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QUASIISOMETRIC MAPPINGS AND THE VARIATIONAL CAPACITY[1]

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1. Introduction. Given a domain D in d-dimensional Euclidean space $\mathbf{R}^d$ $(d \geq 2)$ and given an exponent $1 < p < \infty$, the *Royden p-algebra* $M_p(D)$ of D is defined by $M_p(D) := L^{1,p}(D) \cap L^\infty(D) \cap C(D)$, which is a commutative Banach algebra, i.e. the so-called normed ring, under pointwise addition and multiplication with $\|u; M_p(D)\| := \|u; L^\infty(D)\| + \|\nabla u; L^p(D)\|$ as norm, where $L^{1,p}(D)$ is the Dirichlet space, i.e. the space of locally integrable real valued functions u on D whose distributional gradients ∇u belong to $L^p(D)$. The maximal ideal space D_p^* of $M_p(D)$ is referred to as the *Royden p-compactification* of D, which is also characterized as the compact Hausdorff space containing D as its open and dense subspace such that every function in $M_p(D)$ is continuously extended to D_p^* and $M_p(D)$ is uniformly dense in $C(D_p^*)$.

Suppose that D and D' are domains in $\mathbf{R}^d$ (and more generally that D and D' are Riemannian manifolds of class C^∞ which are orientable and connected). In 1982 , H. Tanaka and the present author [7] showed that D_d^* and $(D')_d^*$ are homeomorphic if and only if there exists an almost quasiconformal mappimg of D onto D'. Here we say that a homeomorphism f of D onto D' is an almost quasiconformal mapping of D onto D' if there exists a compact set $E \subset D$ such that $f|D \setminus E$ is a quasiconformal mapping of $D \setminus E$ onto $D' \setminus f(E)$. Since then it has been an open question what can be said about the above result if the exponent d is replaced by $1 < p < d$. Recently we obtained the following result [6] answering to the above question: when $1 < p < d$, D_p^* and $(D')_p^*$ are homeomorphic if and only if there exists an almost quasiisometric mapping of D onto D'. Here we say that a homeomorphism f of D onto D' is almost quasiisometric mapping if there exists a compact set $E \subset D$ such that $f|D \setminus E$ is a quasiisometric (i.e. bi-Lipschtz with respect to geodesic distances) mapping of $D \setminus E$ onto $D' \setminus f(E)$. The proof of this last result seems to depend essentially on the fact that when $1 \leq p < d$, a homeomorphism which, together with its inverse, does not increase the p-capacity of spherical rings having the p-capacity less than any given fixed positive number by more than a fixed factor is quasiisometric (i.e. locally bi-Lipschitzian).

The purpose of this paper is to give a proof of the above mentioned last fact. This is a slight but nontrivial extension of a part of a beautiful theorem of Gehring [3] appeared

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in 1971 (cf. also Reimann [8]). Our proof is an amelioration of that of Gehring in the above paper and largely mimics it. The reason we do not consider exponents $d < p \leq \infty$ is the following two: first, the formal extention of the Gehring theorem to our present setting is no longer true for $d < p \leq \infty$ and thus we must exclude exponents $d < p \leq \infty$; second, even if D_p^* and $(D')_p^*$ for $d < p \leq \infty$ are homeomorphic, D and D' need not even be homeomorphic and therefore, from this point of view, there is no need to consider the case $d < p \leq \infty$. As for the case $p = d$, we only have to remark that quasiisometric mappings are quasiconformal mappings but there are quasiconformal mappings which are not quasiisometric mappings.

2. Terminology and the main result. We denote by $\mathbf{R}^d$ $(d \geq 2)$ the d-dimensional Euclidean space and by $\overline{\mathbf{R}}^d$ its one point compactification obtained by adding the point ∞ at infinity. Sometimes points in $\mathbf{R}^d$ are denoted by the uppercase letter P or by the lowercase letters x and y. In the latter case the i^{th} component of x will be denoted by x_i. Points in $\mathbf{R}^d$ are also viewed as vectors and the norms of P and x are denoted by $|P|$ and $|x|$.

For each subset $E \subset \mathbf{R}^d$ we denote by ∂E, $\overline{E}$, and E^c respectively the boundary, closure, and complement of E in $\overline{\mathbf{R}}^d$. For each $1 \leq k \leq d$ we denote by m_k the k-dimensional Hausdorff measure in $\mathbf{R}^d$ so normalized that m_d is the Lebesgue volume measure in $\mathbf{R}^d$ and $m_{d-1}(S)$ is the surface area measure of a smooth surface S in $\mathbf{R}^d$. We use the notation

$$\omega_d = m_{d-1}(S^{d-1}), \qquad \tau_d = m_d(B^d),$$

where B^d is the unit ball $\{x : |x| < 1\}$ in $\mathbf{R}^d$ and S^{d-1} the unit sphere ∂B^d so that $\omega_d = d\tau_d$.

We say that a region $R \subset \mathbf{R}^d$ is a *ring* if R^c consists of exactly two components C_0 and C_1. To be definite we always assume that $\infty \in C_1$ so that C_0 is a compact set in $\mathbf{R}^d$. The *p-capacity* $\text{cap}_p R$ of R $(1 \leq p \leq \infty)$ is given by

$$\text{cap}_p R := \inf_{u \in W(R)} \int_R |\nabla u(x)|^p dm_d(x),$$

where ∇u is the (usual or distributional) gradient vector $(\partial u/\partial x_1, \cdots, \partial u/\partial x_d)$ of u and $W(R) := \{u \in C(\overline{\mathbf{R}}^d) \cap ACL(R) : u|C_i = i \ \ (i = 0, 1)\}$, where $ACL(R)$ is the class of real valued functions u on R such that u is absolutely continuous on each component of the intersection of R with almost every straight line perpendicular to each coordinate plane. Here a family F of straight lines l perpendicular to a coordinate plane $\pi_i = \{x \in \mathbf{R}^d : x_i = 0\}$ for some $i = 1, \cdots, d$ is measured by the m_{d-1}-measure of the set $\{\pi_i \cap l : l \in F\}$. As an example of important rings we have a *Teichmüller ring* R_T determined by

$$(R_T)^c = \{te_1 : -1 \leq t \leq 0\} \cup \{te_1 : 1 \leq t \leq \infty\}$$

with $e_1 = (1, 0, \cdots, 0)$. We will use the conformal d-capacity $t_d := \text{cap}_d R_T$. A ring R is said to be a *spherical ring* if

$$R = \{x : a < |x - P| < b\},$$

where $0 < a < b < \infty$ and $P \in \mathbf{R}^d$. In the case of the above spherical ring R it can be easily seen (cf. e.g. p.35 in Heinonen et al. [4] for a proof for the case $1 < p < \infty$; the result for the case of $p = 1$ is indicated in [3]; the cases for $p = 1$ and $p = \infty$ can be shown directly and more easily by the similar way as those for $1 < p < \infty$ (see also the appendix at the end of this paper)) that

$$\text{(3)} \qquad \mathrm{cap}_p R = \begin{cases} \omega_d a^{d-1} & (p = 1), \\ \omega_d \left(\dfrac{b^q - a^q}{q}\right)^{1-p} & (1 < p < \infty,\ p \neq d), \\ \omega_d \left(\log \dfrac{b}{a}\right)^{1-d} & (p = d), \\ \dfrac{1}{b-a} & (p = \infty), \end{cases}$$

where we have set $q = (p-d)/(p-1)$. In passing we remark that when $d < p \leq \infty$ and D is a bounded open set in $\mathbf{R}^d$, as a result of (3), we have $\inf_R \mathrm{cap}_p R > 0$, where the infimum is taken with respect to all spherical rings $R \subset D$.

