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On a property of fuzzy stopping times

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Abstract

This note is concerned with a fuzzy stopping time for a dynamic fuzzy system. A new class of fuzzy stopping times is introduced and constructed by subsets of α-cut for fuzzy states. The results are applied to the optimization of a corresponding problem with an additive weighting function.

Keywords: Fuzzy stopping times; Markov property; α-cuts of fuzzy sets; optimality.

1 Introduction and notations

The stopping time with fuzziness, which is called a fuzzy stopping time, is considered by our previous paper [11] in which optimization of a corresponding fuzzy problem is pursued by the constructive method.

In this note, we introduce a new class of fuzzy stopping times defined by subsets of the α-cuts of fuzzy states and we apply it to a fuzzy stopping problem with additive weighting functions as the scalarization of the fuzzy total rewards. As related works, refer to [1, 5, 6, 7, 15].

In the remainder of this section, a fuzzy stopping time for a fuzzy dynamic system is defined explicitly. A new class of fuzzy stopping time is introduced in Section 2 and its construction is discussed. These results are applied to the 'optimization' of a corresponding fuzzy stopping problem in Section 3. In Section 4, a example is given to illustrate the results.

Let E, E_1, E_2 be convex compact subsets of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. A fuzzy set $\tilde{u} : E \mapsto [0,1]$ is called convex if

$$\tilde{u}(\lambda x+(1-\lambda)y) \geq \tilde{u}(x) \wedge \tilde{u}(y), \quad x, y \in E,\ \lambda \in [0,1],$$

where $a \wedge b := \min\{a, b\}$ for real numbers a, b (c.f. Chen-wei Xu [2]). Also, a fuzzy relation $\tilde{h} : E_1 \times E_2 \mapsto [0,1]$ is called convex if

$$\tilde{h}(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2) \geq \tilde{h}(x_1, y_1) \wedge \tilde{h}(x_2, y_2)$$

for $x_1, x_2 \in E_1$, $y_1, y_2 \in E_2$ and $\lambda \in [0,1]$.

Let $\mathcal{F}(E)$ be the set of all convex fuzzy sets, $\tilde{u}$, on E whose membership functions are upper semi-continuous and have compact supports and the normality condition : $\sup_{x\in E} \tilde{u}(x) = 1$. The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{u}$ is defined by

$$\tilde{u}_\alpha := \{x \in E \mid \tilde{u}(x) \geq \alpha\}\ (\alpha > 0) \quad \text{and} \quad \tilde{u}_0 := \mathrm{cl}\{x \in E \mid \tilde{u}(x) > 0\},$$

where cl denotes the closure of a set. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of E . Clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0,1]$.

Let $\mathbf{R}$ be the set of all real numbers. We see, from the definition, that $\mathcal{C}(\mathbf{R})$ is the set of all bounded closed intervals in $\mathbf{R}$. The elements of $\mathcal{F}(\mathbf{R})$ are called fuzzy numbers. The addition and the scalar multiplication on $\mathcal{F}(\mathbf{R})$ are defined as follows (see Puri and Ralescu [13]): For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$,

$$(\tilde{m}+\tilde{n})(x) := \sup_{x_1, x_2 \in \mathbf{R}:\ x_1+x_2=x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\} \quad (x \in \mathbf{R}) \tag{1.1}$$

and

$$(\lambda\tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ 1_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbf{R}). \tag{1.2}$$

And hence

$$(\tilde{m}+\tilde{n})_\alpha = \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda\tilde{m})_\alpha = \lambda\tilde{m}_\alpha\ (\alpha \in [0,1]),$$

where $A+B := \{x+y \mid x \in A, y \in B\}$, $\lambda A := \{\lambda x \mid x \in A\}$, $A+\emptyset = \emptyset + A := A$ and $\lambda\emptyset := \emptyset$ for any non-empty closed intervals $A, B \in \mathcal{C}(\mathbf{R})$. We use the following lemma.

Lemma 1.1 (Chen-wei Xu [2]).

