数理解析研究所講究録
1015 巻 1997 年 14-23

Sensitivity Analysis in Multiobjective Optimization Problems[1]

Hun Kuk†, Do Sang Kim† and Tetsuzo Tanino‡

† Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Korea

‡ Department of Electronic and Information Systems, Graduate School of Engineering, Osaka University, Osaka 565, Japan

1. Introduction

Stability and sensitivity analysis is not only theoretically interesting but also practically important in optimization theory.

In this paper we provide some theoretical results concerning sensitivity analysis in multiobjective optimization. Since there are three types of solution concepts, i.e., minimality, proper minimality and weak minimality with respect to the ordering cone for a multiobjective optimization problem, we can consider three types of perturbation maps according to those solution concepts for a given family of multiobjective optimization problems that depend on a parameter vector. Each of the perturbation maps, called perturbation map, proper perturbation map and weak perturbation map, is defined as a set-valued map which associates to each parameter value the set of all minimal, properly minimal and weakly minimal points, respectively, of the perturbed feasible set in the objective space with respect to a fixed ordering cone. The behavior of the perturbation maps is analyzed quantitatively by using the concept of contingent derivatives for set-valued maps in finite dimensional Euclidean spaces. Namely, we investigate the relationships between the contingent derivatives of the perturbation maps and those of the feasible set map in the objective space.

2. A Parametrized Family of Nonlinear Multiobjective Optimization Problems and Perturbation Maps

[1]This research was supported in part by BSRI-97-1440 and Center for Applied Mathematics at KAIST

We consider a family of parametrized multiobjective optimization problems

$$\begin{cases} \text{minimize} & f(x,u) \\ \text{subject to} & x \in X(u) \subset R^n \end{cases}$$

where f is a p-dimensional vector-valued function, x is a decision variable in R^n, u is a perturbation parameter vector in R^m, f is a real-valued objective function defined on $R^n \times R^m$ and X is a set-valued function (multifunction) from R^m to R^n.

Let Y be a set-valued map from R^m to R^p defined by

$$Y(u) = \{y = f(x,u) : x \in X(u)\} \quad \text{for each } u \in R^m.$$

Y is considered as the feasible set map in the objective space. In order to define a solution of the multiobjective optimization problem we consider a partial order in the objective space R^p induced by a pointed closed convex cone K with a nonempty interior in R^p, where K is said to be pointed if $K \cap (-K) = \{0\}$. Then we can define the following three sets for a set A in R^p:

$$\begin{aligned} \mathrm{Min}_K A &= \{\hat{y} \in A : \text{there exists no } y \in A \text{ such that } y \leq_K \hat{y}\} \\ &= \{\hat{y} \in A : (A - \hat{y}) \cap (-K) = \{0\}\}, \\ \mathrm{PrMin}_K A &= \{\hat{y} \in A : \text{there exists a cone } C \text{ such that where} \\ &\qquad C \text{ is a convex cone with } C \neq R^p \text{ and } K \setminus \{0\} \subset \operatorname{int} C\}, \\ \mathrm{WMin}_K A &= \{\hat{y} \in A : \text{there exists no } y \in A \text{ such that } y <_K \hat{y}\} \\ &= \{\hat{y} \in A : (A - \hat{y}) \cap (-\operatorname{int} K) = \emptyset\} \end{aligned}$$

We call these three sets the sets of the K-minimal, properly K-minimal, and weakly K-minimal points of A, respectively.

According to these three solution concepts we can define the following three set-valued maps W, G and S from R^m to R^p by

$$W(u) = \mathrm{Min}_K Y(u), \quad \text{for any } u \in R^m,$$

$$G(u) = \mathrm{PrMin}_K Y(u), \quad \text{for any } u \in R^m,$$

and

$$S(u) = \mathrm{WMin}_K Y(u), \quad \text{for any } u \in R^m$$

are called the perturbation map, the proper perturbation map and the weak perturbation map, respectively.

