数理解析研究所講究録
1017巻 1997年 54-69

GENERALIZED KAC-MOODY ALGEBRA に対する HARISH-CHANDRA HOMOMORPHISM について

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INTRODUCTION

Let $\mathfrak{g}(A)$ be the complex contragredient Lie algebra associaed to a symmetrizable real square matrix $A = (a_{ij})_{i,j\in I}$ indexed by a finite set I (see [K1] and [KK] for details). In [K2], Kac introduced a complex associative algebra $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$, which can be thought of as a certain completion of the universal enveloping algebra $U(\mathfrak{g}(A))$ of the contragredient Lie algebra $\mathfrak{g}(A)$. In it he showed that there exists an isomorphism H (called the Harish-Chandra homomorphism) between the center $Z_{\mathcal{F}}$ of the algebra $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$ and the algebra $\mathcal{F}$ of complex-valued fuctions on the set $\mathfrak{h}^* \setminus L$, where L is the union of certain infinitely many affine hyperplanes in the algebraic dual $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$.

Moreover, he studied the "holomorphicity" of the elements of the algebra $Z_{\mathcal{F}}$ as "vector-valued" functions on the interior K of the complexified Tits cone $X_{\mathbb{C}}$ in the case where $\mathfrak{g}(A)$ is the symmetrizable Kac-Moody algebra (i.e., the matrix $A = (a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix).

In this paper, we generalize his results in [K2] to the case where $\mathfrak{g}(A)$ is the symmetrizable generalized Kac-Moody algebra (i.e., the complex contragredient Lie algebra associated to a certain symmetrizable real matrix $A = (a_{ij})_{i,j\in I}$, called a GGCM).

1. Harish-Chandra Homomorphism

In this section we briefly review the setting and some results in [K2], which are valid for arbitrary symmetrizable contragredient Lie algebras over $\mathbb{C}$, hence for symmetrizable generalized Kac-Moody algebras over $\mathbb{C}$.

1.1. A completion of the universal enveloping algebra. Let $\mathfrak{g}(A)$ be the symmetrizable generalized Kac-Moody algebra (GKM algebra for short) over $\mathbb{C}$. Then the Lie algebra $\mathfrak{g}(A)$ is nothing but the contragredient Lie algebra associated to a symmetrizable real matrix $A = (a_{ij})_{i,j\in I}$ (called a GGCM) indexed by a finite set I satisfying the following conditions:

(C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$ for $i \in I$;

(C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ for $j \neq i$ if $a_{ii} = 2$;

(C3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Note that this definition of GKM algebras is due to Kac (see [K1, Chap. 11]), and slightly different from the original one by Borcherds in [B1]). From now on we follow the notation of [K1], and freely use results in it (see also our previous papers [N1] – [N3]).

Let $\mathfrak{h}$ be the Cartan subalgebra of the GKM algebra $\mathfrak{g}(A)$. Then, since we have been assuming that the GGCM $A = (a_{ij})_{i,j\in I}$ is symmetrizable, there exists a nondegenerate symmetric $\mathbb{C}$-bilinear form $(\cdot|\cdot)$ on the dual $\mathfrak{h}^*$ of $\mathfrak{h}$, which is invariant under the action of the Weyl group W. (Here recall that the Weyl group W of the GKM algebra $\mathfrak{g}(A)$ is by definition the subgroup of $GL(\mathfrak{h}^*)$ generated by the fundamental reflections r_i with $a_{ii} = 2$.)

Now, for $\alpha \in Q = \sum_{i\in I} \mathbb{Z}\alpha_i$, we define the affine linear function $T_\alpha(\cdot)$ on $\mathfrak{h}^*$ by: $T_\alpha(\lambda) = 2(\lambda + \rho|\alpha) - (\alpha|\alpha)$ $(\lambda \in \mathfrak{h}^*)$, where $\rho \in \mathfrak{h}^*$ is a fixed element of $\mathfrak{h}^*$ such that

$2(\rho|\alpha_i) = (\alpha_i|\alpha_i)$ for $i \in I$. Then we put

$$L := \bigcup_{\substack{\gamma \in Q \\ \beta \in \Delta_+ \\ n \in \mathbb{Z}_{\geq 1}}} \{\lambda \in \mathfrak{h}^* \mid T_{n\beta}(\lambda + \gamma) = 0\}.$$

Let $\mathcal{F}$ be the algebra of $\mathbb{C}$-valued functions defined on $\mathfrak{h}^* \setminus L$. Because the set $\mathfrak{h}^* \setminus L$ is dense in $\mathfrak{h}^*$ in the usual metric topology, there exists a canonical embedding $\imath \colon S(\mathfrak{h}) \to \mathcal{F}$, where $S(\mathfrak{h})$ is viewed as the algebra of polynomial functions on $\mathfrak{h}^*$. Here we define the action π of the universal enveloping algebra $U(\mathfrak{g}(A))$ of the GKM algebra $\mathfrak{g}(A)$ on the algebra $\mathcal{F}$ by: $\pi(e_\beta)\varphi(\cdot) = \varphi(\cdot + \beta)$ for $\varphi(\cdot) \in \mathcal{F}$ and $e_\beta \in U(\mathfrak{g}(A))_\beta$, where $h(e_\beta) = \beta(h)e_\beta$ $(\beta \in Q, h \in \mathfrak{h})$.

By using the action π of $U(\mathfrak{g}(A))$ on $\mathcal{F}$, we can define the structure of an associative algebra on the vector space $U(\mathfrak{g}(A)) \otimes_{\mathbb{C}} \mathcal{F}$ by:

$$(e_\alpha \otimes \varphi(\cdot))(e_\beta \otimes \psi(\cdot)) := e_\alpha e_\beta \otimes (\pi(e_\beta)\varphi(\cdot))\psi(\cdot),$$

for $\varphi(\cdot), \psi(\cdot) \in \mathcal{F}$ and $e_\alpha, e_\beta \in U(\mathfrak{g}(A))$ with $\alpha, \beta \in Q$. Let $U_{\mathcal{F}}(\mathfrak{g}(A))$ be the quotient algebra of this associative algebra $U(\mathfrak{g}(A)) \otimes_{\mathbb{C}} \mathcal{F}$ by the two-sided ideal generated by the elements $f \otimes 1 - 1 \otimes \imath(f)$ for $f \in S(\mathfrak{h})$. Then the associative algebra $U_{\mathcal{F}}(\mathfrak{g}(A))$ is generated by the algebra $\mathcal{F}$ and $U(\mathfrak{g}(A))$, and the following relation holds in it:

$$\varphi(\cdot)e_\beta - e_\beta\varphi(\cdot) = e_\beta(\varphi(\cdot + \beta) - \varphi(\cdot)),$$

where $\varphi(\cdot) \in \mathcal{F}$ and $e_\beta \in U(\mathfrak{g}(A))_\beta$ with $\beta \in Q$. Moreover this algebra $U_{\mathcal{F}}(\mathfrak{g}(A))$ decomposes into the tensor product of vector spaces as:

$$U_{\mathcal{F}}(\mathfrak{g}(A)) = U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathcal{F} \otimes_{\mathbb{C}} U(\mathfrak{n}_+),$$

and canonically contains the algebra $U(\mathfrak{g}(A)) = U(\mathfrak{n}_-) \otimes_{\mathbb{C}} S(\mathfrak{h}) \otimes_{\mathbb{C}} U(\mathfrak{n}_+)$.

By putting $\deg(e_i) = 1$ and $\deg(f_i) = -1$ for $i \in I$, and $\deg(\mathcal{F}) = 0$, we have a $\mathbb{Z}$-gradation of $U_{\mathcal{F}}(\mathfrak{g}(A))$ as:

$$U_{\mathcal{F}}(\mathfrak{g}(A)) = \bigoplus_{j \in \mathbb{Z}} U_{\mathcal{F}}(\mathfrak{g}(A))_j, \quad U_{\mathcal{F}}(\mathfrak{g}(A))_j := \bigoplus_{\substack{m-k=j \\ k,m \geq 0}} U_{-k}(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathcal{F} \otimes_{\mathbb{C}} U_m(\mathfrak{n}_+),$$

so that we can "complete" it in a canonical way as:

$$\hat{U}_{\mathcal{F}}(\mathfrak{g}(A)) := \bigoplus_{j\in\mathbb{Z}} \hat{U}_{\mathcal{F}}(\mathfrak{g}(A))_j, \ \ \hat{U}_{\mathcal{F}}(\mathfrak{g}(A))_j := \prod_{\substack{m-k=j\\ k,m\geq 0}} U_{-k}(\mathfrak{n}_-)\otimes_{\mathbb{C}} \mathcal{F} \otimes_{\mathbb{C}} U_m(\mathfrak{n}_+),$$

where $U_m(\mathfrak{n}_+)$ (resp. $U_{-k}(\mathfrak{n}_-)$) is the subspace of $U(\mathfrak{n}_+)$ (resp. $U(\mathfrak{n}_-)$) of degree m (resp. $-k$). Note that the multiplication in $U_{\mathcal{F}}(\mathfrak{g}(A))$ extends to $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$, so that $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$ is an associative algebra containing $U_{\mathcal{F}}(\mathfrak{g}(A))$.

Moreover, if $V(\Lambda)$ is a highest weight $\mathfrak{g}(A)$-module with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$, then the action of $U(\mathfrak{g}(A))$ on $V(\Lambda)$ can be extended to the action of the algebra $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$, while the algebra $\mathcal{F}$ acts on $V(\Lambda)$ by:

$$\varphi(\cdot)(v_\tau) = \varphi(\tau)v_\tau,$$

where $\varphi(\cdot) \in \mathcal{F}$ and $v_\tau \in V(\Lambda)_\tau$ is a weight vector of weight $\tau \in \mathfrak{h}^*$.

1.2. Harish-Chandra homomorphism. We denote by $Z_{\mathcal{F}}$ the center of the associative algebra $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$.

Now we prepare some notation. Let $\tilde{\Delta}_+$ be the multiset in which every positive root $\alpha \in \Delta_+$ appears with its multiplicity. For $\beta \in Q_+ = \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$, denote by $\operatorname{Par}\beta$ the set of maps $k\colon \tilde{\Delta}_+ \to \mathbb{Z}_{\geq 0}$ such that $\beta = \sum_{\alpha\in\tilde{\Delta}_+} k(\alpha)\alpha$, and put $\operatorname{Par} := \cup_{\beta\in Q_+}\operatorname{Par}\beta$.

For each $\beta \in Q_+$, we can choose a basis $\{F^k\}_{k\in\operatorname{Par}\beta}$ of the vector space $U(\mathfrak{n}_-)_{-\beta}$ consisting of elements of the form $F^k = \prod_{\alpha\in\tilde{\Delta}_+}\tilde{f}_\alpha^{k(\alpha)}$ (finite product) for $k = (k(\alpha))_{\alpha\in\tilde{\Delta}_+} \in \operatorname{Par}\beta$, where $\tilde{f}_\alpha \in \mathfrak{g}_{-\alpha}$ is a root vector for a root $\alpha \in \tilde{\Delta}_+$ such that $\mathfrak{g}_{-\alpha} = \oplus\mathbb{C}\tilde{f}_\alpha$. Then elements of $\hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$ are expresed in the form

$$\sum_{k,m\in\operatorname{Par}} F^k\varphi_{k,m}\sigma(F^m) \quad \text{(infinite sum)},$$

with $\varphi_{k,m} \in \mathcal{F}$ and $|\deg(F^m) - \deg(F^k)| <$ constant.

In [K2], Kac proved the following theorem. (Here we also record the full proof by Kac for later use.)

Theorem 1 ([K2, Theorem 1]). *Let $\varphi \in \mathcal{F}$ be a function on $\mathfrak{h}^* \setminus L$. Then there exists a unique element $z_\varphi = \sum_{\beta \in Q_+} \sum_{k,m \in \mathrm{Par}\,\beta} F^k \varphi_{k,m} \sigma(F^m)$ in $Z_\mathcal{F}$ with $\varphi_{k,m} \in \mathcal{F}$ such that $\varphi_{0,0} = \varphi$. Here σ is the involutive anti-automorphism of $U(\mathfrak{g}(A))$ determined by $\sigma(e_i) = f_i, \sigma(f_i) = e_i$ for $i \in I$, and $\sigma(h) = h$ for $h \in \mathfrak{h}$.*