(i) *For any $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$, it holds that $\tilde{m} + \tilde{n} \in \mathcal{F}(\mathbf{R})$.*

(ii) *Let $\tilde{u} \in \mathcal{F}(E_1)$ and $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$. Then $\sup_{x \in E_1}\{\tilde{u}(x) \wedge \tilde{p}(x, \cdot)\} \in \mathcal{F}(E_2)$.*

We consider the dynamic fuzzy system([9]), which is denoted by the elements $(S, \tilde{q})$ as follows.

Definition 1.

(i) The state space S is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state and is denoted by an element of $\mathcal{F}(S)$.

(ii) The law of motion for the system is denoted by time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0,1]$, and assume that $\tilde{q} \in \mathcal{F}(S \times S)$.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, the state is moved to a new fuzzy state $Q(\tilde{s})$ after unit time, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ is defined by

$$Q(\tilde{s})(y) := \sup_{x \in S}\{\tilde{s}(x) \wedge \tilde{q}(x,y)\} \quad (y \in S). \tag{1.3}$$

Note that the map Q is well-defined by Lemma 1.1.

For the dynamic fuzzy system $(S, \tilde{q})$ with a given initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^{\infty}$ by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \tag{1.4}$$

A fuzzy stopping time for this sequence $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in the next section. In order to define a fuzzy stopping time, we need the following preliminaries.

Associated with the fuzzy relation $\tilde{q}$, the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ $(\alpha \in [0,1])$ are defined as follows: For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x,y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{y \in S \mid \tilde{q}(x,y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{cases} \tag{1.5}$$

From the assumption on $\tilde{q}$, the maps Q_α is well-defined. The iterates Q_α^t $(t \geq 0)$ are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the α-cuts of $Q(\tilde{s})$ defined by (1.3) is specified using the maps Q_α.

Lemma 1.2 ([9, 10]). *For any $\alpha \in [0,1]$ and $\tilde{s} \in \mathcal{F}(S)$, we have:*

(i) $Q(\tilde{s})_\alpha = Q_\alpha(\tilde{s}_\alpha)$;

(ii) $\tilde{s}_{t,\alpha} = Q_\alpha^{t-1}(\tilde{s}_\alpha)$ $(t \geq 1)$,

where $\tilde{s}_{t,\alpha} := (\tilde{s}_t)_\alpha$ and $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

2 Fuzzy stopping times

In this section, we define a fuzzy stopping time to be discussed here. And a new class of fuzzy stopping times is introduced, which is constructed thorough subsets of α-cuts of fuzzy states.

For the sake of simplicity, denote $\mathcal{F} := \mathcal{F}(S)$. Let $\mathbf{N} = \{1, 2, \cdots\}$ and $\mathcal{F}'$ a subset of $\mathcal{F}$.

Definition 2 (cf.[11]). A *fuzzy stopping time(FST)* on $\mathcal{F}'$ is a fuzzy relation $\tilde{\sigma}$: $\mathcal{F}' \times \mathbf{N} \mapsto [0,1]$ such that, for each fuzzy state $\tilde{s} \in \mathcal{F}'$, $\tilde{\sigma}(\tilde{s}, t)$ is non-increasing in t and there exists a natural number $t(\tilde{s}) \in \mathbf{N}$ with $\tilde{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t(\tilde{s})$.

We note here that 0 represents 'stop' and 1 represents 'continue' in the grade of membership (cf.[11]). An FST $\tilde{\sigma}(\tilde{s}, \cdot)$ means the degree of 'continue' at time t starting at a fuzzy state $\tilde{s} \in \mathcal{F}'$. The set of all FSTs on $\mathcal{F}'$ is denoted by $\Sigma(\mathcal{F}')$. Assuming $Q(\mathcal{F}') \subset \mathcal{F}'$, an FST $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ is called *Markov* if there exist a mapping $\delta : \mathcal{F}' \mapsto [0,1]$ satisfying

(i) $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$, and

(ii) $\tilde{\sigma}(\tilde{s}, t) = \delta(\tilde{s}_t)$ for all $\tilde{s} \in \mathcal{F}'$ and $t \geq 1$,

where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

The above δ is called a *support* of $\tilde{\sigma}$. We consider ourselves with the construction of Markov FSTs. For this purpose, we assume the following condition holds.

