Logics Over MIPC

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1 Introduction

This paper presents a number of simple observations concerning completeness theory for normal logics containing the well-known modal intuitionistic propositional calculus **MIPC** of Prior [12] and Bull [2].

There are at least two main sources of interest to **MIPC** and its extensions. First, as was shown by Prior [12] and Bull [2], **MIPC** may be regarded as the fragment of predicate intuitionistic logic with only one variable. According to Wajsberg [16], the one-variable fragment of predicate classical logic is Lewis' **S5**, while Ono and Suzuki [10, 11, 15] proved that there is a continuum of logics between **MIPC** and **S5** which are the one-variable fragments of predicate intermediate (or superintuitionistic) logics. On the other hand, **MIPC** may be considered as an intuitionistic formalization of **S5** (see e.g. [12, 2, 5]).

The general aim of our current investigation is to develop a completeness theory for logics over **MIPC** comparable to that for intermediate logics or normal extensions of **S4**. Although there is a certain similarity between all these three types of logics, those above **MIPC** turn out to be much more complex, witness, for instance, the fact that the 1-generated universal frame for **MIPC** contains infinitely many points of depth 1 (while finitely generated refined frames for **Int** and even **K4** have only finitely many points of finite depth).

In this paper we report on a few first results of the project. Our main concern here is the relation between the finite model property and completeness of a logic, on the one hand, and the depth and width of its frames, on the other.

2 MIPC

Denote by $\mathcal{L}$ the propositional language with countably many variables $p_1, p_2, \ldots$ and the connectives $\wedge$, $\vee$, $\rightarrow$, $\perp$,[1] and let the modal language $\mathcal{ML}$ result from $\mathcal{L}$ by enriching it with the modal operators $\square$ and $\Diamond$. Intuitionistic modal logic **MIPC** is the minimal set of $\mathcal{ML}$-formulas containing all the axioms of intuitionistic propositional logic **Int**, the modal axioms

(A1) $\square p \rightarrow p$ $\qquad\qquad\qquad\qquad p \rightarrow \Diamond p$

(A2) $\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q)$ $\quad \Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$

(A3) $\Diamond p \rightarrow \square \Diamond p$ $\qquad\qquad\qquad \Diamond \square p \rightarrow \square p$

(A4) $\square(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q)$

and closed under substitution, modus ponens and necessitation $(\varphi / \square\varphi)$.

A set of $\mathcal{ML}$-formulas containing **MIPC** and closed under the above mentioned inference rules will be called in this paper an *intuitionistic modal logic* (*over* **MIPC**) or simply an *im-logic*. NExt**MIPC** (normal extensions of **MIPC**) is the family of all such logics. (More generally, the set of normal extensions of a logic L is denoted by NExt$L$; Ext**Int** is the set of all superintuitionistic propositional logics.) The minimal intuitionistic modal logic containing a logic $L \in$ NExt**MIPC** and a set of $\mathcal{ML}$-formulas Γ is denoted by $L \oplus \Gamma$.

Simple syntactic arguments (the same as in classical modal logic) show that the family NExt**MIPC** with the operations $\cap$ and $\oplus$ is a complete algebraic distributive lattice in which intersection distributes over infinite sums

$$L \cap \bigoplus_{i \in I} L_i = \bigoplus_{i \in I} (L \cap L_i),$$

but sum does not in general distribute over infinite intersections (otherwise all logics in NExt**MIPC** would have the finite model property; see below). Needless to say also that like normal extensions of **S4** all logics $L \in$ NExt**MIPC** admit the deduction theorem in the following form:

$$\Gamma, \varphi \vdash_L^* \psi \text{ iff } \Gamma \vdash_L^* \square\varphi \rightarrow \psi,$$

where $\Delta \vdash_L^* \chi$ means that χ is derivable in L from Δ with the help of modus ponens and necessitation.