Proof. First we note that an element $x \in \hat{U}_\mathcal{F}(\mathfrak{g}(A))$ is zero if and only if it acts as a zero operator on each Verma module $M(\Lambda)$ with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$ (cf. the proof of Proposition 1 below). So the element $z_\varphi \in \hat{U}_\mathcal{F}(\mathfrak{g}(A))$ of the form $z_\varphi = \sum_{\beta \in Q_+} \sum_{k,m \in \mathrm{Par}\,\beta} F^k \varphi_{k,m} \sigma(F^m)$ with $\varphi_{k,m} \in \mathcal{F}$ is in the center $Z_\mathcal{F}$ if z_φ acts as the scalar $\varphi_{0,0}(\Lambda)$ on each Verma module $M(\Lambda)$ with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$. Therefore, we will choose $\varphi_{k,m} \in \mathcal{F}$ with $k, m \in \mathrm{Par}\,\beta$ by induction on $\beta \in Q_+$ in such a way that z_φ acts as the scalar $\varphi_{0,0}(\Lambda) = \varphi(\Lambda)$ on the weight space $M(\Lambda)_{\Lambda-\beta}$ for each $\beta \in Q_+$. Here we use a partial ordering $\leq$ on $\mathfrak{h}^*$ defined by: $\lambda \leq \mu \Leftrightarrow \mu - \lambda \in Q_+$.

We denote by $G^\beta_\gamma(\Lambda)$ the matrix of the oprator $\sum_{k,m \in \mathrm{Par}\,\gamma} F^k \varphi_{k,m} \sigma(F^m)$ on $M(\Lambda)_{\Lambda-\beta}$ in the basis $\{F^s(v_\Lambda)\}_{s \in \mathrm{Par}\,\beta}$ for $\beta, \gamma \in Q_+$, where $v_\Lambda \in M(\Lambda)$ is a highest weight vector of weight $\Lambda \in \mathfrak{h}^* \setminus L$. Let us fix $\beta \in Q_+$. Assume that we have already chosen the functions $\varphi_{k,m}$ with $k, m \in \mathrm{Par}\,\gamma$ for $\gamma < \beta$, so that we know the matrices $G^\beta_\gamma(\Lambda)$ for $\gamma < \beta$ and $\Lambda \in \mathfrak{h}^* \setminus L$. For the matrix $G^\beta_\beta(\Lambda)$, we have that

$$G^\beta_\beta(\Lambda) = \Phi_\beta(\Lambda) B^\Lambda_\beta, \quad \Phi_\beta(\Lambda) := (\varphi_{k,m}(\Lambda))_{k,m \in \mathrm{Par}\,\beta}, \; B^\Lambda_\beta := (B^\Lambda_\beta(F^k, F^m))_{k,m \in \mathrm{Par}\,\beta}.$$

Here $B^\Lambda_\beta(F^k, F^m) \in \mathbb{C}$ is determined by $\sigma(F^k)F^m(v_\Lambda) = B^\Lambda_\beta(F^k, F^m)v_\Lambda$. Moreover, the condition that z_φ acts on $M(\Lambda)_{\Lambda-\beta}$ as the scalar $\varphi(\Lambda)$ can be written as:

$$(*) \qquad \Phi_\beta(\Lambda) B^\Lambda_\beta + \sum_{\gamma < \beta} G^\beta_\gamma(\Lambda) = \varphi(\Lambda)\mathrm{Id},$$

since $G^\beta_\gamma(\Lambda) = 0$ for $\gamma \not\leq \beta$. Here we recall from [KK, Theorem 1] that the determinant $\det B^\Lambda_\beta$ can be writen as:

$$\det B^\Lambda_\beta = \prod_{\alpha \in \bar{\Delta}_+} \prod_{n=1}^{\infty} T_{n\alpha}(\Lambda)^{\#(\mathrm{Par}(\beta - n\alpha))},$$

up to a nonzero constant factor independent of Λ. Because $\Lambda \in \mathfrak{h}^* \setminus L$, we have $\det B_\beta^\Lambda \neq 0$, so that $\varphi_{k,m}(\Lambda)$ for $\Lambda \in \mathfrak{h}^* \setminus L$, $k, m \in \operatorname{Par}\beta$ is determined. $\square$

Conversely we have the following proposition.

Proposition 1. *An element $x \in \hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$ lies in the center $Z_{\mathcal{F}}$ only if it is of the form $x = \sum_{\beta \in Q_+} \sum_{k,m \in \operatorname{Par}\beta} F^k \varphi_{k,m} \sigma(F^m)$ for some $\varphi_{k,m} \in \mathcal{F}$.*

Proof. Let $x = \sum_{k,m \in \operatorname{Par}} F^k \varphi_{k,m} \sigma(F^m)$ with $\varphi_{k,m} \in \mathcal{F}$ and $|\deg(F^m) - \deg(F^k)| <$ constant be an element of the center $Z_{\mathcal{F}}$. It is clear that, for a highest weight vector v_Λ of the Verma module $M(\Lambda)$ with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$, we have $x(v_\Lambda) \in \mathbb{C} v_\Lambda$. So x acts as a scalar on each Verma module $M(\Lambda)$ with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$. Note that, in the summation above for the expression of x, m is an element of the set $\operatorname{Par} = \cup_{\beta \in Q_+} \operatorname{Par}\beta$. We will show by induction on β that if $m \in \operatorname{Par}\beta$, then $\varphi_{k,m} = 0$ for $k \notin \operatorname{Par}\beta$.

Let us fix $\beta \in Q_+$ and $\Lambda \in \mathfrak{h}^* \setminus L$. The element x acts as a scalar (independent of β) on the weight space $M(\Lambda)_{\Lambda-\beta}$. Now fix an arbitrary $m_0 \in \operatorname{Par}\beta$. Because the matrix $B_\beta^\Lambda = (B_\beta^\Lambda(F^k, F^m))_{k,m \in \operatorname{Par}\beta}$ is nonsingular for $\Lambda \in \mathfrak{h}^* \setminus L$, we can choose an element $v \in M(\Lambda)_{\Lambda-\beta}$ such that $\sigma(F^{m_0})(v) = c v_\Lambda$ for some nonzero $c \in \mathbb{C}$, and $\sigma(F^m)(v) = 0$ for any $m \neq m_0 \in \operatorname{Par}\beta$. Then we have

$$M(\Lambda)_{\Lambda-\beta} \supset \mathbb{C}v \ni x(v) = \sum_{k \in \operatorname{Par}} \sum_{\gamma < \beta} \sum_{m \in \operatorname{Par}\gamma} F^k \varphi_{k,m} \sigma(F^m)(v) + \sum_{k \in \operatorname{Par}} c\varphi_{k,m_0}(\Lambda) F^k(v_\Lambda),$$

where $F^k \varphi_{k,m} \sigma(F^m)(v) \in M(\Lambda)_{\Lambda-\beta}$ for $m \in \operatorname{Par}\gamma$ with $\gamma < \beta$ by the inductive assumption. Therefore, we deduce that $\varphi_{k,m_0}(\Lambda) = 0$ for any $k \notin \operatorname{Par}\beta$ since the vectors $\{F^k(v_\Lambda)\}_{k \in \operatorname{Par}}$ are linearly independent. This means that $\varphi_{k,m_0} = 0$ as an element of $\mathcal{F}$ for $k \notin \operatorname{Par}\beta$. $\square$