Hereafter we always assume that D and D' are nonempty open sets in $\mathbf{R}^d$ and that f is a homeomorphism of D onto D'; then f maps each ring R in D onto a ring $f(R) \subset D'$. We say that

$$f \in Q_p(K, \delta)$$

for $0 < K < \infty$ and $0 < \delta \leq \infty$, if the following condition is satisfied:

$$\text{(4)} \qquad \mathrm{cap}_p f(R) \leq K \mathrm{cap}_p R$$

for every *spherical* ring R with $\overline{R} \subset D$ and

$$\text{(5).} \qquad \mathrm{cap}_p R < \delta.$$

Here we do not exclude the $\delta = \infty$ case in which the condition (5) is redundant. Hence of course our main concern is about the class $Q_p(K, \delta)$ for $0 < \delta < \infty$. To be precise we sometimes write $Q_p(K, \delta; D, D')$ to indicate that mappings are from D onto D'.

Next we consider a metric property for homeomorphisms f of D onto D'. For each $P \in D$ we set

$$L(P, f) := \limsup_{r \downarrow 0} \frac{\max_{|x-P|=r} |f(x) - f(P)|}{r}.$$

We say that f is K-*Lipschitzian* on D,

$$f \in Lip(K)$$

or more precisely $f \in Lip(K; D, D')$ in notation, if $L(P, f) \leq K$ $(0 < K < \infty)$ for every $P \in D$. The Riemannian distance $\rho_D(x, y)$ between x and y in D is given by

$$\rho_D(x, y) = \inf_\gamma m_1(\gamma),$$

where γ runs over all connected polygonal lines in D connecting x and y in D; if there is no γ connecting x and y or equivalently if x and y are in different components of D, then we understand that $\rho_D(x, y) = \infty$. A homeomorphism f of D onto D' is referred to as a *K-quasiisometric mapping* of D onto D' if there exists a $K \in [1, \infty)$ such that

$$K^{-1}\rho_D(x, y) \leq \rho_{D'}(f(x), f(y)) \leq K\rho_D(x, y)$$

for every pair of points x and y in D. It is readily seen that a homeomorphism f of D onto D' is a K-quasiisometric mapping if and only if $f \in Lip(K; D, D')$ and $f^{-1} \in Lip(K; D', D)$.

It is easy to see that if a homeomorphism f of D onto D' satisfies $f, f^{-1} \in Lip(K)$, then $f, f^{-1} \in Q_p(K^{p+d}, \delta)$ for every $1 \leq p < \infty$ and $0 < \delta \leq \infty$. In view of the reason mentioned in the introduction we are interested in studying the converse of the above fact only for $1 \leq p < d$. We can state the following rsult.

THE MAIN THEOREM. *Suppose f is a homeomorphism of a nonempty open set D onto D' in d-dimensional Euclidean space $\mathbf{R}^d$ with $d \geq 2$, $1 \leq p < d$, $0 < K < \infty$, and $0 < \delta \leq \infty$. If $f, f^{-1} \in Q_p(K, \delta)$, then $f, f^{-1} \in Lip(K_1)$, where $0 < K_1 < \infty$ depends only on d, p, K, and not dependent on δ; explicitly*

$$K_1 = K^{\frac{1}{d-p}} \exp\left(\left(2^{d+1}\omega_d^{1+\frac{1}{d}} K^{\frac{2(d-1)}{d-p}} t_d^{-\frac{1}{d}}\right)^{\frac{d}{d-1}}\right). \tag{6}$$

Gehring [3] obtained the result that if $f, f^{-1} \in Q_p(K, \infty)$ for $p \in [1, d) \cup (d, \infty)$, then $f, f^{-1} \in Lip(K')$ where $0 < K' < \infty$ depends only on d, p, and K. Our result is a generalization of the above Gehring theorem for $1 \leq p < d$ since $Q_p(K, \delta) \supset Q_p(K, \infty)$ $(0 < \delta \leq \infty)$. Contrary to the above Gehring theorem, our main theorem above cannot be true in general for $d < p \leq \infty$. In fact, if D is bounded, then $\delta := \inf_R \operatorname{cap}_p R > 0$, where the infimum is taken with respect to all spherical rings R in D (cf. the remark right after (3)). Thus taking any homeomorphism f with $f, f^{-1} \notin Lip(K_1)$ for every $0 < K_1 < \infty$, we have $f, f^{-1} \in Q_p(K, \delta)$ for every $0 < K < \infty$, and the invalidity of the above main theorem follows. Nevertheless our proof is entirely based upon the idea of the Gehring proof in the above paper. The proof will be given in §23 after a series of preparatory discussins in sections 7, 11, 14, and 21: in §7 several estimates for the p-capacity are given; in §11 volume distortions under mappings in $Q_p(K, \delta)$ are considered; in §14 area distiortions under mappings in $Q_p(K, \delta)$ are studied; in §21 mappings together with their inverses in $Q_p(K, \delta)$ are shown to be quasiconformal; in §24 there is an appendix in which proofs for the first and the last identity in (3) are given.

In passing we state a remark. Let us say that

$$f \in \tilde{Q}_p(K, \delta)$$

for $0 < K < \infty$ and $0 < \delta < \infty$, if (4) holds for every spherical ring R with $\overline{R} \subset D$ and, instead of (5), $\mathrm{cap}_p R \geq \delta$. Then we can show by using the well known inequality mentioned below that $\tilde{Q}_p(K, \delta) = Q_p(K, \infty)$ $(1 < p < \infty)$ for every $0 < \delta < \infty$: if R_i $(1 \leq i \leq n)$ are disjoint rings each of which separates the boundary components of R, then

$$(\mathrm{cap}_p R)^{\frac{1}{1-p}} \geq \sum_{i=1}^{n} (\mathrm{cap}_p R_i)^{\frac{1}{1-p}},$$

where the equality holds if $\overline{R} = \cup_{i=1}^{n} \overline{R_i}$ and R_i are spherical rings $(1 \leq i \leq n)$. Hence the original Gehring theorem for $p \in (1, \infty)$ and $p \neq d$ may be restated that if, for $p \in (1, \infty)$ and $p \neq d$, $f, f^{-1} \in \tilde{Q}_p(K, \delta)$, $0 < K < \infty$ and $0 < \delta < \infty$, then $f, f^{-1} \in Lip(K')$, where $0 < K' < \infty$ depends only upon d, p, and K and not dependent on δ. On the contrary our main theorem above seems not to be able to be reduced to the Gehring theorem although we closely follow the original proof of Gehring modyfying it here and there to prove our main theorem. In this context we must stress that the class $\tilde{Q}_1(K, \delta)$ is a difficult class to treat since we do not know any $p = 1$ counterpart to the above displayed capacity inequality. Hence there remain an open question to resolve whether $f, f^{-1} \in \tilde{Q}_1(K, \delta)$, $0 < K < \infty$ and $0 < \delta < \infty$, implies the existence of $0 < K' < \infty$ dependent only upon d, p, and K and not dependent on δ such that $f, f^{-1} \in Lip(K')$. Here we add one more comment on $Q_d(K, \delta)$, $0 < K < \infty$ and $0 < \delta \leq \infty$, although it is a subsidiary object in this paper. We can show that if $f \in Q_d(K, \delta)$, then $f \in Q_d(K', \infty)$ (and hence, as is well known, $f^{-1} \in Q_d(K', \infty)$), $0 < K' < \infty$, i.e. f is a quasiconformal mapping. However we can only show that K' depends not only on d and K but also on δ. Thus to determine whether the dependence on δ is essential or not is another open question left.