Condition A1. For each $\alpha \in [0,1]$, there exists a non-empty subset $\mathcal{K}_\alpha$ of $\mathcal{C}(S)$ satisfying

$$Q_\alpha(\mathcal{K}_\alpha) \subset \mathcal{K}_\alpha. \tag{2.1}$$

Using this subset $\mathcal{K}_\alpha$, we define a sequence of subsets $\{\mathcal{K}_\alpha^t\}_{t=1}^{\infty}$ inductively by

$$\mathcal{K}_\alpha^1 := \mathcal{K}_\alpha \tag{2.2}$$

and for each $t \geq 2$,

$$\mathcal{K}_\alpha^t := \{c \in \mathcal{C}(S) \mid Q_\alpha(c) \in \mathcal{K}_\alpha^{t-1}\}. \tag{2.3}$$

Clearly, $\mathcal{K}_\alpha^t = Q_\alpha^{-1}(\mathcal{K}_\alpha^{t-1}) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha)$. Also, it holds from (2.1) that $\mathcal{K}_\alpha^t \subset \mathcal{K}_\alpha^{t+1}$ $(t \geq 1)$.

To simplify our discussion, we assume the following condition holds henceforth.

Condition A2. For all $\alpha \in [0,1]$, it holds that

$$\mathcal{C}(S) = \bigcup_{t=1}^{\infty} \mathcal{K}_\alpha^t.$$

For $c \in \mathcal{C}(S)$ and $\alpha \in [0,1]$, define $\hat{\sigma}_\alpha(c)$ by

$$\hat{\sigma}_\alpha(c) := \min\{t \geq 1 \mid c \in \mathcal{K}_\alpha^t\}. \tag{2.4}$$

That is, it is the first entry time of $c \in \mathcal{C}(S)$ with the grade α. We define a restricted class $\hat{\mathcal{F}} \subset \mathcal{F}$ by

$$\hat{\mathcal{F}} := \{\tilde{s} \in \mathcal{F} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) \text{ is non-increasing in } \alpha \in [0,1]\}. \tag{2.5}$$

Using the class $\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) \mid \alpha \in [0,1]\}$, for the restricted element $\tilde{s} \in \hat{\mathcal{F}}$, let us construct

$$\hat{\sigma}(\tilde{s}, t) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{D_\alpha}(t)\} \quad (t \geq 1), \tag{2.6}$$

where 1_{D_α} is the indicator of a set $D_\alpha = \{t \in \mathbf{N} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. This is the usual technique of constructing a corresponding fuzzy number from the class of level sets. Now let

$$\hat{\sigma}(\tilde{s}, \cdot)_\alpha := \min\{t \in \mathbf{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}. \tag{2.7}$$

Then we obtain the following theorem.

Theorem 2.1.

(i) $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \hat{\sigma}_\alpha(\tilde{s}_\alpha)$, $\quad \tilde{s} \in \hat{\mathcal{F}}$, $\alpha \in [0,1]$;

(ii) $\hat{\sigma}$ *is an FST on* $\hat{\mathcal{F}}$.

Proof. By (2.6) and (2.7), we have that $\hat{\sigma}(\tilde{s},\cdot)_\alpha \le t$ is equivalent to $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \le t$ for all $t \ge 1$. This fact shows (i). From Condition A2, there exists $t^* \in \mathbf{N}$ with $\tilde{s}_0 \in \mathcal{K}_0^{t^*}$. So, $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \le \tilde{s}_0(\tilde{s}_0) \le t^*$ for all $\alpha \in [0,1]$, which shows by (2.5) that $\hat{\sigma}(\tilde{s},t) = 0$ for all $t \ge t^*$. Since $\hat{\sigma}(\tilde{s},t+1) \le \hat{\sigma}(\tilde{s},t)$ holds clearly for $t \ge 1$ from the definition (2.6), we also obtain (ii). *q.e.d.*

In order to show the Markov property of $\hat{\sigma}$, we need the following lemma.

Lemma 2.1. *Let $\tilde{s} \in \hat{\mathcal{F}}$. Then*

(i) $\hat{\sigma}(\tilde{s},t) = \alpha$ *if and only if, for any* $\epsilon > 0$,

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}_{\alpha+\epsilon}^t \quad \textit{and} \quad \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}_{\alpha-\epsilon}^t;$$

(ii) $\tilde{s}_t \in \hat{\mathcal{F}}$ $(t \ge 1)$.