From Theorem 1 and Proposition 1, we see that there exists an algebra isomorphism $H\colon Z_{\mathcal{F}} \to \mathcal{F}$ defined by $z_\varphi \mapsto \varphi = \varphi_{0,0}$. we call this isomrphism H the Harish-Chandra homomorphism.

2. Holomorphicity of the Functions $\varphi_{k,m}$

2.1. The Tits cone of GKM algebras. From now on, we assume that the GKM algebra $\mathfrak{g}(A)$ over $\mathbb{C}$ is the complexification of the GKM algebra $\mathfrak{g}(A)_{\mathbb{R}}$ over $\mathbb{R}$ (i.e., $\mathfrak{g}(A) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}(A)_{\mathbb{R}}$). So the Cartan subalgebra $\mathfrak{h}$ over $\mathbb{C}$ is also the complexification of the Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ (i.e., $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$), and the set of simple roots $\Pi = \{\alpha_i\}_{i \in I}$ is a linearly independent subset of the algebraic dual $\mathfrak{h}_{\mathbb{R}}^*$ of $\mathfrak{h}_{\mathbb{R}}$ over $\mathbb{R}$. Further there exits a nondegenerate W-invariant symmetric $\mathbb{R}$-bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}_{\mathbb{R}}^*$, whose complexification on $\mathfrak{h}^*$ is also denoted by $(\cdot|\cdot)$.

Here we define the fundamental chamber C and the Tits cone X of the GKM algebra $\mathfrak{g}(A)$. We put

$$C := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha_i) \geq 0 \text{ for } i \in I\},$$

and then $X := W \cdot C = \cup_{w \in W} w \cdot C$. We denote by X° (resp. X^-) the interior (resp. the closure) of X in the usual metric topology of $\mathfrak{h}_{\mathbb{R}}^*$.

Remark 1. In [B3] and [K1], the fundamental chamber was defined to be the set

$$C^{re} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha_i) \geq 0 \text{ for } i \in I \text{ with } a_{ii} = 2\},$$

and the the Tits cone was defined to be $X^{re} := W \cdot C^{re}$. However this definition is not appropriate for our purpose here.

The proof of the following lemma is almost the same as in the case of Kac-Moody algebras (see [K1, Chap. 3] and [W, Chap. 4]).

Lemma 1. (1) *The fundamental chamber C is a fundamental domain for the action of W on X, i.e., any orbit $W \cdot \lambda$ of $\lambda \in X$ intersects C in exactly one point. Moreover, W operates simply transitively on chambers.*

(2) $X = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha) < 0$ *for only a finite number of* $\alpha \in \Delta_+\}$. *In particular, X is a convex cone.*

(3) $X^\circ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha) \leq 0$ *for only a finite number of* $\alpha \in \Delta_+\}$.

Here we prepare some more notation for GKM algebras. Let $\Pi^{re} := \{\alpha_i \in \Pi \mid a_{ii} = 2\}$ be the set of real simple roots, and $\Pi^{im} := \{\alpha_i \in \Pi \mid a_{ii} \leq 0\}$ the set of imaginary simple roots, $\Delta^{re} := W \cdot \Pi^{re}$ the set of real roots, and $\Delta^{im} := \Delta \setminus \Delta^{re}$ the set of imaginary roots. We know from [K1, Chap. 11] that $\Delta^{im} \cap \Delta_+ = W \cdot N$, where

$$N = \{\alpha \in Q_+ \setminus \{0\} \mid (\alpha|\alpha_i) \leq 0 \text{ for } i \text{ with } a_{ii} = 2, \text{ and } \mathrm{supp}(\alpha) \text{ is connected}\} \setminus \bigcup_{j \geq 2} j \cdot \Pi^{im}.$$

In particular, the set $\Delta_+^{im} := \Delta_+ \cap \Delta^{im}$ is W-stable.

Now we have the following lemma.

Lemma 2. (1) $X^- \subset \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha) \geq 0$ *for all* $\alpha \in \Delta_+^{im}\}$.

(2) $X^\circ \subset \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha) > 0$ *for all* $\alpha \in \Delta_+^{im}\}$.

Proof. (1) Let $X' := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda|\alpha) \geq 0$ for all $\alpha \in \Delta_+^{im}\}$. Then it is clear that the set X' is a W-stable closed subset of $\mathfrak{h}_{\mathbb{R}}^*$ since Δ_+^{im} is W-stable. Because $C \subset X'$ from the definition, we have $X \subset X'$, so that $X^- \subset X'$.

(2) Put $l := \dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$, and take a basis $\{v_i\}_{i=1}^l$ of $\mathfrak{h}_{\mathbb{R}}^*$. Let $\lambda \in X^\circ$. Then thre exists $\epsilon > 0$ such that $\lambda \pm \epsilon v_i \in X$ for $1 \leq i \leq l$. For any $\alpha \in \Delta_+^{im}$, there exists some v_i such that $(v_i|\alpha) \neq 0$. If $(v_i|\alpha) > 0$, we have $(\lambda|\alpha) \geq \epsilon(v_i|\alpha) > 0$ since $(\lambda - \epsilon v_i|\alpha) \geq 0$ by (1). If $(v_i|\alpha) < 0$, we have $(\lambda|\alpha) \geq -\epsilon(v_i|\alpha) > 0$ since $(\lambda + \epsilon v_i|\alpha) \geq 0$. □

Let $X_{\mathbb{C}} := X + \sqrt{-1}\,\mathfrak{h}_{\mathbb{R}}^* = \{x + \sqrt{-1}\,y \mid x \in X, y \in \mathfrak{h}_{\mathbb{R}}^*\}$ be the complexified Tits cone, and denote by K the interior of $X_{\mathbb{C}}$ in the usual metric topology of $\mathfrak{h}^*$. It is obvious that $K = X^\circ + \sqrt{-1}\,\mathfrak{h}_{\mathbb{R}}^*$.

