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Existence of non-topological solutions to a nonlinear elliptic equation arising in self-dual Chern-Simons-Higgs theory

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1 Introduction

In this paper I will report our recent studies on existence of 0-vortex non-topological solutions to a nonlinear elliptic equation arising in self-dual Chern-Simons-Higgs theory in a general background metric. For physical backgrounds for the Chern-Simons-Higgs theory, see [HKP], [JW], and [Du].

As in Schiff[Sc], the energy for static states in the (2+1)-dimensional relativistic Abelian Chern-Simons Higgs theory under the background metric $g = diag(1, -k(x), -k(x))$ is defined as follows:

$$E = \int \left\{ (|D_1\phi|^2 + |D_2\phi|^2) + \frac{\kappa^2 F_{12}^2}{4k(x)|\phi|^2} + k(x)V(|\phi|) \right\} dx,$$

where ϕ is a complex scalar field, $A_\mu(\mu = 1, 2)$ is a vector field, $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\kappa > 0$ is a coupling constant, and $k(x)$ is a positive function. If we take the special Higgs potential

$$V(|\phi|) = (1/\kappa^2)|\phi|^2(|\phi|^2 - 1)^2.$$

Then we have the following formula under certain decay assumptions to some quantities.

$$E = \int \left\{ |(D_1 \pm iD_2)\phi|^2 + \left| \frac{\kappa F_{12}}{2\sqrt{k(x)}\phi} \pm \frac{\sqrt{k(x)}}{\kappa}\phi^*(|\phi|^2 - 1) \right|^2 \right\} dx \pm \int F_{12} dx. \quad (1)$$

Here, ϕ^* is the complex conjugate of ϕ. Let $\Phi \equiv \int F_{12}\, dx$ be the total magnetic flux and fix it. Then, it follows that (ϕ, A_μ) is the global minimizer of E if and only if (ϕ, A_μ) satisfies

$$(D_1 \pm iD_2)\phi) = 0, \tag{2}$$

$$F_{12} \pm \frac{2k(x)}{\kappa^2}|\phi|^2(|\phi|^2 - 1) = 0. \tag{3}$$

As in self-dual models in many gauge theory, the study of this system can be reduced to the one of an certain second order scalar nonlinear elliptic equation. For simplicity, consider 0-vortex solutions, i.e. $\phi \neq 0$. Writing $\phi = he^{i\omega} = e^{(1/2)u+i\omega}$, (2) yields

$$A_i = -\partial_i\omega \pm \epsilon_{ij}\partial_j(\log h).$$

Thus, $F_{12} = -\{\partial_1(\partial_1 h/h) + \partial_2(\partial_2 h/h)\} = -\Delta(\log h)$. Therefore by (3) we obtain

$$\Delta u = \frac{4k(x)}{\kappa^2}e^u(e^u - 1). \tag{4}$$

We use the notation $\lambda = 4/\kappa^2$ throughout this paper.

Remark 1 *(3) in the case of plus sign yields*

$$\Phi = \int F_{12}\, dx = \frac{1}{2}\int \frac{4k(x)}{\kappa^2}e^u(1 - e^u)\, dx. \tag{5}$$

Later, we will find solutions to (4) (i.e. 0-vortex non-topological solution) in the form $u = u_0 + w, u_0 = \log(1 + |x|^2)^{\alpha/2}$, where w satisfies $\int \Delta w\, dx = 0$ and $w(x)$ tends to a constant at infinity. Since $\int \Delta u_0\, dx = 2\pi\alpha$, we have the relation $\Phi = -\alpha\pi$. So, to prescribe the total magnetic flux Φ is equivalent to prescribe the number α.

