Lemma 3 *If $Game(1_n, A, \min)$ is totally balanced, then the core for $Game(1_m, A, \max)$ is nonempty.*

Proof: To prove that the core of $Game(1_m, A, \max)$ is nonempty, we only have to show by Theorem 1 that $LP(1_m, A, \max)$ has an integer optimal solution.

We prove this by induction on the number m of rows of A. The base case $m = 1$ is trivial, since there is only one variable y_1 in $LP(1_m, A, \max)$ and $y_1 = 1$ is an integer optimal solution to the problem. We then prove the general case of $m > 1$ on the premise that it is true for $m - 1$.

Let x^* be any integer optimal solution of $LP(1_n, A, \min)$. Denote by A^i the submatrix of A excluding the i-th row $A_{i\cdot}$, and let x^{i*} be an optimal solution for $LP(1_n, A^i, \min)$, where $m \geq 2$. Clearly, for each $i = 1, 2, \ldots, m$, $1_n^t x^* - 1 \leq 1_n^t x^{i*} \leq 1_n^t x^*$ and $A_{i\cdot}x^* \geq 1$ hold. We now show that there is an integer optimal solution to $LP(1_m, A, \max)$, as this completes the proof by Theorem 1, by considering the following three cases.

Case-1: $1^t x^{i*} = 1^t x^*$ for some $i \in \{1, 2, \ldots, m\}$. This implies that $LP(1_n, A^i, \min)$ and $LP(1_n, A, \min)$ have the same optimum value. Since $Game(1_n, A, \min)$ is totally balanced, $Game(1_n, A^i, \min)$ is also totally balanced. By inductive hypothesis, $LP(1_{m-1}, A^i, \max)$ has an integer optimal solution $\hat{y} : (M - \{i\}) \to \{0, 1\}$, which can be extended to a feasible solution $\hat{y} : M \to \{0, 1\}$ of $LP(1_m, A, \max)$ by assigning $\hat{y}_i = 0$ for this i. Since $LP(1_n, A^i, \min)$ has the same objective value as $LP(1_n, A, \min)$, this $\hat{y} \in \{0, 1\}^m$ is an optimal solution for $LP(1_m, A, \max)$ which is integer.

Case-2: $A_{i\cdot}x^* > 1$ holds for some $i \in \{1, 2, \ldots, m\}$. We show that x^* is also an integer optimal solution to $LP(1_n, A^i, \min)$ for such i. Let y^* be an optimal solution to $LP(1_m, A, \max)$, and y^{i*} be obtained from y^* by removing its i-th component y_i^*. Clearly, x^* (resp., y^{i*}) is feasible to $LP(1_n, A^i, \min)$ (resp., its dual, $LP(1_{m-1}, A^i, \max)$). Since $A_{i\cdot}x^* > 1$ implies $y_i^* = 0$ by complementary slackness of linear programming, it follows that $1_{m-1}^t y^{i*} = 1_m^t y^* = 1_n^t x^*$. Thus, x^* and y^{i*} are optimal solutions to $LP(1_n, A^i, \min)$ and its dual, respectively, and $LP(1_n, A^i, \min)$ and $LP(1_n, A, \min)$ have the same optimum value. Then, we can apply the same argument as in Case-1 to show that $LP(1_m, A, \max)$ has an integer optimal solution.

Case-3: For any integer optimal solution x^* to $LP(1_n, A, \min)$, $1_n^t x^{i*} = 1_n^t x^* - 1$ and $A_{i\cdot}x^* = 1$ hold for all $i = 1, 2, \ldots, m$. Now let x^* be an integer optimal solution to $LP(1_n, A, \min)$. Renaming the indices if necessary, we may assume that $x_1^* = x_2^* = \cdots = x_p^* = 1$ and $x_{p+1}^* = \cdots = x_n^* = 0$. We will show below that $p = m$ and the submatrix $A_{M,\{1,\ldots,m\}}$ is the identity matrix. Let $I(j) = \{i \mid A_{ij} = 1\}$ for $j = 1, 2, \ldots, n$. Then $I(j) \neq \emptyset$ for all $j = 1, \ldots, p$, since x^* is an optimal solution of $LP(1_n, A, \min)$. Without loss of generality, we can permute the rows of A so that $A_{11} = 1$. We claim the following properties.

1. $\{I(1), I(2), \ldots, I(p)\}$ is a partition of the set M.
2. For all j with $2 \leq j \leq p$, we have $A_{1j} = 0$.
3. For all i with $2 \leq i \leq m$, we have $A_{i1} = 0$.

From the assumption of Case-3, $Ax^* = 1$ holds, which means that $\{I(1), I(2), \ldots, I(p)\}$ is a partition of the set M, i.e., property 1. Clearly, property 1 and $A_{11} = 1$ imply property 2.