3. Contingent Derivatives of Set-Valued Maps

In this section we briefly review the concepts of contingent derivatives and TP-derivatives of set-valued maps and provide some basic properties which are necessary in the following section. The notions of derivatives of set-valued maps are direct generalizations of the point-valued directional derivatives. Throughout this section, let F be a set-valued map from R^m to R^p.

Definition 3.1. ([1, 2]) Let A be a nonempty subset of R^m, and let $\hat{v} \in R^m$. The set $T_A(\hat{v}) \subset R^m$, defined by

$$T_A(\hat{v}) = \{v \in R^m : \exists \{v^k\} \subset R^m, \exists \{h_k\} \subset \text{int}R_+ \text{ such that } v^k \to v, \ h_k \to 0 \text{ and } \forall k, \ \hat{v} + h_k v^k \in A\},$$

is called the contingent cone to A at $\hat{v}$, where intR_+ is the set of all positive real numbers.

The graph

$$\text{graph } F = \{(v, z) : z \in F(v)\} \subset R^m \times R^p.$$

Definition 3.2. ([12]). Let $(\hat{v}, \hat{z})$ be a point in graph F. The set $TP_{\text{graph } F}(\hat{v}, \hat{z}) \subset R^m \times R^p$, defined by

$$TP_{\text{graph } F}(\hat{v}, \hat{z}) = \{(v, z) \in R^m \times R^p : \exists \{(v^k, z^k)\} \subset \text{graph } F, \exists \{h_k\} \subset \text{int}R_+ \text{ such that } v^k \to \hat{v} \text{ and } h_k((v^k - \hat{v}) \to v\}.$$

is called the TP-cone to graph F at $(\hat{v}, \hat{z})$.

It is clear that

$$T_{\text{graph } F}(\hat{v}, \hat{z}) \subset TP_{\text{graph } F}(\hat{v}, \hat{z})$$

with equality holding if graph F is convex.

Now we introduce two concepts of contingent derivatives of the set-valued map F defined by considering the above two cones to graph F, respectively.

Definition 3.3. ([2]) Let $(\hat{v}, \hat{z})$ be a point in graph F. We denote by $DF(\hat{v}, \hat{z})(\cdot)$ the set-valued map from R^m to R^p whose graph is the contingent cone $T_{\text{graph } F}(\hat{v}, \hat{z})$ and call it the contingent derivative of F at $(\hat{v}, \hat{z})$.

Definition 3.4. ([12]) Let $(\hat{v}, \hat{z})$ be a point in graph F. We denote by $PF(\hat{v}, \hat{z})$ the set-valued map from R^m to R^p whose graph is the cone $TP_{\text{graph } F}(\hat{v}, \hat{z})$ and call it the TP-derivative of F at $(\hat{v}, \hat{z})$.

We consider the set-valued map $F + K$ from R^m to R^p defined by

$$(F + K)(v) = F(v) + K, \qquad \text{for all } v \in R^m.$$

Proposition 3.1. ([12]) Assume that

$$PF(\hat{v}, \hat{z})(0) \cap (-K) = \{0\}.$$

Then, for any $v \in R^m$,

$$DF(\hat{v}, \hat{z})(v) + K = D(F + K)(\hat{v}, \hat{z})(v).$$

Theorem 3.1. ([6]) Assume that

$$PF(\hat{v}, \hat{z})(0) \cap (-K) = \{0\}.$$

Then, for any $v \in R^m$,

(i) $\mathrm{PrMin}_K DF(\hat{v}, \hat{z})(v) = \mathrm{PrMin}_K D(F + K)(\hat{v}, \hat{z})(v)$,

(ii) $\mathrm{Min}_K DF(\hat{v}, \hat{z})(v) = \mathrm{Min}_K D(F + K)(\hat{v}, \hat{z})(v)$,

(iii) $\mathrm{WMin}_K DF(\hat{v}, \hat{z})(v) \subset \mathrm{WMin}_K D(F + K)(\hat{v}, \hat{z})(v)$.