7. Estimates for the p-capacity. We state in this section three different type estimates for the p-capacity of rings, where the exponent p is supposed to satisfy $1 \leq p \leq d$ in this section. To begin with we give some extremal length type estimates. For a ring R with complementary components C_0 and C_1 we consider

$$V(R) = m_d(R), \quad A(R) = \inf_{\Sigma} m_{d-1}(\Sigma), \quad L(R) = \mathrm{dist}(C_0, C_1),$$

where the infimum is taken with respect to every polyhedral surfaces Σ in R separating C_0 and C_1. If C_0 and C_1 are nondegenerate,then the following two sided estimates of $\mathrm{cap}_p R$ hold (see Lemma 1 in Gehring [3]):

$$\frac{A(R)^p}{V(R)^{p-1}} \leq \mathrm{cap}_p R \leq \frac{V(R)}{L(R)^p}. \tag{8}$$

Next let R be an arbitrary ring with complementary components C_0 and C_1. If R^* is a spherical ring with complementary components C_0^* and C_1^* such that $m_d(C_0^*) = m_d(C_0)$ and $m_d(C_0^* \cup R^*) = m_d(C_0 \cup R)$, then

$$\text{cap}_p R \geq \text{cap}_p R^*. \tag{9}$$

For a proof of this fact see Gehring [2] (see also Lemma 2 in Gehring [3]).

Finally we recall the extremal length definition of the conformal capacity $\text{cap}_d R$ of rings R. For a ring R and a function φ which is nonnegative and Borel measurable in R we put

$$V(\varphi, R) = \int_R \varphi^d dm_d, \quad A(\varphi, R) = \inf_{\Sigma} \int_\Sigma \varphi^{d-1} dm_{d-1},$$

where the infimum is taken with respect to every polyhedral surface Σ in R separating the complementary components C_0 and C_1. We denote by $\Phi(R)$ the family of nonnegative Borel functions φ on R with $V(\varphi, R)$ and $A(\varphi, R)$ not simultaneously 0 or ∞. It is known (Gehring [1] and Ziemer [10]) that

$$\text{cap}_d R = \sup_{\varphi \in \Phi(R)} \frac{A(\varphi, R)^d}{V(\varphi, R)^{d-1}}.$$

We wish to replace $\Phi(R)$ in the above identity by its subfamily $\Psi(R)$ of nonnegative *continuou* functions ψ on R with $V(\psi, R) < \infty$ and $A(\psi, R) > 0$. To do this we need to requir for R to be *approximable from outside* in the sense that there is a sequence of rings R_i containing $\overline{R}$ such that $\overline{R}$ separates the complementary components of each R_i and

$$\text{cap}_d R = \lim_{i \to \infty} \text{cap}_d R_i.$$

Under the condition that R is approximable from outside we have the following extremal length expression of the conformal capacity $\text{cap}_d R$ of R (cf. Lemma 5 in Gehring [3]):

$$\text{cap}_d R = \sup_{\psi \in \Psi(R)} \frac{A(\psi, R)^d}{V(\psi, R)^{d-1}}. \tag{10}$$

11. Volume distortions. We next study analytic properties of mappings in $Q_p(K, \delta)$. First we consider how volumes of sets are distorted under the mappings in $Q_p(K, \delta)$. The exponent p is supposed to satisfy $1 \leq p < d$ in this section. The study of volume distortions under f is based upon the following quantity associated with f, so to speak , a kind of Jacobian of f:

$$J(P, f) := \limsup_{r \downarrow 0} \frac{m_d(f(B(P, r)))}{m_d(B(P, r))}$$

for any $P \in D$ and for any homeomorphism f of D onto D', where $B(P, r)$ is the open ball of radius $0 < r \leq \infty$ centered at P: $B(P, r) := \{x : |x - P| < r\}$. First we maintain that

$$J(P, f) \leq K^{\frac{d}{d-p}} \tag{12}$$

for every $P \in D$ if $f \in Q_p(K,\delta)$ for any $0 < \delta \le \infty$. The following proof is a minor modification of that for Lemma 6 in Gehring [3] in which the above assertion for $\delta = \infty$ is stated.

PROOF OF (12): Let $b = \operatorname{dist}(P, D^c)/2$ and take an a arbitrarily in $(0,b)$. Consider spherical rings $R = \{x : a < |x-P| < b\}$ and $R^* = \{x : a^* < |x| < b^*\}$ with complementary components C_0, C_1 and C_0^*, C_1^*, respectively, where a^* and b^* are chosen so as to satisfy $m_d(C_0^*) = m_d(f(C_0))$ and $m_d(C_0^* \cup R^*) = m_d(f(C_0 \cup R))$. Observe that b^* is a fixed number determined by f, P, and b; $a^* \downarrow 0$ along with $a \downarrow 0$. Recall that, for $1 < p < d$,

$$\operatorname{cap}_p R = \omega_d \left(\frac{b^q - a^q}{q}\right)^{1-p} = \omega_d r^{p-1}\left(\frac{a^r b^r}{b^r - a^r}\right)^{p-1},$$

where $q := (p-d)/(p-1)$ and $r = |q|$; for $p = 1$,

$$\operatorname{cap}_1 R = \omega_d a^{d-1}.$$

Hence $\operatorname{cap}_p R \downarrow 0$ as $a \downarrow 0$ so that there is an $a_1 \in (0,b)$ such that $\operatorname{cap}_p R < \delta$ for $a \in (0,a_1)$. Since $f \in Q_p(K,\delta)$, we have $\operatorname{cap}_p f(R) \le K \operatorname{cap}_p R$ for every $a \in (0,a_1)$. On the other hand, by (9), we see that $\operatorname{cap}_p R^* \le \operatorname{cap}_p f(R)$. Thus we deduce

$$\operatorname{cap}_p R^* \le K \operatorname{cap}_p R$$

for every $a \in (0,a_1)$. In terms of a, b, a^*, and b^*, the above displayed inequality takes the form

$$\left(\frac{(b^*)^q - (a^*)^q}{q}\right)^{1-p} \le K\left(\frac{b^q - a^q}{q}\right)^{1-p}$$