From the lemmas above, we get the following lemma which will be used later.

Lemma 3. (1) *Let* $\alpha \in \Delta_+^{im}$ *and* $n \in \mathbb{Z}_{\geq 1}$. *Then the affine hyperplane* $T_{n\alpha}(\cdot) = 0$ *does not intersect the domain* $-\rho + K$.

(2) *Let* $\alpha \in \Delta_+^{re}$ *and* $n \in \mathbb{Z}_{\geq 1}$. *If* $\lambda \in -\rho + K$ *and* $T_{n\alpha}(\lambda) = 0$, *then* $\lambda - n\alpha \in -\rho + K$.

Proof. (1) Let $\lambda \in -\rho + K$, and suppose that $2(\lambda + \rho|\alpha) = n(\alpha|\alpha)$. Obviously we may assume that $\lambda \in -\rho + X^\circ$. We show that $(\alpha|\alpha) \leq 0$. Because $\Delta_+^{im} = W \cdot N$, we may assume that $\alpha = \sum_{i \in I} k_i \alpha_i \in N \subset Q_+$. Then we have $(\alpha|\alpha) = \sum_{i \in I} k_i(\alpha|\alpha_i) \leq 0$, since $(\alpha|\alpha_i) \leq 0$ for $\alpha_i \in \Pi^{re}$ by the definition of N and $(\alpha_j|\alpha_i) \leq 0$ $(j \in I)$ for $\alpha_i \in \Pi^{im}$. Now the equaity above contradicts part (2) of Lemma 2.

(2) Because $\alpha \in \Delta^{re} = W \cdot \Pi^{re}$, we can write $\alpha = w \cdot \alpha_i$ for some $w \in W$ and $\alpha_i \in \Pi^{re}$. In particular $(\alpha|\alpha) = (\alpha_i|\alpha_i) > 0$. Here note that the reflection r_α of $\mathfrak{h}^*$ with respect to α is defined by $r_\alpha(\lambda) := \lambda - (2(\lambda|\alpha)/(\alpha|\alpha))\alpha$ for $\lambda \in \mathfrak{h}^*$ and can be written as $r_\alpha = w r_i w^{-1}$, so that $r_\alpha \in W$. Now we have $r_\alpha(\lambda + \rho) = \lambda + \rho - (2(\lambda + \rho|\alpha)/(\alpha|\alpha))\alpha = \lambda + \rho - n\alpha$ by the assumption. Since K is W-stable, we deduce that $\lambda - n\alpha \in -\rho + K$. $\square$

2.2 Holomorphicity of the functions $\varphi_{k,m}$ on the domain $-\rho + K$. We first recall the following elementary lemma in [K2].

Lemma 4 ([K2, Lemma 2]). *Let $B = (b_{ij})$ and $C = (c_{ij})$ be two $N \times N$-marices, where b_{ij} and c_{ij} are holomorphic functions in the variables $z_1, \ldots, z_N$ on some neighborhood U of the origin 0. Put $V := U \cap \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid z_1 = 0\}$. Suppose that B is invertible on $U \setminus V$ and that on V one has:*

(a) $\det B$ *has zero of multiplicity* $s \in \mathbb{Z}_{\geq 1}$;

(b) $\dim(\operatorname{Ker} B) \equiv s$;

(c) $\operatorname{Ker} B \subset \operatorname{Ker} C$.

Here $\operatorname{Ker} B = \{x \in \mathbb{C}^N \mid Bx = 0\}$ *(which, in general, depends on $(z_1, \ldots, z_N) \in \mathbb{C}^N$). Then the entries of the matrix CB^{-1} can be extended to holomorphic functions on U.*

We remark that the classification theorem ([K1, Theorem 4.3]) holds also in the case of indecomposable GGCMs:

(1) GGCMs of finite type are exactly GCMs of finite type;

(2) GGCMs of affine type are GCMs of affine type plus the zero 1×1 matrix.

(3) If $A = (a_{ij})_{i,j \in I}$ is a GGCM of indefinite type, then there exists a positive imaginary root $\alpha = \sum_{i \in I} k_i \alpha_i$ such that $k_i > 0$ and $(\alpha|\alpha_i) < 0$ for all $i \in I$ for the GKM algebra $\mathfrak{g}(A)$ (cf. the proof of [K1, Theorem 5.6 c)]).

From now on we assume that the GGCM $A = (a_{ij})_{i,j\in I}$ is indecomposable, hence is either a GCM of finite type, a GCM of affine type, the zero 1×1 matrix, or a GGCM (possibly GCM) of indefinite type.

Here we recall the following well-known facts about the (ordinary) Kac-Moody algebras $\mathfrak{g}(A)$ associated to a GCM $A = (a_{ij})_{i,j\in I}$:

(1) if A is a GCM of finite type, then $X = \mathfrak{h}^*_{\mathbb{R}}$;

(2) if A is a GCM of affine type, then $X^\circ = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid (\lambda|\delta) > 0\}$, where δ is the unique (up to a constant factor) element of Q such that $(\delta|\alpha_i) = 0$ for all $i \in I$. In particular, we have $K - Q_+ = K$ in both of these cases.

In addition, if $\mathfrak{g}(A)$ is the GKM algebra associated to a GGCM $A = (a_{ij})_{i,j\in I}$ such that $a_{ii} \leq 0$ for all $i \in I$, then obviously we have $X - \beta \subset X$ for $\beta \in Q_+$ since $X = C$, $W = \{1\}$, and $Q_+ = \sum_{\alpha_i\in\Pi^{im}} \mathbb{Z}_{\geq 0}\alpha_i$. Hence we have $K - \beta \subset K$ for $\beta \in Q_+$, so that $K - Q_+ = K$ in this case (including the case where A is the zero 1×1 matrix).

We are now in a position to state our main theorem (compare with [K2, Theorem 2]).