Here are some important principles of **MIPC** which it shares with **S4**:

$$\square(p \wedge q) \leftrightarrow \square p \wedge \square q, \quad \square p \leftrightarrow \square\square p, \quad \Diamond p \leftrightarrow \Diamond\Diamond p, \quad (\Diamond p \rightarrow \square q) \rightarrow \square(p \rightarrow q).$$

However, unlike **S4**, the necessity and possibility operators are not dual in **MIPC**:

$$\neg\Diamond\neg p \rightarrow \square p \notin \textbf{MIPC}, \quad \neg\square\neg p \rightarrow \Diamond p \notin \textbf{MIPC}.$$

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[1]As usual, $\leftrightarrow$ and $\neg$ are defined as abbreviations: $\varphi \leftrightarrow \psi$ stands for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\neg\varphi$ for $\varphi \rightarrow \perp$; $\top$ is an abbreviation for $\perp \rightarrow \perp$.

Note that actually we have

$$\mathbf{MIPC} \oplus (\neg\Diamond\neg p \to \Box p) = \mathbf{S5}.$$

On the other hand,

$$\mathbf{MIPC} \oplus (\neg\Box\neg p \to \Diamond p) \subset \mathbf{S5}.$$

It is of interest that $\mathbf{MIPC} \oplus (\neg\Box\neg p \to \Diamond p)$ and $\mathbf{S5}$ contain exactly the same completely modalized formulas. More precisely, we call a formula φ *completely modalized* if every occurrence of a propositional variable in φ is in the scope of some $\Box$ or $\Diamond$.

Proposition 1 *For every completely modalized formula φ,*

$$\varphi \in \mathbf{S5} \text{ iff } \varphi \in \mathbf{MIPC} \oplus (\neg\Box\neg p \to \Diamond p).$$

For this reason we call $\mathbf{MIPC} \oplus (\neg\Box\neg p \to \Diamond p)$ *weak* $\mathbf{S5}$ and denote it by $\mathbf{WS5}$. It is worth noting that the cardinality of $[\mathbf{WS5},\mathbf{S5}]$ is that of continuum. For the map $f : \text{ExtInt} \to [\mathbf{WS5},\mathbf{S5}]$ defined by $f(L) = \mathbf{WS5} \oplus L$ can be shown to be an injection.

3 Semantics

Two equivalent Kripke-type semantics for $\mathbf{MIPC}$ were introduced by Ono [10], Esakia (unpublished) and Fischer Servi [5]. Recall that a triple $\mathcal{F} = \langle W, R, Q \rangle$ is called an *Ono frame* if R is a partial order on $W \neq \emptyset$ and Q a quasi-order on W such that $R \subseteq Q$ and

$$\forall x, y \in W \ (xQy \Rightarrow \exists z \in W \ (xRz \ \& \ zE_Q y)), \tag{1}$$

where $zE_Q y$ iff zQy and yQz, i.e., z and y belong to the same Q-cluster. The cluster generated by x will be denoted by $C(x)$. In other words condition (1) means that if a point x has a Q-successor (in particular, an R-successor) y then every point in $C(x)$ has an R-successor in $C(y)$.

The relation R in an Ono frame $\mathcal{F} = \langle W, R, Q \rangle$ is intended to interpret the intuitionistic connectives, while Q and E_Q interpret the modal ones. More precisely, having fixed an *intuitionistic* valuation in $\mathcal{F}$, i.e., an assignment of R-cones (see below) to propositional variables, we define the truth-relation $\models$ for $\wedge$ and $\vee$ in the classical way and for $\to$, $\Box$ and $\Diamond$ put:

- $w \models \varphi \to \psi$ iff $\forall v \in W \ (wRv \ \& \ v \models \varphi \Rightarrow v \models \psi)$;

- $w \models \Box\varphi$ iff $\forall v \in W \ (wQv \Rightarrow v \models \varphi)$;

- $w \models \Diamond\varphi$ iff $\exists v \in C(w) \ v \models \varphi$.

As we shall see below, there exist im-logics that are not complete with respect to Ono frames. General frames for a wide class of intuitionistic modal logics were first introduced in [17]. For logics over **MIPC** they can be defined in the following way.