Proof. By (2.6), $\hat{\sigma}(\tilde{s},t) = \sup\{\alpha \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. So, (i) follows from (2.4). From Lemma 1.2(ii), for $l \ge 1$, $\hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) = \hat{\sigma}_\alpha(\tilde{s}_{l,\alpha}) = \hat{\sigma}_\alpha(Q_\alpha^{l-1}(\tilde{s}_\alpha))$. So, by (2.3) and (2.4),

$$\begin{aligned}
\hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) &= \min\{t \ge 1 \mid Q_\alpha^{l-1}(\tilde{s}_\alpha) \in \mathcal{K}_\alpha^t\} \\
&= \min\{t \ge 1 \mid \tilde{s}_\alpha \in \mathcal{K}_\alpha^{t+l-1}\} \\
&= \max\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) - (l-1), 1\},
\end{aligned}$$

and it is non-increasing in $\alpha \in [0,1]$ since $\tilde{s} \in \hat{\mathcal{F}}$. Therefore we obtain (ii). *q.e.d.*

Theorem 2.2. *Let $\tilde{s} \in \hat{\mathcal{F}}$. Then, $\hat{\sigma}$ is a Markov FST with $\tilde{s}$.*

Proof. Let $\{\tilde{s}_t\}_{t=1}^\infty$ be defined by (1.4) with $\tilde{s}_1 = \tilde{s}$. First, we prove

$$\hat{\sigma}(\tilde{s},t+r) = \hat{\sigma}(\tilde{s},t) \wedge \hat{\sigma}(\tilde{s}_{t+1},r) \quad \text{for } t,r \in \mathbf{N}. \tag{2.8}$$

Note that $\hat{\sigma}(\tilde{s}_{t+1},r)$ is well-defined from Lemma 2.1(ii). Let $\alpha = \hat{\sigma}(\tilde{s},t+r)$. From Lemma 2.1(i), we have

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}_{\alpha+\epsilon}^{t+r} \quad \text{and} \quad \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}_{\alpha-\epsilon}^{t+r} \quad \text{for any } \epsilon > 0.$$

Noting $Q_\alpha^t(\mathcal{K}_\alpha^l) = \mathcal{K}_\alpha^{l-t}$ $(1 \le t < l)$ and Lemma 1.2(ii), we obtain

$$\tilde{s}_{t+1,\alpha+\epsilon} = Q_{\alpha+\epsilon}^t(\tilde{s}_{\alpha+\epsilon}) \in Q_{\alpha+\epsilon}^t(\mathcal{K}_{\alpha+\epsilon}^{t+r}) = \mathcal{K}_{\alpha+\epsilon}^r \tag{2.9}$$

and

$$\tilde{s}_{t+1,\alpha-\epsilon} = Q_{\alpha-\epsilon}^t(\tilde{s}_{\alpha-\epsilon}) \notin Q_{\alpha-\epsilon}^t(\mathcal{K}_{\alpha-\epsilon}^{t+r}) = \mathcal{K}_{\alpha-\epsilon}^r. \tag{2.10}$$

Therefore, we get $\hat{\sigma}(\tilde{s}_{t+1},r) = \alpha$ from Lemma 2.1(i). Namely, $\hat{\sigma}(\tilde{s},t+r) = \hat{\sigma}(\tilde{s}_{t+1},r)$. Since $\hat{\sigma}(\tilde{s},t+r) \le \hat{\sigma}(\tilde{s},t)$ from Theorem 2.1(ii), we obtain $\hat{\sigma}(\tilde{s},t) \wedge \hat{\sigma}(\tilde{s}_{t+1},r) = \alpha$, and so (2.8) holds.