Moreover, if $\tilde{K}$ is a closed convex cone with $\tilde{K} \subset \text{int } K \cup \{0\}$, then

(iii)' $\mathrm{WMin}_K DF(\hat{v}, \hat{z})(v) = \mathrm{WMin}_K D(F + \tilde{K})(\hat{v}, \hat{z})(v)$.

4. Contingent Derivatives of Perturbation Maps

In this section we provide some relationships between the contingent derivative DY of Y and the contingent derivatives DG, DW and DS of

G, W and S, respectively. Throughout this section, when speaking of the perturbation maps G, W and S, we fix the nominal value of u as $\hat{u}$ and consider a point $\hat{y}$ belonging to $G(\hat{u})$, $W(\hat{u})$ and $S(\hat{u})$, respectively. A cone $\tilde{K}$ is assumed to be a closed convex cone contained in $(\text{int } K) \cup \{0\}$.

Definition 4.1. We say that
(i) Y is K-dominated by W near $\hat{u}$ if

$$Y(u) \subset W(u) + K, \quad \text{for any } u \in N_{\hat{u}},$$

(ii) Y is K-dominated by S near $\hat{u}$ if

$$Y(u) \subset S(u) + K, \quad \text{for any } u \in N_{\hat{u}},$$

where $N_{\hat{u}}$ is some neighborhood of $\hat{u}$ in R^m.

Remark 4.1. Since $W(u) \subset S(u)$, if Y is K-dominated by W near $\hat{u}$, then Y is K-dominated by S near $\hat{u}$. Moreover, when $\overset{\circ}{K} = (\text{int } K) \cup \{0\}$, if Y is K-dominated by W near $\hat{u}$, then Y is $\overset{\circ}{K}$-dominated by S near $\hat{u}$.

Theorem 4.1. ([6]) Assume that

$$PY(\hat{u}, \hat{y})(0) \cap (-K) = \{0\}.$$

(i) If Y is K-dominated by W near $\hat{u}$, then

$$\text{Min}_K DY(\hat{u}, \hat{y})(u) \subset DW(\hat{u}, \hat{y})(u), \qquad \text{for any } u \in R^m.$$

(ii) If Y is $\tilde{K}$-dominated by S near $\hat{u}$, then

$$\text{WMin}_K DY(\hat{u}, \hat{y})(u) \subset DS(\hat{u}, \hat{y})(u), \qquad \text{for any } u \in R^m.$$

In order to obtain the relationship between DY and DG, we introduce the concepts of cone closedness and cone boundedness.

Definition 4.2. Let A be a nonempty set in R^p, and let D be a cone in R^p. Then A is said to be
(i) D-closed if $A + D$ is closed, and
(ii) D-bounded if $A^+ \cap (-D) = \{0\}$,
where

$$A^+ = \{y \in R^p : \exists \{h_k\} \subset \text{int} R_+, \exists \{y^k\} \subset A \text{ such that } h_k \to 0 \text{ and } h_k y^k \to y\}.$$

Remark 4.2. Let D be a pointed cone. If there exists $y^\circ \in R^p$ such that $\Gamma \subset y^\circ +$ cl D, then it is clear that Γ is D-bounded.

Remark 4.3. Let Y be a set-valued map from R^m to R^p. A sufficient condition for Y to be K-dominated by W near $\hat{u}$ is that the set $Y(u)$ be a nonempty K-bounded, K-closed set for any $u \in N_{\hat{u}}$.