A set $X \subseteq W$ in a quasi-order $\langle W, R \rangle$ is said to be an *R-cone* if

$$\forall x, y \in W \ (x \in X \ \& \ xRy \ \Rightarrow \ y \in X).$$

For all $x \in W$ and $X \subseteq W$, let

$$x{\uparrow}_R = \{y \in W : xRy\}, \quad X{\uparrow}_R = \bigcup_{x \in X} x{\uparrow}_R, \quad x{\downarrow}_R = \{y \in W : yRx\}, \quad X{\downarrow}_R = \bigcup_{x \in X} x{\downarrow}_R.$$

A *general frame* for **MIPC** is a quadruple $\langle W, R, Q, \mathcal{P} \rangle$, where $\langle W, R, Q \rangle$ is an Ono frame and $\mathcal{P}$ a family of R-cones in W which contains $\emptyset$ and is closed under $\cup$, $\cap$, and the operations:

$$X \to Y = \{w \in W : \forall v \in W \ (wRv \ \& \ v \in X \Rightarrow v \in Y)\} = -((X - Y){\downarrow}_R), \qquad (2)$$

$$\Box X = \{w \in W : \forall v \in W \ (wQv \Rightarrow v \in X)\} = -((-X){\downarrow}_Q), \qquad (3)$$

$$\Diamond X = \{w \in W : \exists v \in C(w) \ v \in X\} = X{\uparrow}_Q. \qquad (4)$$

¿From now on we will call general frames for **MIPC** simply *frames*, while Ono frames, i.e., frames $\mathcal{F} = \langle W, R, Q, \mathcal{P} \rangle$ with $\mathcal{P}$ containing all R-cones in $\mathcal{F}$, will be called *full frames*. Unlike full frames, valuations in general frames are restricted only to those R-cones that belong to the set of possible values $\mathcal{P}$.

A frame $\mathcal{F} = \langle W, R, Q, \mathcal{P} \rangle$ is *refined* when $\neg(xRy)$ only if there is $X \in \mathcal{P}$ such that $x \in X$ and $y \notin X$, and $\neg(xQy)$ only if there is a Q-cone $X \in \mathcal{P}$ such that $x \in X$ and $y \notin X$. In this case we call R and Q *refined*. $\mathcal{F}$ is *compact* if for all $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{-X : X \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$. Finally, refined and compact frames are called *descriptive*.

It follows from [17] that we have

Theorem 2 *Every im-logic L is characterized by the class of descriptive frames for L.*

A frame $\mathcal{F}' = \langle W', R', Q', \mathcal{P}' \rangle$ is called a *generated subframe* of a frame $\mathcal{F} = \langle W, R, Q, \mathcal{P} \rangle$ if W' is a Q-cone of W, R' and Q' are the restrictions of R and Q to W', respectively, and $\mathcal{P}' = \{X \cap W' : X \in \mathcal{P}\}$. Note that all generated subframes of $\mathcal{F} = \langle W, R, Q, \mathcal{P} \rangle$ can be obtained in the following way: we take some $X \subseteq W$, form its Q-closure $W' = X{\uparrow}_Q$, define R' and Q' to be the restrictions of R and Q to W', respectively, and put $\mathcal{P}' = \{Y \cap W' : Y \in \mathcal{P}\}$. The resultant frame $\mathcal{F}' = \langle W', R', Q', \mathcal{P}' \rangle$ is called the *subframe of $\mathcal{F}$ generated by X*. A frame $\mathcal{F}$ is said to be *rooted* if it coincides with its subframe generated by a single point, a *root* of $\mathcal{F}$.

Given two frames $\mathcal{F} = \langle W, R, Q, \mathcal{P} \rangle$ and $\mathcal{F}' = \langle W', R', Q', \mathcal{P}' \rangle$, a function f from W into W' is called a *p-morphism* from $\mathcal{F}$ into $\mathcal{F}'$ if for all $x \in W$ and $y \in W'$,

- $f(x)R'y$ iff $\exists z \in W$ $(xRz$ & $f(z) = y)$;

- $f(x)Q'y$ iff $\exists z \in W$ $(xQz$ & $f(z) = y)$;

- $yQ'f(x)$ implies $\exists z \in W$ $(zQx$ & $yR'f(z))$;

- $X \in \mathcal{P}'$ implies $f^{-1}(X) \in \mathcal{P}$.

A surjective p-morphism is known also as a *reduction*. If f is injective and for every $X \in \mathcal{P}$ there is $Y \in \mathcal{P}'$ such that $f(X) = f(W) \cap Y$ then $\mathcal{F}$ is (isomorphic to) the subframe of $\mathcal{F}'$ generated by $f(W)$.