for $1 < p < d$ and

$$(a^*)^{d-1} \le K a^{d-1}$$

for $p = 1$. Since $(a^*)^r \le (a^*)^r (b^*)^r/((b^*)^r - (a^*)^r)$, the first of the above inequalities implies

$$\left(\frac{a^*}{a}\right)^d \le K^{\frac{d}{d-p}}\left(1 - \left(\frac{a}{b}\right)^r\right)^{\frac{d(1-p)}{d-p}}$$

and the second of the above inequalities implies

$$\left(\frac{a^*}{a}\right)^d \le K^{\frac{d}{d-1}}$$

for every $a \in (0,a_1)$. By $m_d(f(B(P,a)))/m_d(B(P,a)) = (a^*/a)^d$, we conclude that

$$J(P,f) = \limsup_{a \downarrow 0}\left(\frac{a^*}{a}\right)^d \le K^{\frac{d}{d-p}}$$

as desired. □

For any homeomorphism f of D onto D' satisfying (12) for every $P \in D$, the following volume distortion inequality holds:

$$m_d(f(E)) \le K^{\frac{d}{d-p}} m_d(E) \tag{13}$$

for every Borel set E in D. In particular, (13) holds for every Borel set E in D if $f \in Q_p(K,\delta)$. Although the proof for Lemma 7 in Gehring [3] works in essence as that for (13) for any f satisfying (12), we repeat it here in the form to suit the present situation only for the sake of convenience and completeness.

PROOF OF (13): The inequality (13) is trivially true if $m_d(E) = \infty$ and thus we assume that $m_d(E) < \infty$. Given an arbitrary positive number $\varepsilon > 0$. By the regularity of m_d we can find an open set $G \supset E$ such that

$$m_d(G) \le m_d(E) + \varepsilon.$$

Since $E \subset D \cap G$, by replacing G by $D \cap G$, we can assume that $E \subset G \subset D$.

In view of (12) we can find a sequence $(r_j(P))_{j\ge 1}$ for each $P \in G$ with the following two conditions: first

$$0 < r_j(P) < \min\{\mathrm{dist}(P, \partial G), 1\}/10j \quad (j = 1, 2, \cdots);$$

second on setting $U = B(P, r_j(P))$

$$m_d(f(kU)) \le (K^{\frac{d}{d-p}} + \varepsilon) m_d(kU) \quad (k = 1, 5),$$

where, in general, for balls $B(x,r)$ we write $aB(x,r) = B(x,ar)$ for any positive number $a > 0$. Then we consider the collection $\mathcal{U} := \{B(P, r_j(P)) : P \in G, j = 0, 1, \cdots\}$.

We first consider the case $m_d(E) = 0$. By the 5r-covering theorem (cf. e.g. p.23 of Mattila [5]) we can find a coutable subcollection of $\mathcal{U}$ consisting mutually disjoint balls $U_i \in \mathcal{U}$ such that

$$G = \bigcup_{U \in \mathcal{U}} U \subset \bigcup_i 5U_i.$$

Since $E \subset G$

$$\begin{aligned} m_d(f(E)) &\le m_d\left(f\left(\bigcup_i 5U_i\right)\right) = m_d\left(\bigcup_i f(5U_i)\right) \\ &\le \sum_i m_d(f(5U_i)) \le \sum_i (K^{\frac{d}{d-p}} + \varepsilon) m_d(5U_i) \\ &= \sum_i (K^{\frac{d}{d-p}} + \varepsilon) 5^d m_d(U_i) = 5^d (K^{\frac{d}{d-p}} + \varepsilon) m_d\left(\bigcup_i U_i\right) \\ &\le 5^d (K^{\frac{d}{d-p}} + \varepsilon) m_d(G) \le 5^d (K^{\frac{d}{d-p}} + \varepsilon)(m_d(E) + \varepsilon). \end{aligned}$$

Hence by $m_d(E) = 0$ we obtain

$$m_d(f(E)) \le 5^d (K^{\frac{d}{d-p}} + \varepsilon)\varepsilon.$$

On letting $\varepsilon \downarrow 0$ we conclude that $m_d(f(E)) = 0$ and therefore (13) holds.

Next we consider the case $m_d(E) > 0$. By the Vitali covering theorem we can find a countable subcollection of $\mathcal{U}$ consisting of mutually disjoint balls $U_i \in \mathcal{U}$ such that

$$m_d(F) = 0 \qquad \left(F := E \setminus \bigcup_i U_i\right).$$

As we have seen above $m_d(F) = 0$ implies $m_d(f(F)) = 0$. Since $f(E) = f(\cup_i U_i) \cup f(F)$, we have

$$\begin{aligned} m_d(f(E)) &= m_d\left(f\left(\bigcup_i U_i\right)\right) = m_d\left(\bigcup_i f(U_i)\right) \\ &= \sum_i m_d(f(U_i)) \le (K^{\frac{d}{d-p}} + \varepsilon) \sum_i m_d(U_i) \\ &= (K^{\frac{1}{d-p}} + \varepsilon) m_d\left(\bigcup_i U_i\right) \le (K^{\frac{d}{d-p}} + \varepsilon) m_d(G). \end{aligned}$$

Hence we conclude that

$$m_d(f(E)) \le (K^{\frac{d}{d-p}} + \varepsilon)(m_d(E) + \varepsilon).$$

On letting $\varepsilon \downarrow 0$ in the above inequality, the validity of (13) follows. □

14. Area distortions. We continue to study analytic properties of mappings in $Q_p(K, \delta)$. Here we consider how the $(d-1)$-dimensional measures or areas are distorted under mappings in $Q_p(K, \delta)$. For any positive number $\varepsilon > 0$ and any set $X \subset \mathbf{R}^d$ we use the notation $(X)_\varepsilon$ for the set $\{x : \operatorname{dist}(x, X) < \varepsilon\}$.

15. LEMMA. *Given an $f \in Q_p(K, \delta)$, $(1 \le p < d)$, any relatively compact open set W with $\overline{W} \subset D$, and an arbitrary positive number $\eta > 0$. Then there exists a positive number $\alpha > 0$ with the followng properties: for any $P \in W$ and any $a \in (0, \alpha)$, $\overline{U} \subset D$ $(U = B(P, a))$ and there exists an open polyhedron G' such that first*

$$f(U) \subset G' \subset (f(U))_\eta \tag{16}$$

and second

$$m_{d-1}(\partial G') \le c_1 K^{\frac{d-1}{d-p}} m_{d-1}(\partial U) \qquad (c_1 := 2^d/d). \tag{17}$$

PROOF: We choose a postive number $\alpha > 0$ so small that the following three conditions are fulfilled. First we requir α to satisfy

$$0 < \alpha < \operatorname{dist}(W, \partial D)/2.$$

As a result of this we have $\overline{B}(P, 2a) \subset D$ for any $P \in W$ as far as $a \in (0, \alpha)$. By the uniform continuity of f on $\overline{W}$, we can choose $\alpha > 0$ small so as to satisfy

$$f(B(P, 2\alpha)) \subset B(f(P), \eta)$$

for every $P \in \overline{W}$. Let $R = \{x : a < |x - P| < 2a\}$ for a positive number $a > 0$. Then $\mathrm{cap}_p R = \omega_d |(2^q - 1)/q|^{1-p} a^{d-p}$ $(q = (p-d)/(p-1))$ for $1 < p < d$ and $\mathrm{cap}_1 R = \omega_d a^{d-1}$. Then we can choose $\alpha > 0$ so small that

$$\mathrm{cap}_p R < \delta$$

for any P and for every $a \in (0, \alpha)$. Thus we can fix a positive nunber $\alpha > 0$ satisfying simultaneously the above three displayed conditions.