Next, we put $\delta(\tilde{s}) = \hat{\sigma}(\tilde{s},1)$ for $\tilde{s} \in \hat{\mathcal{F}}$. From (2.8), we get

$$\begin{aligned}
\hat{\sigma}(\tilde{s},t) &= \hat{\sigma}(\tilde{s},1) \wedge \hat{\sigma}(\tilde{s}_2,t-1) \\
&= \hat{\sigma}(\tilde{s},1) \wedge \hat{\sigma}(\tilde{s}_2,1) \wedge \hat{\sigma}(\tilde{s}_3,t-2) \\
&= \cdots \\
&= \bigwedge_{l=1}^t \hat{\sigma}(\tilde{s}_l,1) \\
&= \bigwedge_{l=1}^t \delta(\tilde{s}_l) \\
&= \delta(\tilde{s}_t) \quad \text{for } t \in \mathbf{N}.
\end{aligned}$$

Since we also have $\delta(Q(\tilde{s})) \le \delta(\tilde{s})$ from Theorem 2.1(ii), $\hat{\sigma}$ is a Markov FST with $\tilde{s}$. *q.e.d.*

3 Applications to fuzzy stopping problem

In this section, applying the results in the previous section, we obtain the optimal FST for a fuzzy dynamic system with fussy rewards([10]) when the weighting function is additive.

Firstly, we will formulate the stopping problem to be considered here. Let $\tilde{r} : S \times \mathbf{R} \mapsto [0, 1]$ be a fuzzy relation satisfying $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$. If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}$, the following fuzzy reward is earned:

$$R(\tilde{s})(z) := \sup_{x \in S}\{\tilde{s}(x) \vee \tilde{r}(x, z)\}, \quad z \in \mathbf{R}.$$

Then we can define a sequence of fuzzy rewards $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$, where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in (1.4) with the initial fuzzy state $\tilde{s}_1 = \tilde{s}$. Let

$$\varphi(\tilde{s}, t) := \sum_{l=1}^{t-1} R(\tilde{s}_l) \quad \text{for } t \in \mathbf{N}. \tag{3.1}$$

We need the following lemma, which is proved in [9].

Lemma 3.1 ([9, 10]). *For $t \in \mathbf{N}$ and $\alpha \geq 0$,*

$$\varphi(\tilde{s}, t)_\alpha = \sum_{l=1}^{t-1} R_\alpha(\tilde{s}_{l,\alpha})$$

holds, where

$$R_\alpha(\tilde{s}_{l,\alpha}) := \begin{cases} \{z \in \mathbf{R} \mid \tilde{r}(x, z) \geq \alpha \text{ for some } z \in \tilde{s}_{l,\alpha}\} & \text{for } \alpha > 0 \\ cl\{z \in \mathbf{R} \mid \tilde{r}(x, z) > 0 \text{ for some } z \in \tilde{s}_{l,\alpha}\} & \text{for } \alpha = 0. \end{cases} \tag{3.2}$$

Let $g : C(\mathbf{R}) \mapsto \mathbf{R}$ be any additive map with $g(\phi) = 0$, that is,

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S).$$

Adapting this g for a weighting function(see [4]), when an FST $\hat{\sigma} \in \Sigma(\hat{\mathcal{F}})$ and an initial fuzzy state $\tilde{s} \in \hat{\mathcal{F}}$ are used, the scalarization of the total fuzzy reward is given by

$$\begin{aligned} G(\tilde{s}, \hat{\sigma}) &= \int_0^1 g\left(\varphi(\tilde{s}, \hat{\sigma}_\alpha)_\alpha\right) d\alpha \\ &= \int_0^1 g\left(\sum_{t=1}^{\hat{\sigma}_\alpha - 1} R_\alpha(\tilde{s}_{t,\alpha})\right) d\alpha, \end{aligned} \tag{3.3}$$

where $\sum_{t=1}^{0} R_\alpha(\tilde{s}_{t,\alpha}) = \phi$ and $\hat{\sigma}_\alpha$ means $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \min\{t \in \mathbf{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}$ for simplicity. Since $\varphi(\tilde{s}, \hat{\sigma}_\alpha) \in C(\mathbf{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \hat{\sigma}_\alpha)_\alpha)$ is left-continuous in $\alpha \in (0, 1]$, therefore the right-hand integral of (3.3) is well-defined. For a given $\mathcal{F}' \subset \mathcal{F}$, our objective is to maximize (3.3) over all FSTs $\hat{\sigma} \in \Sigma(\mathcal{F}')$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}'$.