To show the property 3, we extend the optimal solution x^{1*} of $LP(1_n, A^1, \min)$ to an integer solution x^{1o} of $LP(1_n, A, \min)$ by assigning $x_1^{1o} = 1$ and $x_j^{1o} = x_j^{1*}$, $2 \leq j \leq n$. Clearly, x^{1o} is feasible in $LP(1_n, A, \min)$, and $1_n^t x^{1o} \leq 1_n^t x^{1*} + 1 \leq 1_n^t x^*$ holds by the above assumption $1_n^t x^{1*} \leq 1_n^t x^* - 1$ of Case-3. Then, we see that $1_n^t x^{1o} = 1_n^t x^{1*} + 1 = 1_n^t x^*$ and $x_1^{1*} = 0$ must hold. Therefore, x_1^{1o} is an integer optimal solution of $LP(1_n, A, \min)$, and we can assume that $Ax^{1o} = 1$ holds (otherwise, Case-2 can be applied to this x^{1o}), from which $\{I(j) \mid x_j^{1o} = 1\}$ is a partition of M. By the feasibility of x^{*1}, we have $\cup_{x_j^{1*}=1} I(j) = M - \{1\}$, and $\cup_{x_j^{1*}=1} I(j)$ is also a partition of $M - \{1\}$ (since $\{I(j) \mid x_j^{1o} = 1\}$ is a partition of M). Therefore, $x_1^{1*} = 0$ implies that $I(1) - \{1\} = \emptyset$ (otherwise, for an $i \in I(1) - \{1\}$, $A_{1\cdot}x^{1*} = 0$ would result). This proves property 3.

By applying this to other indices j with $2 \leq j \leq p$, we see that the j-th column $A_{\cdot j}$ contains exactly one nonzero entry for each $j = 1, \dots, p$. From this and property 1, we have $p = m$ and hence $A_{M,\{1,\dots,m\}}$ is the identity matrix.

This property and the fact that x^* is an optimal solution of $LP(1, A, \min)$ imply that every column $A_{\cdot j}$ for $m < j \leq n$ also contains at most one nonzero entry. Therefore, the vector $y^* = 1_m$ is feasible, and hence, optimal to $LP(1_m, A, \max)$. □

The condition that $Game(1, A, \min)$ is totally balanced cannot be relaxed to the nonemptiness of the core of $Game(1, A, \min)$ as shown by the following example.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

To see this, first note that $y^* = (1/2, 1/2, 1/2, 1/2)^t$ and $x^* = (1, 0, 0, 0, 0, 1)^t$ are optimal solutions to $LP(1_4, A, \max)$ and $LP(1_6, A, \min)$, respectively, because they satisfy the complementary slackness condition of linear programming. Since x^* is integer, $Game(1_6, A, \min)$ has a nonempty core by Theorem 2. However, $LP(1_4, A, \max)$ cannot have an integer optimal solution x whose entries consist of two ones and two zeros, because, for any choice of two entries, the corresponding entries in some row of A are ones. Then, by Theorem 1, the core for $Game(1_4, A, \max)$ is empty.

However, we can make the conclusion stronger.

Theorem 4 *If $Game(1_n, A, \min)$ is totally balanced, then $Game(1_m, A, \max)$ is also totally balanced.*

Proof: To show that $Game(1_m, A, \max)$ is totally balanced, it is enough to show that the following linear programs have integer optimal solutions for all $S \subseteq N$.

$$\begin{array}{ll} \max & y^t 1_m \\ s.t. & y^t A_{M,S} \leq 1^t_{|S|}, \quad y^t A_{M,\overline{S}} \leq 0^t_{n-|S|}, \\ & y \geq 0_m, \end{array}$$

where $\overline{S} = N - S$. Let $T = \{i \in M \mid A_{ij} = 0 \text{ for all } j \in \overline{S}\}$, $\overline{T} = M - T = \{i \in M \mid A_{ij} = 1 \text{ for some } j \in \overline{S}\}$. For this, consider the following linear program:

$$\begin{array}{lll} LP(1_{|T|}, A_{T,S}, \max): & \max & y_T^t 1_{|T|} \\ & s.t. & y_T^t A_{T,S} \leq 1^t_{|S|}, \quad y_T \geq 0_{|T|}. \end{array}$$

Given any optimal solution $y_T \in \{0,1\}^{|T|}$ to this linear program, the vector y^* defined by $y^*_i = 0$ for all $i \in \overline{T}$ and $y^*_i = (y_T)_i$, $i \in T$ is an optimal solution tom $LP(1_m, A_{M,S}, \max)$. Therefore, it is sufficient to show that y_T can be chosen as an integer optimal solution. For this, consider its dual $DLP(1_{|T|}, A_{T,S}, \max)$ which is described as follows:

$$\begin{array}{lll} LP(1_{|S|}, A_{T,S}, \min): & \min & 1^t_{|S|} x_S \\ & s.t. & A_{T,S} x_S \geq 1^t_{|T|}, x_S \geq 0_{|S|}. \end{array}$$

This can be rewritten as follows, since $A_{T,\overline{S}}$ has all zero entries.

$$\begin{array}{rll} LP(1_n, A_{T,N}, \min): & \min & 1_n^t x \\ & s.t. & A_{T,S}x_S + A_{T,\overline{S}}x_{\overline{S}} \geq 1_{|T|}^t, \quad x \geq 0_n. \end{array}$$