Since $V(R) = \tau_d((2a)^d - a^d)$, $A(R) = \omega_d a^{d-1}$, and $L(R) = a$, We have $V(R)/A(R)L(R) = \tau_d a^d (2^d - 1)/(\omega_d a^{d-1} \cdot a) = (\tau_d/\omega_d)(2^d - 1) = (2^d - 1)/d =: c_0$. Hence we see that

$$L(R) = V(R)/c_0 A(R).$$

Next by the right hand side inequality in (8), we see that $\mathrm{cap}_p R \leq V(R)/L(R)^p = V(R)/(V(R)/c_0 A(R))^p = c_0^p A(R)^p / V(R)^{p-1}$, i.e.

$$\mathrm{cap}_p R \leq c_0^p \frac{A(R)^p}{V(R)^{p-1}}.$$

Observe that $A(f(R))^p/V(R)^{p-1} = (V(f(R))/V(R))^{p-1} \cdot (A(f(R))^p / V(f(R))^{p-1})$. The first factor on the right is dominated by $(K^{d/(d-p)})^{p-1}$ in view of (13) and the second factor on the right is dominated by $\mathrm{cap}_p f(R)$ by the left hand side inequality of (8). Thus

$$\frac{A(f(R))^p}{V(R)^{p-1}} \leq K^{\frac{d(p-1)}{d-p}} \mathrm{cap}_p f(R).$$

Since $\mathrm{cap}_p R < \delta$, $f \in Q_p(K, \delta)$ implies that $\mathrm{cap}_p f(R) \leq K \mathrm{cap}_p R$. Hence $A(f(R))^p/V(R)^{p-1} \leq K^{d(p-1)/(d-p)} \cdot K \mathrm{cap}_p R = K^{p(d-1)/(d-p)} \mathrm{cap}_p R \leq K^{p(d-1)/(d-p)} c_0^p A(R)^p / V(R)^{p-1}$. Then $A(f(R))^p \leq c_0^p (K^{(d-1)/(d-p)})^p A(R)^p$ so that

$$A(f(R)) \leq c_0 K^{\frac{d-1}{d-p}} A(R).$$

By the definition of $A(f(R))$ and by $c_0 < c_1$ we can find a polyhedral surface Σ' in $f(R)$ separating $f(C_0)$ and $f(C_1)$, where C_0 and C_1 are complementary components of R^c, such that

$$m_{d-1}(\Sigma') \leq c_1 K^{\frac{d-1}{d-p}} A(R) = c_1 K^{\frac{d-1}{d-p}} m_{d-1}(\partial U).$$

Let G' be the complementary component of Σ' containing $f(U)$. Then it is easily seen that G' satisfies (16) and (17). $\square$

18. LEMMA. *Given an $f \in Q_p(K,\delta)$ $(1 \le p < d)$, any compact set $E \subset D$ with $m_{d-1}(E) > 0$, and any positive number $\varepsilon > 0$. Then there exists an open polyhedron G' such that first*

$$f(E) \subset G' \subset (f(E))_\varepsilon \tag{19}$$

and second

$$m_{d-1}(\partial G') \le c_2 K^{\frac{d-1}{d-p}} m_{d-1}(E) \quad (c_2 := 2c_1\omega_d\tau_{d-1}^{-1}). \tag{20}$$

PROOF: By the local uniform continuity of f and by the compactness of E, we can find a relatively compact open set W with $E \subset W \subset \overline{W} \subset D$ such that

$$f(E) \subset f(W) \subset \overline{f(W)} \subset (f(E))_\varepsilon.$$

Fix positive numbers $\beta > 0$ and $\eta \in (0,\varepsilon)$ so as to satisfy $(E)_{2\beta} \subset W$ and $(f(W))_\eta \subset (f(E))_\varepsilon$. Then choose a positive number $\alpha > 0$ determined by Lemma 15 with respect to the above W and η. Finally let $\gamma := \min(\alpha, \beta)$.

Since $m_{d-1}(E) > 0$ and E is compact, we can find a finite collection $(U_i)_i$ of open balls $U_i := B(P_i, a_i)$ such that $0 < a_i < \gamma$, $E \subset \cup_i U_i$, and

$$\sum_i \tau_{d-1} a_i^{d-1} < 2m_{d-1}(E).$$

Here we can clearly assume that $U_i \cap E \ne \emptyset$ so that by $a_i \le \beta$ we have $U_i = B(P_i, a_i) \subset (E)_{2\beta} \subset W$, i.e. $U_i \subset W \subset D$. Then, by Lemma 15, there exists an open polyhedoron G'_i such that

$$f(U_i) \subset G'_i \subset (f(U_i))_\eta$$

and

$$m_{d-1}(\partial G'_i) \le c_1 K^{\frac{d-1}{d-p}} m_{d-1}(\partial U_i).$$

Set $G' := \cup_i G'_i$ and then $\partial G' \subset \cup_i \partial G'_i$. We have

$$\begin{aligned} m_{d-1}(\partial G') &\le \sum_i m_{d-1}(\partial G'_i) \le c_1 K^{\frac{d-1}{d-p}} \sum_i m_{d-1}(\partial U_i) \\ &= c_1 K^{\frac{d-1}{d-p}} \sum_i \omega_d a_i^{d-1} = c_1 K^{\frac{d-1}{d-p}} \omega_d \tau_{d-1}^{-1} \sum_i \tau_{d-1} a_i^{d-1} \\ &\le c_1 K^{\frac{d-1}{d-p}} \omega_d \tau_{d-1}^{-1} \cdot 2m_{d-1}(E) = (2c_1\omega_d\tau_{d-1}^{-1}) K^{\frac{d-1}{d-p}} m_{d-1}(E), \end{aligned}$$

which is nothing but (20). To see G' satisfies (19) we proceed as follows:

$$f(E) \subset f\left(\bigcup_i U_i\right) = \bigcup_i f(U_i) \subset \bigcup_i G'_i$$

$$\subset \bigcup_i (f(U_i))_\eta \subset (f(W))_\eta \subset (f(E))_\varepsilon$$

and thus $G' = \cup_i G'_i$ satisfies (19), as desired. □

21. Quasiconformality. Our main concern is about the class $Q_p(K,\delta)$ $(1 \le p < d)$. However we need to consider the class $Q_d(K,\infty)$ as an auxiliary class to clarify $Q_p(K,\delta)$ $(1 \le p < d)$. It is known that a homeomorphism f of D onto D' is a quasiconformal mapping if and only if $f \in Q_d(K,\infty)$ for some $K > 0$. Often this fact itself is taken as a definition of quasiconformal mappings (cf. e.g. Väisälä [9]). In this section we prove the following result.