Definition 3. An FST $\hat{\sigma}^*$ with $\tilde{s} \in \mathcal{F}'$ is called an $\tilde{s}$-*optimal* if

$$G(\tilde{s}, \hat{\sigma}) \leq G(\tilde{s}, \hat{\sigma}^*) \quad \text{for all } \hat{\sigma} \in \Sigma(\mathcal{F}').$$

If $\hat{\sigma}^*$ is $\tilde{s}$-optimal for all $\tilde{s} \in \mathcal{F}'$, $\hat{\sigma}^*$ is called *optimal* in $\mathcal{F}'$.

Now we will seek a $\tilde{s}$-optimal or an optimal FST by using the results in the previous sections. For each $\alpha \in [0, 1]$, let

$$\mathcal{K}_\alpha(g) := \{c \in C(S) \mid g(R_\alpha(c)) \leq 0\}. \tag{3.4}$$

Here we need the following Assumptions B1 and B2, which are assumed to hold henceforth.

Assumption B1 (Closedness).

$$Q_\alpha(\mathcal{K}_\alpha(g)) \subset \mathcal{K}_\alpha(g) \quad \text{for all } \alpha \in [0, 1]$$

Now we define the sequence $\{\mathcal{K}^t_\alpha(g)\}_{t=1}^\infty$ by (2.2) – (2.3), that is,

$$\mathcal{K}^t_\alpha(g) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) \quad \text{for } t \geq 1. \tag{3.5}$$

Assumption B2. For all $\alpha \in [0,1]$, it holds that

$$C(S) = \bigcup_{t=1}^\infty \mathcal{K}^t_\alpha(g).$$

Using the sequence $\{\mathcal{K}^t_\alpha(g)\}_{t=1}^\infty$ given in (3.5), we define $\hat{\sigma}_\alpha$, $\hat{\mathcal{F}}$, $\hat{\sigma}$ and $\hat{\sigma}(\tilde{s},\cdot)_\alpha$, respectively, by (2.4), (2.5), (2.6) and (2.7). Then, from Theorems 2.1 and 2.2, $\hat{\sigma}$ is a Markov FST on $\hat{\mathcal{F}}$.

The following theorem will be proved by applying the idea of the one-step look ahead(OLA) policy([3, 8, 14]) for stochastic stopping problems.

Theorem 3.1. *Under Assumptions B1 and B2, $\hat{\sigma}$ is optimal in $\hat{\mathcal{F}}$.*

Proof. Firstly, condsider the deterministic stopping problem which maximizes $g(\varphi(\tilde{s},t)_\alpha)$ over $t \geq 1$. As g is additive, $g(\varphi(\tilde{s},t)_\alpha) = \Sigma_{l=1}^{t-1} g\left(R_\alpha(\tilde{s}_{l,\alpha})\right)$. Therefore $g(\varphi(\tilde{s},t)_\alpha) \geq g(\varphi(\tilde{s},t+1)_\alpha)$ if and only if $\tilde{s}_{t,\alpha} \in K_\alpha(g)$. By the assumption B1, $\tilde{s}_{t,\alpha} \in K_\alpha(g)$ implies $g(\varphi(\tilde{s},t)_\alpha) \geq g(\varphi(\tilde{s},l)_\alpha)$ for all $l \geq t$. Thus, since $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \hat{\sigma}(\tilde{s},\cdot)_\alpha$ by Theorem 2.1, we can show

$$g\left(\varphi(\tilde{s},\hat{\sigma}(\tilde{s},\cdot)_\alpha)\right) \geq g\left(\varphi(\tilde{s},\tilde{\sigma}(\tilde{s},\cdot)_\alpha)\right)$$

for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ and $\alpha \in [0,1]$. This implies that $G(\tilde{s},\hat{\sigma}) \geq G(\tilde{s},\tilde{\sigma})$ for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ by using (3.3). This complete the proof. *q.e.d.*

4 A numerical example

An example is given to illustrate the previous results of fuzzy stopping problem in this section.

Let $S := [0,1]$. The fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by

$$\tilde{q}(x,y) = \begin{cases} 1 & \text{if } y = \beta x \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{r}(x,z) = \begin{cases} 1 & \text{if } z = x - \lambda \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is an observation cost and $0 < \beta < 1$ for $x, y, z \in [0,1]$ and $z \in \mathbf{R}$. Then, Q_α and R_α defined by (1.5) and (3.2) are independent of α and are calculated as follows:

$$Q_\alpha([a,b]) = \beta[a,b] \quad \text{and} \quad R_\alpha([a,b]) = [a-\lambda, b-\lambda]$$

for $0 \leq a \leq b \leq 1$.