Since $Game(1_n, A, \min)$ is totally balanced, so is subgame $Game(1_n, A_{T,N}, \min)$. By Theorem 2 and Lemma 3, this implies that the following linear program has an integer optimal solution.

$$\begin{array}{rll} LP(1_{|T|}, A_{T,N}, \max): & \max & y_T^t 1_{|T|} \\ & s.t. & y_T^t A_{T,N} \leq 1_n^t, \quad y_T \geq 0_{|T|}. \end{array}$$

Obviously, any feasible solution y_T to $LP(1_{|T|}, A_{T,S}, \max)$ is feasible to $LP(1_{|T|}, A_{T,N}, \max)$, since $y_T^t A_{T,\overline{S}} \leq 1_{|\overline{S}|}^t$ is automatically satisfied by the fact that $A_{T,\overline{S}}$ is of all zero entries. This proves that $LP(1_{|T|}, A_{T,S}, \max)$ has an integer optimal solution. □

5 Matching and Vertex Cover Games on Graphs

We shall now exemplify the duality properties between packing and covering games, proved in the previous section, for a pair of games on a graph. Given a graph $G = (V, E)$, the maximum matching game has players on vertices and the game value $v(S)$ for $S \subseteq V$ defined by the maximum matching size in the induced subgraph $G[S]$. Similarly, the minimum vertex cover game has players on edges and the game value $v(S)$ for $S \subseteq E$ defined by the size of a minimum vertex cover in the subgraph $G[S] = (V, S)$. These games are formulated by packing game $Game(1_{|E|}, A, \max)$ and covering game $Game(1_{|V|}, A, \min)$, respectively, where the constraint matrix A is the incidence matrix of G in which rows correspond to edges E and columns correspond to vertices V; $A_{ij} = 1$ if and only if edge i and vertex j are incident.

By Lemma 1, an imputation z is in the core of the matching game if and only if $z(u) + z(u') \geq 1$ holds for all edges $(u, u') \in E$. Based on this observation, we can easily find two classes of graphs for which the cores are always nonempty: The class of graphs for which the size of a minimum vertex cover is the same as the size of a maximum matching, and the class of graphs with a perfect matching. For a graph $G = (V, E)$ in the first class, we assign $z(v) = 1$ if v is in the minimum vertex cover and $z(v) = 0$ otherwise. It is easy to see that this z is indeed in the core. For a graph $G = (V, E)$ in the second class, we assign every vertex $v \in V$ with $z(v) = 0.5$. Then $z(V) = |V|/2 = v(E)$, since G has a perfect matching. Furthermore, since the size of a maximum matching in any subgraph $G[S]$ induced by $S \subseteq V$ is no more than $|S|/2$, this z is indeed in the core.

Furthermore, one can easily construct other graphs with non-empty cores, which are not in the above two classes. For example, take two graphs one from each of the above classes, and connect them with edges between the vertices in the minimum cover and the vertices in the perfect matching. However, the next theorem says that these are essentially all graphs which have nonempty cores for the maximum matching game.

Theorem 5 *An undirected graph $G = (V, E)$ has a nonempty core for the maximum matching game if and only if there exists a subset $V_1 \subseteq V$ such that*

1. the subgraph $G_1 = G[V_1]$ induced by V_1 has a minimum vertex cover W with the same size as its maximum matching,

2. the subgraph $G_2 = G[V - V_1]$ induced by $V - V_1$ has a perfect matching,

3. all the remaining edges $(u, u') \in E$ between G_1 and G_2 satisfy $u \in W$ for the vertex cover W in 1.

By elaborating the proof of this theorem, we can also show the next corollary.

Corollary 1 *The maximum matching game is totally balanced if and only if graph $G = (V, E)$ is bipartite.*

In the minimum vertex cover game $Game(1_{|V|}, A, \min)$, the players are on edges and the value of a subset $S \subseteq V$ is the minimum vertex cover in the induced subgraph $G[S]$. The matrix A for this game is the same as the maximum matching game. By Lemma 2, an imputation is in the core if and only if there is no vertex u such that the sum of the imputation over the edges incident with u is more than one.

Theorem 6 *The core for the minimum vertex cover game on graph $G = (V, E)$ is nonempty if and only if the size of a maximum matching is equal to the size of a minimum vertex cover.*

We note here that the condition in the above theorem holds if G is a bipartite graph by König's theorem. The next corollary can also be proved by elaborating the proof of the above theorem.

Corollary 2 *The minimum vertex cover game is totally balanced if and only if graph $G = (V, E)$ is bipartite.*

It is now evident that Corollaries 1, 2 are consistent with the duality properties stated as Theorems 3, 4 in Section 4.

6 Conclusion

Based on the previous results [6], we examined duality properties that hold between minimization game and maximization game, and show some asymmetry in terms of totally balancedness in these games.

Many open problems result from our approach: Would our model help in study of other solution concepts for cooperative games? Can these nice results about cores be extended to larger classes of combinatorial optimization games? The mathematical formulation of many combinatorial optimization games will be on hypergraphs instead of graphs. Can our methodology still work for hypergraphs? One may observe that totally balanced games and balanced matrices are somewhat related. Can we completely understand the relationship between totally balanced games and balanced matrices for our model?

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