From the energy finiteness of solutions, the following two type of solution can be considered: the one is $|\phi| \to 1$ as $|x| \to \infty$ (which is called topological solutions), the other is $|\phi| \to 0$ as $|x| \to \infty$ (which is called non-topological solutions). Hence, for solution u to (4), we call u is topological iff $u(x) \to 0$ as $|x| \to \infty$; u is non-topological iff $u(x) \to -\infty$ as $|x| \to \infty$. We also say ϕ is a N-Vortex solution ($N \in 0 \cup \mathbf{N}$) for prescribed points $\{p_i\}_{i=1}^l$, if

$$|\phi| \sim c_i|x - p_i|^{n_i}(|x - p_i| \to 0), c_i > 0, N = \sum_{i=1}^{l} n_i.$$

For a $N-$vortex solution, the system (2)-(3) can be reduced into the study of

$$\Delta v = \frac{4k(x)}{\kappa^2} e^{u_0+v}(e^{u_o+v} - 1) + 4\sum_{i=1}^{l} n_i \frac{1}{(1+|x-p_i|^2)^2},$$

where $u_0(x) = -\sum_{i=1}^{l} n_i \log(1+|x-p_i|^{-2})$ (see e.g. [SpYa2]), with the asymptotic behaviour $v(x) \to 0$ for a topological solution, $v(x) \to -\infty$ for a non-topological solution, respectively. We recall several known results on existence of non-topological and topological solutions. First, for the case $g =$ diag(1,-1,-1)(i.e., $k(x) \equiv 1$), existence of arbitrary N-vortex topological solution was shown by Wang[Wa], Spruck-Yang[SpYa2], and Φ, Q(total charge), E are all quantized:

$$\Phi = 2\pi N, Q = 2\pi N\kappa, E = 2\pi N.$$

They also proved the asymptotic behaviour $(|\phi|^2 - 1), |F_{12}| \sim O(e^{-(c/|\kappa|)|x|})$ at infinity, The existence of radially symmetric N-voretex non-topological solutions was proved by Spruck-Yang[SpYa1], Chen-Hastings-McLeod-Yang, especially in [CHMY] they showed that for every $\beta > 2N+4, N \geq 0$, there exist a solution s.t. $|\phi|^2, |F_{12}| \sim O(|x|^{-\beta}), |D_j\phi|^2 = O(|x|^{-(2+\beta)})$ at infinity. In this case, we have

$$\Phi = 2\pi N + \pi\beta, Q = \Phi\kappa, , E = \Phi.$$

Next, Schiff studied self-dual Chern-Simon-Higgs theory in a general bachground metric and proved that for the case $4k(|x|)/\kappa^2 = \beta^2/|x|^2, \beta > 0$, $u(|x|) = -\log(\lambda|x|^\beta + 1), \lambda > 0$ is a solution of (4) (β-vortex non-topological solution). In [CHMY] they alos studied for certain $k(x) = k(|x|)$ the uniqueness of N- vortex topological radially symmetric solution for the prescribed N; existence of N-vortex non-topological radially symmetric solutions for ceratin range for β.

The purpose of this paper is to study (4) for certain general, not necessary radially symmetric, $k(x)$ and show existence of 0-vortex non-topological solutions via a variational method or a fixed point theorem.

2 Main Results

Throughout this paper, we assume $k(x) \not\equiv 0$ and $k(x)$ is a non-negative Hölder continuous function. Our main result is as follows.

Theorem 1 *Suppose $k(x)$ satisfies $k(x) = O(1/|x|^l)$ as $|x| \to \infty$ for some $l > 2$. Let $-4 < \alpha < \min(0, l-4)$. Then there exists a constant $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ (4) has a solution u satisfying*

$$\lim_{|x|\to\infty} (u(x) - \alpha \log |x|) = C_0$$

for some constant C_0.

Remark 2 *Actually, we can prove the following: there exists a critical parameter $\lambda_c \geq (-8\pi\alpha)/\int k(x)\,dx$ s.t. there exists a solution for every $\lambda > \lambda_c$ and no solution for $0 < \lambda < \lambda_c$. We can prove this result by combining a subsolution-supersolution method with Theorem 1. It will be published in elsewhere. Our method can be applied to obtain a 1-vortex non-topological solution under certain conditions.*

We can show Theorem 1 via a variational method based on several results on the weighted Sobolev spaces $W^2_{s,\delta}$(see [Mc]). (4) has some similarity to the Gauss curvature equation, but a difficult problem to determine the sign of the Lagrange multiplier occurs due to the nonlinearity in (4). We overcome this dificulty by using the idea of Caffarelli and Yang, in [CaYa] they employed their idea to periodic problem.