22. Lemma. *If f and f^{-1} belong to $Q_p(K,\delta)$ $(1 \le p < d)$, then f and f^{-1} belong to $Q_d(K_0,\infty)$, where $K_0 = (2^{d+1}\omega_d d^{-1}\tau_{d-1}^{-1}K^{2(d-1)/(d-p)})^d$ is a constant depending only on d, p, and K.*

Obseve that K_0 does not depend on $\delta > 0$. This result is obtained for the class $Q_p(K,\infty)$ not only for $1 \le p < d$ but also for $d < p < \infty$ in Gehring [3] as Theorem 1. As already stated we are not interested in the $d < p \le \infty$ case from the view point of the application of the main theorem to our final purpose and actually the above lemma is not true in general for the $d < p \le \infty$ case.

Proof: We only have to show that $f \in Q_d(K_0,\infty)$ since the proof for $f^{-1} \in Q_d(K_0,\infty)$ is identical by symmetry. Hence, for any spherical ring $R = \{x : a < |x-P| < b\}$ with $\overline{R} \subset D$, we only have to prove that $\mathrm{cap}_d f(R) \le K_0 \mathrm{cap}_d R$. As an obvious result of the fact that R is a spherical ring, it is seen that not only R but also $f(R)$ are approximable from outside in the sense of Section 7 (cf. e.g. Lemma 6 in Gehring [3]; see also Chapter 2 in Heinonen et al. [4]). Hence by (10) we have

$$\mathrm{cap}_d f(R) = \sup_{\phi \in \Psi(f(R))} \frac{A(\phi, f(R))^d}{V(\phi, f(R))^{d-1}}.$$

With each $\psi \in \Psi(R)$ we associate $\phi \in \Psi(f(R))$ by $\psi = \phi \circ f$. Then $\Psi(R) = \Psi(f(R)) \circ f$. We will evaluate $A(\phi, f(R))$ from above by $A(\psi, R)$ and $V(\phi, f(R))$ from below by $V(\psi, R)$. We start with $A(\phi, f(R))$. Take an arbitrary positive number $\varepsilon > 0$ and any polyhedral surface $\Sigma \subset R$ separating two complementary components C_0 and C_1 of R. By the uniform continuity of ψ on Σ, we can express Σ as a finite union of Borel sets E_i, i.e. $\Sigma = \cup_i E_i$, such that $m_{d-1}(E_i) > 0$, $\mathrm{Osc}_{E_i}\psi^{d-1} < \varepsilon$, and $m_{d-1}(E_i \cap E_j) = 0$ $(i \ne j)$. Clearly $\mathrm{Osc}_{f(E_i)}\phi^{d-1} < \varepsilon$. We apply Lemma 18 to each $f(E_i)$. Then there exists an open polyhedron G'_i such that $f(E_i) \subset G'_i \subset \overline{G'_i} \subset f(R)$, $\mathrm{Osc}_{\overline{G'_i}}\phi^{d-1} < \varepsilon$, and $m_{d-1}(\partial G'_i) \le c_2 K^{(d-1)/(d-p)} m_{d-1}(E_i)$. Obviously $\cup_i \partial G'_i$ contains a polyhedral surface $\Sigma' \subset f(R)$ separating $f(C_0)$ and $f(C_1)$.

Then

$$A(\phi, f(R)) \leq \int_{\Sigma'} \phi^{d-1} dm_{d-1} \leq \sum_i \int_{\partial G_i'} \phi^{d-1} dm_{d-1}$$

$$\leq \sum_i (\psi^{d-1}(P_i) + \varepsilon) m_{d-1}(\partial G_i') \leq c_2 K^{\frac{d-1}{d-p}} \sum_i (\psi(P_i)^{d-1} + \varepsilon) m_{d-1}(E_i)$$

$$\leq c_2 K^{\frac{d-1}{d-p}} \sum_i \int_{E_i} (\psi^{d-1} + 2\varepsilon) dm_{d-1} = c_2 K^{\frac{d-1}{d-p}} \int_{\Sigma} (\psi^{d-1} + 2\varepsilon) dm_{d-1},$$

where $P_i \in E_i$ is an arbitrarily fixed point for each i. Since $\varepsilon > 0$ and Σ are arbitrary, we deduce

$$A(\phi, f(R)) \leq c_2 K^{\frac{d-1}{d-p}} A(\psi, R).$$

Next we turn to $V(\psi, f(R))$. Take an arbitrary positive number $\varepsilon > 0$. By the local uniform continuity of ψ on R we can express R as a countable union of Borel sets E_i, i.e. $R = \cup_i E_i$, such that $m_d(E_i) > 0$, $\mathrm{Osc}_{E_i}\psi^d < \varepsilon$, and $m_d(E_i \cap E_j) = 0$ $(i \neq j)$. Clearly $\mathrm{Osc}_{E_i}\psi^d < \varepsilon$ implies $\mathrm{Osc}_{f(E_i)}\phi^d < \varepsilon$. By (13), $m_d(E_i \cap E_j) = 0$ $(i \neq j)$ assures that $m_d(f(E_i) \cap f(E_j)) = 0$ $(i \neq j)$. Observe that, since f and $f^{-1} \in Q_p(K, \delta)$, the inequality (13) valid for $f \in Q_p(K, \delta)$ and $E \subset D$ must also be valid for $f^{-1} \in Q_p(K, \delta)$ and $f(E) \subset f(D) = D'$ so that we also have $m_d(E) \leq K^{d/(d-p)} m_d(f(E))$. Using this inequality we proceed as follows:

$$V(\psi, R) = \sum_i \int_{E_i} \psi^d dm_d \leq \sum_i (\psi(P_i)^d + \varepsilon) m_d(E_i)$$

$$\leq K^{\frac{d}{d-p}} \sum_i (\phi(f(P_i))^d + \varepsilon) m_d(f(E_i))$$

$$\leq K^{\frac{d}{d-p}} \sum_i \int_{f(E_i)} (\phi^d + 2\varepsilon) dm_d = K^{\frac{d}{d-p}} \int_{f(R)} (\phi^d + 2\epsilon) dm_d,$$

where $P_i \in E_i$ is an arbitrarily fixed point for each i. Since $\varepsilon > 0$ is arbitrary, we deduce

$$V(\psi, R) \leq K^{\frac{d}{d-p}} V(\phi, f(R)).$$

Fix an arbitrary $\phi \in \Psi(f(R))$. Then

$$\frac{A(\phi, f(R))^d}{V(\phi, f(R))^{d-1}} \leq \frac{\left(c_2 K^{\frac{d-1}{d-p}} A(\psi, R)\right)^d}{\left(K^{\frac{-d}{d-p}} V(\psi, R)\right)^{d-1}}$$

$$= c_2^d K^{\frac{2d(d-1)}{d-p}} \frac{A(\psi, R)^d}{V(\psi, R)^{d-1}} \leq c_2^d K^{\frac{2d(d-1)}{d-p}} \mathrm{cap}_d R,$$

where the last inequality follows from (10). Taking the supremum of the leftmost side term in the above inequality with respect to $\phi \in \Psi(f(R))$, we deduce, by (10), that

$$\mathrm{cap}_d f(R) \leq K_0 \mathrm{cap}_d R,$$

where the constant K_0 is given by $K_0 = c_2^d K^{2d(d-1)/(d-p)} = (2c_1\omega_d\tau_{d-1}^{-1}K^{2(d-1)/(d-p)})^d = (2^{d+1}\omega_d d^{-1}\tau_{d-1}^{-1}K^{2(d-1)/(d-p)})^d$. $\square$

23. The proof for the main result. Compiling the results stated thus far in the preceding sections we can now prove the main result of this paper. For the purpose we need to consider one more quantity $H(P,f)$ associated with any homeomorphism f of D onto D' and any point P in D:

$$H(P,f) := \limsup_{r\downarrow 0} \frac{\max_{|x-P|=r} |f(x)-f(P)|}{\min_{|x-P|=r} |f(x)-f(P)|}.$$

It is known (cf. e.g. §22 in Väisälä [9]) that f is a quasiconformal mapping if and only if $\sup_{P\in D} H(P,f) < \infty$.