The disjoint unions of frames are defined in exactly the same manner as in the intuitionistic or classical modal cases.

For a frame $\mathcal{F}$ denote by $Log(\mathcal{F})$ the set of all $\mathcal{ML}$-formulas that are valid in $\mathcal{F}$. It is easily checked that $Log(\mathcal{F})$ is an im-logic, and using standard arguments one can prove the following

Theorem 3 (i) *If $\mathcal{F}'$ is a generated subframe of $\mathcal{F}$, then $Log(\mathcal{F}) \subseteq Log(\mathcal{F}')$.*

(ii) *If $\mathcal{F}'$ is a reduct of $\mathcal{F}$, then $Log(\mathcal{F}) \subseteq Log(\mathcal{F}')$.*

(iii) *If $\mathcal{F}$ is the disjoint union of a family $\{\mathcal{F}_i\}_{i \in I}$, then $Log(\mathcal{F}) = \bigcap_{i \in I} Log(\mathcal{F}_i)$.*

We conclude this section with a few remarks on the semantics of some logics used in the paper. First, let us recall that **S5** is complete with respect to those Ono frames in which R is equality. **WS5** is characterized by the class of Ono frames with $Q = E_Q$. The formula $\Diamond p \rightarrow \Box p$ is valid in $\mathcal{F} = \langle W, R, Q \rangle$ iff $R = Q$. $\mathcal{F} \models \Box(\Box p \vee q) \rightarrow (\Box p \vee \Box q)$ iff $xQy \Rightarrow \exists z \in X(xE_Qz$ & $zRy)$. The formula $\Box \neg\neg p \rightarrow \neg\neg\Box p$ is valid in a descriptive frame $\mathcal{F}$ iff $C(x)$ is R-discrete for every $x \in maxW$, that is $y \in C(x)$ and yRx imply $y = x$ for every $x \in maxW$. Here $maxW$ denotes the set of all R-maximal points of W^2. (Since $\mathcal{F}$ is descriptive, $maxW \neq \emptyset$). As was proved by Esakia, the formula $\Box((p \rightarrow \Box p) \rightarrow \Box p) \rightarrow \Box p$ is valid in a descriptive $\mathcal{F}$ iff $X \subseteq (maxX \cap maxX{\downarrow}_Q){\downarrow}_Q$ for every $X \in \mathcal{B}(\mathcal{P})$. Here $\mathcal{B}(\mathcal{P})$ denotes the Boolean closure of $\mathcal{P}$.

4 Logics of finite depth and width

According to Segerberg's Theorem from [13], all classical normal modal logics containing **K4** and characterized by frames of finite depth have the finite model property (FMP, for short) in the sense that all of them are complete with respect to classes of finite frames. The same result holds for finite depth logics in ExtInt; this was proved by Kuznetsov [8] or [9] and Komori [7]. Another interesting semantical parameter guaranteeing Kripke completeness of classical modal and intermediate logics is the width of rooted frames. As was shown by Fine [4], all logics in NExt**K4** characterized by rooted frames of finite width are Kripke complete (though not necessarily have FMP).

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[2] Recall that a point $x \in W$ is said to be *R-maximal* if xRy implies $x = y$ for every $y \in W$.

The same concerns the intermediate finite width logics, as was observed by Sobolev [14].

The aim of this section is to introduce notions of depth and width of **MIPC**-frames and find out what we can say about FMP and completeness of the corresponding finite depth and finite width logics. Of course, in this case we are talking about completeness with respect to full frames. Denote by $Fr(L)$ the class of all frames validating L.

Since every **MIPC**-frame $\mathcal{F} = \langle W, R, Q, \mathcal{P} \rangle$ has two relations R and Q, there are two possibilities of defining its depth. Say that $\mathcal{F}$ is of R-*depth* n if it contains an R-chain $x_1 R \ldots R x_n$ of n distinct points but no R-chain of greater length. $\mathcal{F}$ is of Q-*depth* n if it contains a Q-chain $x_1 Q \ldots Q x_n$ of n points from distinct Q-clusters but no such chain of greater length. An im-logic L is said to be of R-*depth* n if there exists a frame $\mathcal{F} \in Fr(L)$ of R-depth n but there is no $\mathcal{F} \in Fr(L)$ of greater depth. L is called a *finite R-depth logic* if L is a logic of R-depth n, for some $n < \omega$. Finite Q-depth logics are defined analogously.