Let $g([a,b]) := (a+2b)/3$ for $0 \leq a \leq b \leq 1$, which is additive. Then, $\mathcal{K}_\alpha(g)$ is given as

$$\mathcal{K}_\alpha(g) = \{[a,b] \in C(S) \mid a + 2b \leq 0\},$$

So $\mathcal{K}^t_\alpha(g) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) = \{[a,b] \in C(S) \mid a + 2b \leq 3\lambda\beta^{1-t}\}$. Since $\mathcal{K}^t_\alpha(g)$ is independent of α, we see that $Q_\alpha(\mathcal{K}_\alpha(g)) = \{\beta[a,b] \mid [a,b] \in \mathcal{K}_\alpha(g)\} \subset \mathcal{K}_\alpha(g)$ and $\bigcup_{t=1}^\infty \mathcal{K}^t(g) = C(S)$. Thus Assumptions B1 and B2 in Section 3 are satisfied in this example.

Let the initial fuzzy state be

$$\tilde{s}(x) := (1 - \mid 8x - 4 \mid) \vee 0 \quad \text{for } x \in [0,1].$$

For the stopping time $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ given in (2.4), we easily obtain that $\tilde{s}_\alpha = [(3+\alpha)/8, (5-\alpha)/8]$ and $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \min\{t \geq 1 \mid 13 - \alpha \leq 24\lambda\beta^{1-t}\}$. Thus, as $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$, we have $\tilde{s} \in \hat{\mathcal{F}}$.

Since $\hat{\sigma}_\alpha(\hat{\tilde{s}}_\alpha) \in \mathcal{K}^t(g)$ means $13 - \alpha \leq 24\lambda\beta^{1-t}$, then

$$\hat{\sigma}(\tilde{s}, t) = 1 \wedge \left((13 - 24\lambda)\beta^{1-t} \vee 0\right).$$

The numerical value of $\hat{\sigma}$ is given in Table 1.

Table 1. An $\tilde{s}$-optimal FST ($\lambda = 0.48$, β =0.98).

t	1	2	3	4	5	6	7	8	$\cdots$
$\hat{\sigma}(\tilde{s}, t)$	1	1	1	.7603	.5108	.2552	.00	.00	$\cdots$

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.

[2] Chen-wei Xu, On convexity of fuzzy sets and fuzzy relations, *Information Science* **59** (1992) 92-102.

[3] Y.S.Chow, H.Robbins and D.Siegmund, *The theory of optimal stopping: Great expectations*, Houghton Mifflin Company, New York, 1971.

[4] P.Fortemps and M.Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82** (1996) 319-330.

[5] J.Kacprzyk, Control of a non-fuzzy system in a fuzzy environment with fuzzy termination time, *System Sciences* **3** (1977) 325-341.

[6] J.Kacprzyk, Decision making in a fuzzy environment with fuzzy termination time, *Fuzzy Sets and Systems* **1** (1978) 169-179.

[7] J.Kacprzyk and A.O.Esogbue, Fuzzy dynamic programming: Main developments and applications, *Fuzzy Sets and Systems* **81** (1996) 31-48.

[8] Y.Kadota, M.Kurano and M.Yasuda, Utility-Optimal Stopping in a Denumerable Markov Chain, *Bull. Infor. Cyber. Res. Ass. Stat. Sci., Kyushu University* **28** (1996) 15-21.

[9] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[10] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, Markov-type fuzzy decision processes with a discounted reward on a closed interval, *European Journal of Operational Research* **92** (1996) 649-662.

[11] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, An approach to stopping problems of a dynamic fuzzy system, *preprint*.

[12] V.Novák, *Fuzzy Sets and Their Applications* (Adam Hilder, Bristol-Boston, 1989).

[13] M.L.Puri and D.A.Ralescu, The concept of normality for fuzzy random variables, *Ann. Prob.* **13** (1985) 1373-1379.

[14] S.M.Ross, *Applied Probability Models with Optimization Applications*, Holden-Day, 1970.

[15] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66** (1994) 39-57.

[16] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.