PROOF OF THE MAIN THEOREM. By Lemma 22 we have $f \in Q_d(K_0,\infty)$. Hence we see that

$$H(P,f) \le \exp\left(\left(K_0\frac{\omega_d}{t_d}\right)^{\frac{1}{d-1}}\right) =: H \qquad (P\in D)$$

(see e.g. §22 in Väisälä [9]), where we recall that t_d is the capacity of the Teichmüller ring R_T introduced in Section 2. By (12) we have

$$\sup_{P\in D} J(P,f) \le K^{\frac{1}{d-p}}.$$

With each $r>0$ satisfying $\overline{B}(P,r)\subset D$ for any fixed $P\in D$ we associate a positive number $b(r)>0$ such that

$$m_d(f(B(P,r))) = \tau_d b(r)^d.$$

Clearly $(\min_{|x-P|} |f(x)-f(P)|)/b(r) \le 1$. Therefore we have

$$\frac{\max_{|x-P|=r} |f(x)-f(P)|}{r} \le \frac{\max_{|x-P|=r} |f(x)-f(P)|}{\min_{|x-P|=r} |f(x)-f(P)|}\cdot\left(\frac{\tau_d b(r)^d}{\tau_d r^d}\right)^{\frac{1}{d}}$$

and a fortiori, by taking the superior limit as $r\downarrow 0$ of both sides of the above inequality, we deduce

$$L(P,f) \le H(P,f)\cdot J(P,f)^{\frac{1}{d}}.$$

Hence we conclude that

$$L(P,f) \le H\cdot K^{\frac{1}{d-p}} =: K_1$$

on D and K_1 can be explicitly given as in the statement of the main theorem. Repeating the same argument for f^{-1} instead of f we can deduce $f^{-1}\in Lip(K_1)$ in addition to $f\in Lip(K_1)$.

The proof of the main theorem is herewith complete. $\square$

24. Appendix. For convenience we record here the proofs of the first and the last identity in (3) for $R = \{x : a < |x| < b\}$ $(0 < a < b < \infty)$.

As is well known, if $u \in ACL(R) \cap C(R)$, then the usual gradient ∇u of u exists m_d-a.e. on R and $|\nabla u|$ is a Borel function on R. By the standard mollifier method, it is shown that, if $|\nabla u| \in L^p(R)$ $(1 \le p < \infty)$ in addition to that $u \in ACL(R) \cap C(R)$, then there exists a sequence $(u_i)_{i \ge 1}$ in $C^\infty(R)$ such that

$$\|u - u_i; L^\infty(R)\| + \|\nabla u - \nabla u_i; L^p(R)\| \to 0 \quad (i \to \infty). \tag{25}$$

Based on this fact, the class $W(R)$ of competing functions for calculating $\mathrm{cap}_p R$ may be replaced by $W(R) \cap C^\infty(R)$, which often makes the computation for $\mathrm{cap}_p R$ much easier $(1 \le p < \infty)$. In this regard we must be careful. The fact (25) is no longer true for $p = \infty$ so that $W(R) \cap C^\infty(R)$ is not dense in $W(R)$ with respect to the norm $\|v; L^\infty\| + \|\nabla v; L^\infty\|$ for $v \in W(R)$.

However (25) for $1 \le p < \infty$ is still useful even for $p = \infty$ in the following sense. Let $u \in ACL(R) \cap C(R)$ with $|\nabla u| \in L^1_{loc}(R)$. We denote by $x = r\xi$ the polar coordinate expression of $x \in \mathbf{R}^d \setminus \{0\}$: $r = |x|$ and $\xi = x/|x| \in S^{d-1}$. In terms of $x = r\xi$, we have the identity

$$|\nabla u(r\xi)|^2 = |u_r(r\xi)|^2 + \frac{1}{r^2}|\nabla_\xi u(r\xi)|^2, \tag{26}$$

where $u_r = \partial u / \partial r$ and ∇_ξ indicates the gradient operation for functions on S^{d-1} with respect to the natural Riemannian metric on S^{d-1}. Using the localized version of (25) we can show that $r \mapsto u(r\xi)$ is absolutely continuous on (a, b) for m_{d-1}-a.e. $\xi \in S^{d-1}$. Therefore we can use tha following fact in the proofs below:

FACT. *If $u \in W(R)$ satisfies $|\nabla u| \in L^p(R)$ $(p = 1$ or $p = \infty)$, then $r \mapsto u(r\xi)$ is absolutely continuous on $[a, b]$ for m_{d-1}-a.e. $\xi \in S^{d-1}$.*

This is of course true for every $1 \le p \le \infty$.

We now proceed to the proof of the first identity in (3), i.e. we will prove

$$\mathrm{cap}_1 R = \omega_d a^{d-1}. \tag{27}$$

For the purpose we first show that $\mathrm{cap}_1 R \ge \omega_d a^{d-1}$. Take an arbitrary $u \in W(R)$ and we will show that $\omega_d a^{d-1} \le \|\nabla u; L^1(R)\| \le \infty$. We may then suppose that $\nabla u \in L^1(R)$. By the fact mentioned above, there exists a subset $E_u \subset S^{d-1}$ with $m_{d-1}(E_u) = 0$ such that $r \mapsto u_r(r\xi)$ is absolutely continuous on $[a, b]$ for every fixed $\xi \in S^{d-1} \setminus E_u$ so that we see, by (26), that

$$1 = u(b\xi) - u(a\xi) = \int_a^b u_r(r\xi)dr$$

$$\le \int_a^b |u_r(r\xi)|dr \le \int_a^b |\nabla u(r\xi)|dr$$
$$= \int_a^b \frac{1}{r^{d-1}}|\nabla u(r\xi)|r^{d-1}dr \le \int_a^b \frac{1}{a^{d-1}}|\nabla u(r\xi)|r^{d-1}dr$$

since $1/r^{d-1} \le 1/a^{d-1}$ for every $r \in [a,b]$. Thus we have shown that

$$a^{d-1} \le \int_a^b |\nabla u(r\xi)|r^{d-1}dr$$

for m_{d-1}-a.e. $\xi \in S^{d-1}$. Integrating both sides of the above inequality over S^{d-1} with respect to $dm_{d-1}(\xi)$, we obtain

$$\int_{S^{d-1}} a^{d-1}dm_{d-1}(\xi) \le \int_{S^{d-1}} \left(\int_a^b |\nabla u(r\xi)|r^{d-1}dr\right) dm_{d-1}(\xi).$$

Applying the Fubini theorem to the right hand side term of the above and changing the polar coordinate to the Cartesian coordinate, we deduce

$$\omega_d a^{d-1} \le \int_R |\nabla u(x)|dm_d(x).$$

Hence $\omega_d a^{d-1} \le \inf_{u\in W(R)} \int_R |\nabla u(x)|dm_d(x) = \mathrm{cap}_1 R$ as desired.