Let us consider two sequences of formulas:

$$\boldsymbol{bd}_0 = \bot;$$

$$\boldsymbol{bd}_n = q_n \vee (q_n \to \boldsymbol{bd}_{n-1}), \text{ for } n \geq 1.$$

$$\boldsymbol{bd}_0^\square = \bot;$$

$$\boldsymbol{bd}_n^\square = \square q_n \vee (\square q_n \to \boldsymbol{bd}_{n-1}^\square), \text{ for } n \geq 1.$$

The following syntactical characterization of logics of R-depth (Q-depth) n holds

Proposition 4 (i) *An im-logic L is of R-depth n iff $\boldsymbol{bd}_n \in L$ and $\boldsymbol{bd}_{n-1} \notin L$.*

(ii) *An im-logic L is of Q-depth n iff $\boldsymbol{bd}_n^\square \in L$ and $\boldsymbol{bd}_{n-1}^\square \notin L$.*

Denote by $\mathcal{D}_{fin}^R$ and $\mathcal{D}_{fin}^Q$ the classes of all finite R-depth and finite Q-depth im-logics, respectively. It should be clear that $\mathcal{D}_{fin}^R \subset \mathcal{D}_{fin}^Q$. As in the case of NExt**K4**, one can show that every logic in $\mathcal{D}_{fin}^R$ enjoys FMP. However, this is not the case for logics of finite Q-depth, even for those of Q-depth 1. Indeed, all logics in NExt**WS5** are of Q-depth 1. It is not hard to see now that if L is Kripke incomplete superintuitionistic logic then **WS5** $\oplus$ L is not complete with respect to the Ono semantics (and in particular does not have FMP). Moreover, we have the following

Theorem 5 (i) *If a logic $L \in$ ExtInt does not have FMP then no logic in the interval* [**MIPC** $\oplus$ L, **WS5** $\oplus$ L] *has FMP.*

(ii) *If a logic $L \in$ ExtInt is Kripke incomplete, then no logic in the interval* [**MIPC** $\oplus$ L, **WS5** $\oplus$ L] *is complete with respect to the Ono semantics.*

In particular, the logics **MIPC** $\oplus$ L $\oplus$ $\boldsymbol{bd}_n^\square$ are incomplete (do not have FMP) if $L \in$ ExtInt is incomplete (does not have FMP). Consequently, we obtain

Theorem 6 (i) *Every im-logic of finite R-depth enjoys FMP, and hence, is Ono complete.*

(ii) *For every $n \leq \omega$, there exists a continuum of logics of Q-depth n that are not complete with respect to the Ono semantics (in particular, do not have FMP).*

Now let us turn to im-logics of finite width. Say that a frame $\mathcal{F}$ is of R-*width* (Q-*width*) n, if there is an R-rooted (Q-rooted) subframe of $\mathcal{F}$ containing an R-antichain (Q-antichain) of n points, but no R-rooted (Q-rooted) subframe of $\mathcal{F}$ contains an R-antichain (Q-antichain) with $\geq n$ points. An im-logic is said to be of R-*width* (Q-*width*) n if there exists a frame $\mathcal{F} \in Fr(L)$ of R-width (Q-width) n but there is no $\mathcal{F} \in Fr(L)$ of greater width. L is called a *finite R-width logic* if L is a logic of R-width n, for some $n < \omega$. Finite Q-width logics are defined analogously.

Consider two sequences of formulas:

$$\boldsymbol{bw}_n = \bigvee_{i=0}^{n}(p_i \rightarrow \bigvee_{j \neq i} p_j), \text{ for } n \geq 1,$$

$$\boldsymbol{bw}_n^\square = \bigvee_{i=0}^{n}(\square p_i \rightarrow \bigvee_{j \neq i} \square p_j), \text{ for } n \geq 1.$$

The following syntactical characterization of im-logics of R-width (Q-width) n holds.