Theorem 2. *Let $\varphi \in \mathcal{F}$ be a function that can be extended to a holomorphic function on the domain $-\rho + K$, and $z_\varphi = \sum_{\beta\in Q_+} \sum_{k,m\in \mathrm{Par}\,\beta} F^k \varphi_{k,m}\sigma(F^m)$ be the (unique) element of the center $Z_{\mathcal{F}}$ such that $H(z_\varphi) = \varphi$.*

(1) If all the functions $\varphi_{k,m}$ can be extended to holomorphic functions on the domain $-\rho + K - Q_+ = \cup_{\beta\in Q_+}(-\rho + K - \beta)$, then we have for $\alpha \in \Delta^{re}_+$ and $n \in \mathbb{Z}_{\geq 1}$,

$$T_{n\alpha}(\lambda) = 0 \text{ with } \lambda \in -\rho + K \text{ implies } \varphi(\lambda) = \varphi(\lambda - n\alpha).$$

(2) Let the function φ satisfy the condition that for $\alpha \in \Delta^{re}_+$ and $n \in \mathbb{Z}_{\geq 1}$,

$$T_{n\alpha}(\lambda) = 0 \text{ with } \lambda \in -\rho + K \text{ implies } \varphi(\lambda) = \varphi(\lambda - n\alpha).$$

Then, for each $\beta \in Q_+$, there exists a nonempty domain $M_\beta \subset K$ such that the functions $\varphi_{k,m} \in \mathcal{F}$ with $k, m \in \mathrm{Par}\,\beta$ can be extended to holomorphic functions on the domain

$-\rho + M_\beta$. *If the GGCM A is of finite or affine type, then we can take $M_\beta = K$ for all $\beta \in Q_+$. In the case of indefinite type, as M_β, we can take a domain of the form $\mu_\beta + K \subset K$ for some $\mu_\beta \in V := K \cap (-\sum_{\alpha_i \in \Pi^{re}} \mathbb{R}_{>0}\alpha_i)$.*

Proof. (1) First note that if the GGCM A is not of indefinite type, then we have $K - Q_+ = K$ from the remarks above. Second we remark that even in the case of indefinite type, the set $-\rho + K - Q_+$ is really a connected open set in $\mathfrak{h}^*$. In fact it is obvious that $-\rho + K - Q_+$ is an open set since it is the union of open sets $-\rho + K - \beta$ ($\beta \in Q_+$). The connectedness of $-\rho + K - Q_+$ follows from the connectedness of K itself and the fact that $K \cap (K - \beta) \neq \emptyset$ for any $\beta \in Q_+$. The latter fact is because K is an open convex cone in $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{R}} + \sqrt{-1}\,\mathfrak{h}^*_{\mathbb{R}}$.

Let $\lambda \in -\rho + K$. We will show that the element $z_\varphi \in \hat{U}_{\mathcal{F}}(\mathfrak{g}(A))$ can act on the Verma module $M(\lambda)$ with highest weight λ as the scalar $\varphi(\lambda)$, or equivalently, that z_φ acts as the scalar $\varphi(\lambda)$ on each weight space $M(\lambda)_{\lambda-\beta}$ for $\beta \in Q_+$. It clearly suffices to show that the equation $(*)$ (is well-defined and) holds for this $\lambda \in -\rho + K$ (see the proof of Theorem 1).

Here the entries of the matrix $\Phi_\beta(\cdot) = (\varphi_{k,m}(\cdot))_{k,m \in \mathrm{Par}\,\beta}$ are holomorphic on $-\rho + K$ by assumption, so are the entries of the matrix $G^\beta_\beta(\cdot) = \Phi_\beta(\cdot)B^{\cdot}_\beta$. Moreover we show that for any $\gamma < \beta$, the entries of the matrix $G^\beta_\gamma(\cdot)$ are holomorphic on $-\rho + K$ above. Let $\lambda \in -\rho + K$, $v \in M(\lambda)_{\lambda-\beta}$, and $s, t \in \mathrm{Par}\,\gamma$. Then we have $\sigma(F^t)v \in M(\lambda)_{\lambda-(\beta-\gamma)}$, so that $F^s \varphi_{s,t}\sigma(F^t)v = \varphi_{s,t}(\lambda - (\beta - \gamma))F^s\sigma(F^t)v$, where $\lambda - (\beta - \gamma) \in -\rho + K - Q_+$. Because the functions $\varphi_{s,t}(\cdot)$ are holomorphic on $-\rho + K - Q_+$ by assumption, the entries of the matrix $G^\beta_\gamma(\cdot)$ are holomorphic at any $\lambda \in -\rho + K$.

On the other hand, for each $\lambda \in \mathfrak{h}^* \setminus L$, the equation $(*)$ holds by (the proof of) Theorem 1. Since the set $\mathfrak{h}^* \setminus L$ is dense in $\mathfrak{h}^*$, we can take a sequence $\{\lambda_m\}_{m=1}^\infty$ in $(\mathfrak{h}^* \setminus L) \cap (-\rho + K)$ such that $\lim_{m\to\infty} \lambda_m = \lambda$ for each $\lambda \in -\rho + K$. Because all the entries of the matrices $G^\beta_\beta(\cdot)$, $G^\beta_\gamma(\cdot)$ are holomorphic at $\lambda \in -\rho + K$, by taking the limit as $m \to \infty$, we have the equation $(*)$ for this $\lambda \in -\rho + K$.

Now let $\Lambda \in -\rho + K$ be such that $T_{n\alpha}(\Lambda) = 0$ for some $\alpha \in \Delta_+^{re}$ and $n \in \mathbb{Z}_{\geq 1}$. Then we have an embedding $M(\Lambda - n\alpha) \hookrightarrow M(\Lambda)$ by [KK, Prop. 4.1 (b)]. The element z_φ obviously acts on the highest weight vector $v_{\Lambda - n\alpha} \neq 0 \in M(\Lambda - n\alpha)$ as the scalar $\varphi(\Lambda - n\alpha)$. Thus we have the equality $\varphi(\Lambda) = \varphi(\Lambda - n\alpha)$ for $\Lambda \in -\rho + K$ with $T_{n\alpha}(\Lambda) = 0$.

(2) First of all we remark that, in the case of indefinite type, $V \neq \emptyset$ since there exists $\alpha = \sum_{i \in I} k_i \alpha_i \in \Delta_+^{im}$ such that $k_i > 0$ and $(\alpha|\alpha_i) < 0$ for all $i \in I$ (see the comment above for the classification theorem of GGCMs).