To complete the proof of (27) we have to show that $\mathrm{cap}_1 R \le \omega_d a^{d-1}$. For the purpose take an arbitrary $c \in (a,b)$ which will be ultimately made to tend to a and define a function $u(x) = u_c(x)$ on $\overline{\mathbf{R}}^d$ by

$$u(x) := \begin{cases} 0 & (|x| \le a), \\ \dfrac{|x|-a}{c-a} & (a < |x| < c), \\ 1 & (c \le |x| \le \infty). \end{cases}$$

It is easy to see that $u \in W(R)$ and

$$|\nabla u(r\xi)| = |u_r(r\xi)| = \begin{cases} \dfrac{1}{c-a} & (a < r < c), \\ 0 & (c \le r < \infty) \end{cases}$$

for every $\xi \in S^{d-1}$. Hence we have

$$\int_R |\nabla u(x)|dm_d(x) = \int_{S^{d-1}} \left(\int_a^b |\nabla u(r\xi)|r^{d-1}dr\right) dm_{d-1}(\xi)$$
$$= \omega_d \int_a^c \frac{1}{c-a} r^{d-1}dr = \omega_d \frac{1}{d}\frac{c^d - a^d}{c-a}.$$

Since $\mathrm{cap}_1 R \leq \int_R |\nabla u(x)| dm_d(x)$, we conclude that

$$\mathrm{cap}_1 R \leq \omega_d \frac{1}{d}\frac{c^d - a^d}{c - d}$$

for every $c \in (a, b)$. On letting $c \downarrow a$ in the above inequality, $\lim_{c \downarrow a}(c^d - a^d)/(c - a) = da^{d-1}$ implies that $\mathrm{cap}_1 R \leq \omega_d a^{d-1}$, which completes the proof of (27). □

We next give the proof for the last identity (3), i.e. we will prove

$$\mathrm{cap}_\infty R = \frac{1}{b-a}. \tag{28}$$

First we show that $\mathrm{cap}_\infty R \geq 1/(b-a)$. In order to do this we only have to prove that $\|\nabla u; L^\infty(R)\| \geq 1/(b-a)$ for every $u \in W(R)$. Contrary to the assertion, assume the existence of a $u \in W(R)$ such that

$$\|\nabla u; L^\infty(R)\| < \frac{1}{b-a}.$$

On setting $\delta := (1/(b-a) - \|\nabla u; L^\infty(R)\|)/2 > 0$, we consider the set

$$X := \{x \in R : |\nabla u(x)| < \frac{1}{b-a} - \delta\},$$

which is a Borel subset of R since $|\nabla u|$ is a Borel function on R. By the definition of δ, $\|\nabla u; L^\infty(R)\| < 1/(b-a) - \delta$, which implies that $m_d(R \setminus X) = 0$. Let X_ξ be the ξ-section of X in the porlar coordinate for each $\xi \in S^{d-1}$:

$$X_\xi := \{r : a \leq r \leq b,\ r\xi \in X\} \quad (\xi \in S^{d-1}).$$

Observe that the ξ-section $(R \setminus X)_\xi$ of $R \setminus X$ is $R_\xi \setminus X_\xi = [a, b] \setminus X_\xi$. By the Fubini theorem, X_ξ is Borel measurable for m_d-a.e. $\xi \in S^{d-1}$ and

$$0 = m_d(R \setminus X) = \int_{R \setminus X} dm_d(x) = \int_{S^{d-1}} \left(\int_{(R\setminus X)_\xi} r^{d-1} dr \right) dm_{d-1}(\xi),$$

that is, we obtain

$$\int_{S^{d-1}} \left(\int_{[a,b]\setminus X_\xi} r^{d-1} dr \right) dm_{d-1}(\xi) = 0,$$

which implies that

$$\int_{[a,b]\setminus X_\xi} r^{d-1} dr = 0$$

for m_{d-1}-a.e. $\xi \in S^{d-1}$. Since $r^{d-1} \geq a^{d-1} > 0$, the above identity yields

$$m_1([a, b] \setminus X_\xi) = 0$$

for m_{d-1}-a.e. $\xi \in S^{d-1}$. By the fact mentioned before (27), the function $r \mapsto u(r\xi)$ is absolutely continuous on $[a,b]$ for m_{d-1}-a.e. $\xi \in S^{d-1}$. Hence there exists at least one $\eta \in S^{d-1}$ with the following three properties: X_η is a Borel subset of $[a,b]$;

$$m_1([a,b] \setminus X_\eta) = 0;$$

$r \mapsto u(r\eta)$ is absolutely continuous on the interval $[a,b]$. Then we have, by (26),

$$\begin{aligned} 1 = u(b\eta) - u(a\eta) = \int_a^b u_r(r\eta)dr = \int_{X_\eta} u_r(r\eta)dr \\ \le \int_{X_\eta} |u_r(r\eta)|dr \le \int_{X_\eta} |\nabla u(r\eta)|dr \le \int_{X_\eta} \left(\frac{1}{b-a} - \delta\right) dr \\ = \int_a^b \left(\frac{1}{b-a} - \delta\right) dr = 1 - \delta(b-a), \end{aligned}$$

i.e. $1 \le 1 - \delta(b-a)$, which is clearly a contradiction. Thus the proof of $\mathrm{cap}_\infty R \ge 1/(b-a)$ is herewith complete.

Finally we need to show $\mathrm{cap}_\infty R \le 1/(b-a)$ to complete the proof of (28). This is an easy task compared with the latter half of the proof of (27) since we only have to consider the function $u(x)$ on $\overline{\mathbf{R}}^d$ given by

$$u(x) := \begin{cases} 0 & (|x| \le a), \\ \dfrac{|x|-a}{b-a} & (a < |x| < b), \\ 1 & (b \le |x| \le \infty). \end{cases}$$

Clearly $u \in W(R)$ and, for $x = r\xi$,

$$|\nabla u(x)| = |\nabla u(r\xi)| = |u_r(r\xi)| = \begin{cases} \dfrac{1}{b-a} & (a < r < b), \\ 0 & (0 \le r < a,\ b < r < \infty) \end{cases}$$

for every $\xi \in S^{d-1} = 1/(b-a)$ and for every $x \in \mathbf{R}^d \setminus \partial R$. Hence we have $\|\nabla u; L^\infty(R)\| = 1/(b-a)$. This with $\mathrm{cap}_\infty R \le \|\nabla u; L^\infty(R)\|$ yields $\mathrm{cap}_\infty R \le 1/(b-a)$, which completes the whole proof of the identity (28). □

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