Theorem 7 (i) *An im-logic L is of R-width n iff $\boldsymbol{bw}_n \in L$ and $\boldsymbol{bw}_{n-1} \notin L$.*

(ii) *An im-logic L is of Q-width n iff $\boldsymbol{bw}_n^\square \in L$ and $\boldsymbol{bw}_{n-1}^\square \notin L$.*

Denote by $\mathcal{W}_{fin}^R$ and $\mathcal{W}_{fin}^Q$ the classes of all finite R-width logics and all finite Q-width logics, respectively. Clearly, $\mathcal{W}_{fin}^R \subset \mathcal{W}_{fin}^Q$.

As was shown by Sobolev [14], there exits a logic $L \in \text{ExtInt}$ of R-width 2 which does not have FMP. Since the logic $\textbf{MIPC} \oplus L \oplus (\Diamond p \rightarrow \square p)$ does not have FMP (is Ono incomplete) whenever L does not have FMP (is Kripke incomplete), we can conclude that not every im-logic of finite R-width enjoys FMP. Moreover, we have the following

Theorem 8 (i) *If a logic $L \in \text{ExtInt}$ does not have FMP then no logic in the interval*

$$[\textbf{MIPC} \oplus L, \textbf{MIPC} \oplus L \oplus (\Diamond p \rightarrow \square p)]$$

has FMP.

(ii) *If a logic $L \in \text{ExtInt}$ is Kripke incomplete then no logic in the interval*

$$[\textbf{MIPC} \oplus L, \textbf{MIPC} \oplus L \oplus (\Diamond p \rightarrow \square p)]$$

is Ono complete.

In particular, the logics

$$\textbf{MIPC} \oplus L \oplus \square \neg\neg p \rightarrow \neg\neg\square p$$

and

$$\mathbf{MIPC} \oplus L \oplus \Box((p \to \Box p) \to \Box p) \to \Box p$$

are Ono incomplete (do not enjoy FMP) whenever $L \in \mathrm{ExtInt}$ is Kripke incomplete (does not have FMP). Note that the axiom of the former logic is the modal analogue of Kuroda's formula

$$\forall x \neg\neg P(x) \to \neg\neg\forall x P(x),$$

and the axiom of the latter is the modal analogue of Casari's formula

$$\forall x((P(x) \to \forall x P(x)) \to \forall x P(x)) \to \forall x P(x).$$

Returning to im-logics of Q-width n, we clearly have that every logic in NExtWS5 is of Q-width 1. So there is a continuum of im-logics of Q-width 1 which are Ono incomplete.

To sum up, we have the following

Theorem 9 (i) *For every $n \geq 2$, there exists an im-logic of R-width n without FMP.*

(ii) *For every $n \leq \omega$, there exists a continuum of im-logics of Q-width n which are not complete with respect to the Ono semantics.*

It would be of great interest to know whether all finite R-width logics are Ono complete. We are working on this problem now, and our conjecture is that all of them are complete. Another important problem is to generalize the Bull–Fine Theorem on the finite model property and decidability of logics in ExtS4.3 to im-logics of R-width 1.

We conclude this note with a few related completeness results. They are established by the canonical model and filtration techniques. We remind the reader that an im-logic L is called *d-persistent*, if for every descriptive frame $\mathcal{F}$ for L, the underlying full frame of $\mathcal{F}$ also validates L. Clearly, every d-persistent logic is Ono complete. It should be clear also that d-persistence is preserved under sums of logics.

Proposition 10 (i) **MIPC** *is d-persistent.*

(ii) *For every d-persistent logic $L \in \mathrm{ExtInt}$,* **MIPC** $\oplus L$ *is d-persistent too. In particular, the logics* **MIPC** $\oplus bw_n$ *are d-persistent and complete.*

(iii) **MIPC** $\oplus \Box(\Box p \vee q) \to (\Box p \vee \Box q)$ *is d-persistent (the axiom of this logic is the modal analogue of the constant domain axiom $\forall x(P(x) \vee Q) \to (\forall x P(x) \vee Q))$.*

(iv) **MIPC** $\oplus \Box\neg\neg p \to \neg\neg\Box p$ *is d-persistent.*

(v) **MIPC** $\oplus y_n$ *is d-persistent for every $n \geq 1$, where $y_0 = \bot$, $y_n = \Box(p_n \vee (p_n \to y_{n-1}))$ are the modal analogues of Yokota's formulas $P_0^+ = \bot$, $P_n^+ = \forall x(p_n(x) \vee (p_n(x) \to P_{n-1}^+))$.*[3]

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[3]The formula y_n is valid in an Ono frame iff it is of R-depth n and all Q-clusters in it are R-discrete, i.e., $y \in C(x)$ and yRx imply $y = x$.