Lemma 4.1. ([6]) Let $Y(u)$ be a K-bounded, K-closed set for any $u \in N_{\hat{u}}$. Then, for any $u \in U$,

$$DG(\hat{u}, \hat{y})(u) = DW(\hat{u}, \hat{y})(u).$$

Theorem 4.2. ([6]) Assume that

$$PY(\hat{u}, \hat{y})(0) \cap (-K) = \{0\}.$$

If $Y(u)$ is K-bounded and K-closed for any $u \in N_{\hat{u}}$, then

$$\mathrm{PrMin}_K DY(\hat{u}, \hat{y})(u) \subset DG(\hat{u}, \hat{y})(u), \quad \text{for any } u \in U.$$

Now we introduce here the notions of Dini upper and lower derivatives from Penot [9]. Let F be a set-valued map from R^m to R^p.

Definition 4.3. Let $u \in R^m$, the Dini upper and lower derivatives of F at $(\hat{u}, \hat{y}) \in R^m \times R^p$ in direction u are given respectively by

$$D_{upp}F(\hat{u}, \hat{y})(u) = \limsup_{(h,v) \to (0^+, u)} (F(\hat{u} + hv) - \hat{y})/h,$$

$$D_{low}F(\hat{u}, \hat{y})(u) = \liminf_{(h,v) \to (0^+, u)} (F(\hat{u} + hv) - \hat{y})/h.$$

It is clear that $D_{low}F(\hat{u}, \hat{y})(u) \subset D_{upp}F(\hat{u}, \hat{y})(u) = DF(\hat{u}, \hat{y})(u)$ and these sets are closed.

Definition 4.4. The set-valued map F is said to be semi-differentiable at $(\hat{u}, \hat{y}) \in$ graph F if $D_{low}F(\hat{u}, \hat{y}) = D_{upp}F(\hat{u}, \hat{y})$.

Remark 4.4. Let graph F be a convex set and let $(\hat{u}, \hat{y}) \in$ graph F. A sufficient condition for F to be semi-differentiable at $(\hat{u}, \hat{y})$ is that (i) graph F has nonempty interior or (ii) graph F is closed, $\hat{u} \in$ int (dom F), where dom $F = \{u \in R^m : F(u) \neq \emptyset\}$ ([9]).

Theorem 4.3. ([6]) If Y is semi-differentiable at $(\hat{u}, \hat{y})$, then

$$DW(\hat{u}, \hat{y})(u) \subset \mathrm{WMin}_K DY(\hat{u}, \hat{y})(u), \quad \text{for any } u \in U.$$

5. Contingent Derivatives of Perturbation Maps under Convexity Assumptions

Throughout this section we impose the following convexity assumptions on the feasible decision set map X and the objective function f.

Convexity assumption (CA)

(1) The set-valued map X is convex, i.e. the graph of X

$$\text{graph } X = \{(u, x) \in R^m \times R^n : x \in X(u)\}$$

is a convex set in $R^m \times R^n$. In other words, for any $u^1, u^2 \in R^m$ and any α, $0 \leq \alpha \leq 1$,

$$\alpha X(u^1) + (1 - \alpha) X(u^2) \subset X(\alpha u^1 + (1 - \alpha) u^2).$$

(2) The function f is K-convex, i.e., for any $(x^1, u^1), (x^2, u^2) \in R^n \times R^m$ and any α, $0 \leq \alpha \leq 1$,

$$\alpha f(x^1, u^1) + (1 - \alpha) f(x^2, u^2) \in f(\alpha x^1 + (1 - \alpha) x^2, \alpha u^1 + (1 - \alpha) u^2) + K.$$

Proposition 5.1. ([15]) Under the convexity assumption (CA), the set-valued map Y defined by

$$Y(u) = \{y = f(x, u) : x \in X(u)\} \quad u \in R^m$$

is K-convex, i.e., for any $u^1, u^2 \in R^m$ and any α, $0 \leq \alpha \leq 1$,

$$\alpha Y(u^1) + (1 - \alpha) Y(u^2) \subset Y(\alpha u^1 + (1 - \alpha) u^2) + K.$$

In other words, graph $(Y + K)$ is convex.