Now we will take domain M_β by induction on $\beta \in Q_+$. We first take $M_0 = K$. Note that $K - \alpha_j \subset K$ for $\alpha_j \in \Pi^{im}$ by part (3) of Lemma 1. Let us take $\beta \neq 0 \in Q_+$. Suppose that we have already taken domains $M_\gamma = \mu_\gamma + K \subset K$ with $\mu_\gamma \in V = K \cap (-\sum_{\alpha_i \in \Pi^{re}} \mathbb{R}_{>0}\alpha_i)$ such that $M_\gamma - \alpha_j \subset M_\gamma$ $(\alpha_j \in \Pi^{im})$ for $Q_+ \ni \gamma < \beta$. Put

$$M'_\beta := \bigcap_{\substack{\gamma < \beta \\ \gamma \in Q_+}} \bigcap_{\substack{\eta \leq \beta \\ \eta \in \sum_{\alpha_i \in \Pi^{re}} \mathbb{Z}_{\geq 0}\alpha_i}} (M_\gamma + \eta).$$

For $\alpha_j \in \Pi^{im}$, we have $M'_\beta - \alpha_j \subset M'_\beta$ since $M_\gamma - \alpha_j \subset M_\gamma$ for $\gamma < \beta$ by the inductive assumption. For $\eta \in \sum_{\alpha_i \in \Pi^{re}} \mathbb{Z}_{\geq 0}\alpha_i$ with $\eta \leq \beta$, we obviously have $M'_\beta - \eta \subset M_\gamma$ for any $\gamma < \beta$. Hence we have $M'_\beta - \eta \subset M_\gamma$ for any $Q_+ \ni \gamma < \beta$ and $Q_+ \ni \eta \leq \beta$. We write $M'_\beta = \cap_{i=1}^m (v_i + K)$ for $v_i \in \sum_{\alpha_i \in \Pi^{re}} \mathbb{R}\alpha_i$. Because the set $V = K \cap (-\sum_{\alpha_i \in \Pi^{re}} \mathbb{R}_{>0}\alpha_i)$ is an open convex cone in $\sum_{\alpha_i \in \Pi^{re}} \mathbb{R}\alpha_i$, we can write $v_i = x_i - y_i$ with $x_i, y_i \in V$ for each i, since $V - V = \sum_{\alpha_i \in \Pi^{re}} \mathbb{R}\alpha_i$. Then we have

$$M'_\beta = \cap_{i=1}^m (v_i + K) \supset \cap_{i=1}^m (x_i + K) \supset K + \sum_{i=1}^m x_i,$$

since $K (\supset V)$ is a convex set. So we put $\mu_\beta := \sum_{i=1}^m x_i \in V$, and $M_\beta := \mu_\beta + K \subset K$. It is obvious that the set M_β is really a nonempty open connected set in $\mathfrak{h}^*$.

We proceed by induction on $\beta \in Q_+$. Let us fix $\beta \in Q_+$ and show that the functions $\varphi_{k,m} \in \mathcal{F}$ with $k, m \in \operatorname{Par}\beta$ can be extended to holomorphic functions on the domain $-\rho + M_\beta$. We have $M_\beta - \eta \subset M_\gamma$ for any $\gamma < \beta$ and $\eta \leq \beta$. Therefore the entries of the matrices $G_\gamma^\beta(\cdot)$ for $\gamma<\beta$ are holomorphic on $-\rho+M_\beta$, since the functions $\varphi_{s,t}(\cdot)$ with $s,t\in\operatorname{Par}\gamma$ are holomorphic on $-\rho+M_\gamma$ for $\gamma<\beta$. Hence, by the equation $(*)$ in the proof of Theorem 1, we have only to show that the functions $\varphi_{k,m}$ with $k,m\in\operatorname{Par}\beta$ can be holomorphically extended on $-\rho+M_\beta$ across the finitely many affine hyperplanes $T_{n\alpha}(\cdot)=0$ for $\alpha\in\Delta_+$, $n\in\mathbb{Z}_{\geq 1}$ with $n\alpha\leq\beta$. Furthermore, by part (1) of Lemma 3, we may assume that $\alpha\in\Delta_+^{re}$.

Let us fix arbitrary $\alpha\in\Delta_+^{re}$ and $n\in\mathbb{Z}_{\geq 1}$ with $n\alpha\leq\beta$, and consider the set $\{\Lambda\in-\rho+M_\beta \mid T_{n\alpha}(\Lambda)=0\}$. We now want to apply Lemma 4 to the case where $B=B_\beta^\Lambda$ and $C=\varphi(\Lambda)I_N-\sum_{\gamma<\beta}G_\gamma^\beta(\Lambda)$ with $N=\dim_{\mathbb{C}}M(\Lambda)_{\Lambda-\beta}$ and $s=\#(\operatorname{Par}(\beta-n\alpha))$ (remark that $\dim_{\mathbb{C}}\mathfrak{g}_\alpha=1$ for $\alpha\in\Delta_+^{re}=W\cdot\Pi^{re}$). So we will show that for any $\Lambda\in-\rho+M_\beta$ with $T_{n\alpha}(\Lambda)=0$, we have

$$\varphi(\Lambda)I_N=\sum_{\gamma<\beta}G_\gamma^\beta(\Lambda).$$

Because the entries of the matrices $G_\gamma^\beta(\cdot)$ with $\gamma<\beta$ are holomorphic on $-\rho+M_\beta\subset-\rho+K$, we may assume that $T_{m\alpha'}(\Lambda)\neq 0$ for all $\alpha'\neq\alpha\in\Delta_+$ and $m\in\mathbb{Z}_{\geq 1}$ (recall that $\mathfrak{h}^*\setminus L$ is dense in $\mathfrak{h}^*$). Then, by [KK, Prop. 4.1 (b) and the formula (4.2) on p. 106], we can deduce that the kernel $J(\Lambda)$ of the contravariant bilinear form $B^\Lambda(\cdot,\cdot)$ on the Verma module $M(\Lambda)$ is isomorphic to $M(\Lambda-n\alpha)$, where $B^\Lambda(F^kv_\Lambda,F^mv_\Lambda)=\delta_{\beta,\gamma}B_\beta^\Lambda(F^k,F^m)$ for $k\in\operatorname{Par}\beta$, $m\in\operatorname{Par}\gamma$. Let $R:=M(\Lambda)_{\Lambda-\beta}\cap J(\Lambda)\cong M(\Lambda-n\alpha)_{(\Lambda-n\alpha)-(\beta-n\alpha)}$. Since $J(\Lambda)$ is the kernel of the contravariant bilinear form $B^\Lambda(\cdot,\cdot)$ on $M(\Lambda)$, the matrix of the operator z_φ on R is $\sum_{\gamma<\beta}G_\gamma^\beta(\Lambda)$. We will show that the operator acts as the scalar $\varphi(\Lambda-n\alpha)$ on R. As in the proof of part (1), it suffices to show that the following equation (is well-defined and) holds for this $\Lambda\in-\rho+M_\beta$:

$$(**)\qquad \Phi_{\beta-n\alpha}(\Lambda-n\alpha)B_{\beta-n\alpha}^{\Lambda-n\alpha}+\sum_{\gamma<\beta-n\alpha}G_\gamma^{\beta-n\alpha}(\Lambda-n\alpha)=\varphi(\Lambda-n\alpha)\mathrm{Id}.$$