An example of a logic that is not d-persistent is $\mathbf{MIPC} \oplus \Box((p \to \Box p) \to \Box p) \to \Box p$, as was observed (but not published) by Esakia.

Using a combination of the filtration methods for superintuitionistic logics and classical modal logics (for an exposition of various filtration techniques consult [3]) one can show the following

Theorem 11 *If a logic* $L \in$ ExtInt *admits filtration, then the logics* **MIPC** $\oplus$ L, **MIPC** $\oplus L \oplus \Box(\Box p \vee q) \to (\Box p \vee \Box q)$, *and* **WS5** $\oplus L$ *admit filtration as well.*

One of consequences of this theorem is that **MIPC** $\oplus bw_n$ has FMP.

We are also tempted to conjecture that if $L \in$ ExtInt has FMP (is Kripke complete) then **MIPC** $\oplus L$ also enjoys FMP (respectively, is Ono complete).

References

[1] G. Bezhanishvili, *Modal intuitionistic logics and predicate superintuitionistic logics: correspondence theory*, Proceedings of New Aspects in Non-Classical Logics and Their Kripke Semantics, RIMS Kokyuroku 1010, Kyoto University, 1997, pp. 1-6.

[2] R.A. Bull, *MIPC as the formalization of an intuitionist concept of modality*, Journal of Symbolic Logic 31 (1966), pp. 609-616.

[3] A. Chagrov and M. Zakharyaschev, *Modal Logic*, Clarendon Press, Oxford, 1997.

[4] K. Fine, *Logics containing K4, part I*, Journal of Symbolic Logic 39 (1974), pp. 31-42.

[5] G. Fischer Servi, *Semantics for a class of intuitionistic modal calculi*, in M. L. Dalla Chiara, editor, Italian Studies in the Philosophy of Science, Reidel, Dordrecht, 1980, pp. 59-72.

[6] G. Fischer Servi, *Axiomatizations for some intuitionistic modal logics*, Rend. Sem. Math. Univers. Polit. 42, 1984, pp. 179-194.

[7] Y. Komori, *The finite model property of the intermediate propositional logics on finite slices*, Journal of Faculty of Science, University of Tokyo, Sec. I 22 (1975), pp. 117-120.

[8] A.V. Kuznetsov, *Some properties of the structure of varieties of pseudo-Boolean algebras* [in Russian]. In Proceedings of the 11th USSR Algebraic Colloquium, Kishinev, 1971, pp. 1255-1256.

[9] A.V. Kuznetsov, *Finitely generated pseudo-Boolean algebras and finitely approximated varieties* [in Russian]. In Proceedings of the 12th USSR Algebraic Colloquium, Sverdlovsk, 1973, p. 281.

[10] H. Ono, *On some intuitionistic modal logics*, Publications of Research Institute for Mathematical Sciences, Kyoto University 13 (1977), pp. 687-722.

[11] H. Ono and N.-Y. Suzuki, *Relations between intuitionistic modal logics and intermediate predicate logics*, Reports on Mathematical Logic 22 (1988), pp. 65-87.

[12] A. Prior, *Time and Modality*, Clarendon Press, Oxford, 1957.

[13] K. Segerberg, *An essay in classical modal logic*, Philosophical Studies, Uppsala, 13, 1971.

[14] S.K. Sobolev, *On finite-dimensional superintuitionistic logics*, Mathematics of the USSR, Izvestiya, 11 (1977), pp. 909-935.

[15] N.-Y. Suzuki, *An algebraic approach to intuitionistic modal logics in connection with intermediate predicate logics*, Studia Logica 48 (1989), pp. 141-155.

[16] M. Wajsberg, *Ein erweiterter Klassenkalkül*, Monatshefte für Mathematik und Physik 40 (1933), 113-126.

[17] F. Wolter and M. Zakharyaschev, *On the relation between intuitionistic and classical modal logics*, Algebra and Logic, 36 (1997), pp.73–92.