We can omit the condition

$$PY(\hat{u}, \hat{y})(0) \cap (-K) = \{0\}$$

under the convexity assumption (CA).

Theorem 5.1.
(i) If Y is K-dominated by W near $\hat{u}$, then

$$\mathrm{Min}_K DY(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u), \qquad \text{for any } u \in R^m.$$

(ii) If Y is $\tilde{K}$-dominated by S near $\hat{u}$, then

$$\mathrm{WMin}_K DY(\hat{u},\hat{y})(u) \subset DS(\hat{u},\hat{y})(u), \qquad \text{for any } u \in R^m.$$

Next we consider sufficient conditions for the converse inclusion of the above theorem.

Definition 5.1. ([2]) Let A be a nonempty set in R^p and $\hat{y} \in R^p$. The normal cone $N_A(\hat{y})$ to A at $\hat{y}$ is the negative polar cone of the tangent cone $T_A(\hat{y})$, i.e.,

$$N_A(\hat{y}) = \{T_A(\hat{y})\}^\circ = \{\mu \in R^p : \langle \mu, y\rangle \leq 0 \ \ \forall y \in T_A(\hat{y})\}.$$

When A is a convex set and $\hat{y} \in A$,

$$N_A(\hat{y}) = \{\mu \in R^p : \langle \mu, \hat{y}\rangle \geq \langle \mu, y\rangle \ \ \forall y \in A\}.$$

Definition 5.2. ([15]) Let A be a nonempty K-convex set in R^p. If a point $\hat{y} \in \mathrm{Min}_K A$ satisfies the condition

$$N_{A+K}(\hat{y}) \subset \mathrm{int}\, K^\circ \cup \{0\},$$

then $\hat{y}$ is called the normally K-minimal point of A.

Remark 5.1. A point $\hat{y} \in A$ is said to be a porperly K-minimal point of a convex set A if

$$T_{A+K}(\hat{y}) \cap (-K) = \{0\}.$$

In this case, there exists a vector $\mu \in N_{A+K}(\hat{y}) \cap \mathrm{int}\, K^\circ$. Thus, the normal K-mirmality is a stronger concept than the proper K-minimality. From the geometric viewpoint, the latter implies the existence of the supporting hyperplane to A at $\hat{y}$ with the normal vector $\mu \in \mathrm{int} K^\circ$ and, on the other hand, the former implies that all the normal vectors of the supporting hyperplanes to A at $\hat{y}$ belong to $\mathrm{int} P^\circ$.

Theorem 5.2. ([15]) If $\hat{u} \in \text{int}(\text{dom } Y)$ and $\hat{y}$ is a normally K-minimal point of $Y(\hat{u})$, then

$$DW(\hat{u}, \hat{y})(u) \subset \text{Min}_K DY(\hat{u}, \hat{y})(u) \quad \forall u \in R^m,$$

where

$$\text{dom } Y = \{u \in R^p : Y(u) \neq \emptyset\}.$$

Theorem 5.3. If $\hat{u} \in \text{int}(\text{dom } Y)$, then

$$DW(\hat{u}, \hat{y})(u) \subset DS(\hat{u}, \hat{y})(u) \subset \text{WMin}_K DY(\hat{u}, \hat{y})(u) \quad \forall u \in R^m.$$

Theorem 5.4. If $\hat{u} \in \text{int}(\text{dom } Y)$, $\hat{y} \in G(\hat{u})$ and Y is $\tilde{K}$-dominated by S near $\hat{u}$ for a closed convex cone $\tilde{K}$ such that $\tilde{K} \subset \text{int } K \cup \{0\}$, then

$$DW(\hat{u}, \hat{y})(u) = DS(\hat{u}, \hat{y})(u) = \text{WMin}_K DY(\hat{u}, \hat{y})(u) \quad \forall u \in R^m.$$

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