(Note that $(\Lambda-n\alpha)-(\beta-n\alpha)=\Lambda-\beta$.) Here we have $F^s\varphi_{s,t}\sigma(F^t)v=\varphi_{s,t}(\lambda-(\beta-n\alpha)+\gamma)F^s\sigma(F^t)v$ for $v\in M(\lambda)_{\lambda-(\beta-n\alpha)}$ with $\lambda\in-\rho-n\alpha+M_\beta$ and $s,t\in\operatorname{Par}\gamma$

with $\gamma \leq \beta - n\alpha$. So, for each $\gamma \leq \beta - n\alpha < \beta$, the entries of the matrix $G_\gamma^{\beta-n\alpha}(\cdot)$ (including $\Phi_{\beta-n\alpha}(\cdot)$) are holomorphic on $-\rho - n\alpha + M_\beta$ by the inductive assumption, since $\lambda \in -\rho - n\alpha + M_\beta$ implies $\lambda - (\beta - n\alpha) + \gamma = \lambda + n\alpha - (\beta - \gamma) \in -\rho + M_\gamma$. On the other hand, for each $\lambda \in \mathfrak{h}^* \setminus L$, the equation (**) with Λ replaced with λ holds by (the proof of) Theorem 1. Hence, by taking the limit, we have the equation (**) for Λ above. Thus the operator z_φ acts on $R \cong M(\Lambda - n\alpha)_{\Lambda-\beta}$ as the scalar $\varphi(\Lambda - n\alpha)$.

Due to Lemma 4 above, we deduce that the functions $\varphi_{k,m}$ with $k, m \in \operatorname{Par}\beta$ have a removable singularity at any $\Lambda \in \{\Lambda \in -\rho + M_\beta \mid T_{n\alpha}(\Lambda) = 0$, and $T_{m\alpha'}(\Lambda) \neq 0$ for $\alpha' \neq \alpha \in \Delta_+^{re}, m \in \mathbb{Z}_{\geq 1}$ with $m\alpha' \leq \beta\}$. Then we quote the theorem (cf. [GR, Theorem 1.8]) which asserts that a function of at least two complex variables can be holomorphically extended across the intersection of finitely many (but at least two) affine hyperplanes. Therefore we have proved that the functions $\varphi_{k,m}$ with $k, m \in \operatorname{Par}\beta$ can be extended to holomorphic functions on $-\rho + M_\beta$. $\square$

Remark 2. Let $f \in S(\mathfrak{h})$ be W-invariant. Then the function $\varphi(\cdot) \in \mathcal{F}$ defined by $\varphi(\lambda) := f(\lambda + \rho)$ $(\lambda \in \mathfrak{h}^*)$ satisfies the conditions of Theorem 2 (see the proof of part (2) of Lemma 3).

Finally we consider the domain $-\rho + K - Q_+$ in part (1) and the domain $-\rho + \cap_{\beta \in Q_+} M_\beta$ in part (2) of Theorem 2 above in the case of indefinite type.

We prepare the following lemma, which can be proved almost in the same way as in the case of Kac-Moody algebras (cf. the proof of [K1, Proposition 5.8 c)]).

Lemma 5. *Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM of indefinite type. Then we have*

$$X^- = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid (\lambda|\alpha) \geq 0 \textit{ for all } \alpha \in \Delta_+^{im}\}.$$

We now have the following proposition.

Proposition 2. *Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM $A = (a_{ij})_{i,j \in I}$ of indefinite type with $a_{ii} = 2$ for some $i \in I$. Then we have $K \subsetneq K - Q_+$, and $\cap_{\beta \in Q_+} M_\beta = \emptyset$.*

Proof. We first show that there exists a positive imaginary root $\alpha \in \Delta_+^{im}$ and a real simple root $\alpha_{i_0} \in \Pi^{re}$ such that $(\alpha|\alpha_{i_0}) > 0$. We know that there exists $\alpha' \in \Delta_+^{im}$ such that $(\alpha'|\alpha_i) < 0$ for all $i \in I$. Take $i_0 \in I$ with $a_{i_0 i_0} = 2$, and put $\alpha := r_{i_0}(\alpha')$. We have $(\alpha|\alpha_{i_0}) = (r_{i_0}(\alpha')|\alpha_{i_0}) = -(\alpha'|\alpha_{i_0}) > 0$, and $\alpha \in \Delta_+^{im}$ since the set Δ_+^{im} is W-stable.

If $K - \alpha_{i_0} \subset K$ for this α_{i_0}, then we obviously have $X^\circ - \alpha_{i_0} \subset X^\circ$ since $K = X^\circ + \sqrt{-1}\,\mathfrak{h}_{\mathbb{R}}^*$. Then we have $X^- - \alpha_{i_0} \subset X^-$ since $(X^\circ)^- = X^-$ from the convexity of the set X. Because $0 \in X^-$, we get $-\alpha_{i_0} \in X^-$, so that $(-\alpha_{i_0}|\alpha) \geq 0$ by Lemma 5. This is a contradiction. Hence we have $K - \alpha_{i_0} \not\subset K$, so that $K \subsetneqq K - Q_+$.

Let $x \in \cap_{\beta \in Q_+} M_\beta$. Then we have $x - \beta \in M_\beta - \beta \subset K$ for all $\beta \in Q_+$. Because $K \ni x$ is an open convex cone, we can easily deduce that $K - \beta \subset K$ for all $\beta \in Q_+$, which contradicts the fact that $K \subsetneqq K - Q_+$ just proved above. □

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