Theorem 2 *Suppose $k(x)$ satisfies $k(x) = O(1/|x|^l)$ as $|x| \to \infty$ for some $l > 2$. Fix $\lambda > 0$ in (4). Then, there exists sufficientlly small constant $\alpha_0 > 0$ s.t. for any $\alpha \in (-\alpha_0, 0)$, (4) has a solution u which satisfies*

$$\alpha \log |x| - C_1 \leq u(x) \leq \alpha \log |x| + C_2$$

near infinity, where C_1, C_2 are positive constants.

Theorem 2 is proved by using the Leray-Schauder's fixed point theorem on a weighted Sobolev space $W^p_{s,\delta}$. In this paper, we only give a sketch of the proof of Theorem 1 (see [Ku] for the details). See [Ma] for the proof of Theorem 2.

We can also show an existence theorem under somewhat mild condition for the decay on $k(x)$ via subsolution-supersolution method(see [Ma]) for certain α. However, to author's knowledge, it is an open problem to obtain an existence theorem for non-topological solutions under slow dacay general (not radially symmetirc) background metric.

Remark 3 *Recently, the results on the periodic problem in [CaYa] is sharpend by Tarantello[Ta]. As an analogy to [Ta], we have a following conjecture: there exist two solutions* $\underline{u}_\lambda, \overline{u}_\lambda$ *s.t.* $\underline{u}_\lambda \to 0$ *as* $\lambda \to +\infty$, $\overline{u}_\lambda = w_\lambda + c_\lambda, c_\lambda$ *is a constant, which satisfy* $w_\lambda \to w_0$ *in* $\mathcal{H}$ *and* $c_\lambda \to -\infty$ *as* $\lambda \to +\infty$. *Moreover,* w_0 *satisfies the following equation:*

$$-\Delta w_0 = (-2\pi\alpha)\frac{k(x)e^{u_0+w_0}}{\int_{\mathbf{R}^2} ke^{u_0+w_0}\,dx} - f, \quad \int w_0 f\,dx = 0,$$

where $f = -\Delta u_0$ *and* $u_0 = \log(1+|x|^2)^{\alpha/2}$.

3 Preliminaries

In this section, we recall several known results on weighted Sobolev spaces and Moser-Trudinger's inequality and Poincaré's inequality adapted in this setting. The weighted Sobolev spaces $W^2_{s,\delta}$ are defined as the closure of C_0^∞ with respect to the norm:

$$\|u\|^2_{W^2_{s,\delta}} = \sum_{|\beta|\le s} \|(1+|x|)^{|\beta|+\delta}|D_x^\beta u|\|^2_{L^2}$$

We use the notation $L^2_\delta = W^2_{0,\delta}$. The following results are well-known (see e.g. [Mc]).

(i) If $s' > s, \delta' > \delta$, then we have a compact embedding: $W^2_{s',\delta'} \subset W^2_{s,\delta}$.

(ii) If $s > 1, \delta > -1$, then $W^2_{s,\delta} \subset C_0(\mathbf{R}^2)$.

(iii) $u \in L^2_\delta, \Delta u \in L^2_{\delta+2}$ implies $u \in W^2_{2,\delta}$.

(iv) Let $-1 < \delta < 0$. Then $\Delta : W^2_{2,\delta} \to L^2_{\delta+2}$ is the bijiection to the range $\{f \in L^2_{\delta+2}; \int f\,dx = 0\}$. We also need the follwoing two technical lemmas.

Lemma 1 *Let* $d\mu = h(x)\,dx$ *with* $h(x) \sim (1+|x|)^{-(2+\epsilon)}, \epsilon > 0$, *and* $0 < \beta <$ $\min(4\pi, 2\pi\epsilon)$. *Then we have*

$$\int e^{a|\nu|}\,d\mu \le C\exp(\frac{a^2}{4\beta}\|\nabla\nu\|^2_{L^2}), \ \nu \in \widetilde{\mathcal{H}}.$$

Here $\mathcal{H}$ *is the closure of* C_0^∞ *w.r.t.* $\|\nu\|^2_{\mathcal{H}} = \int |\nabla\nu|^2\,dx + \int \nu^2\,d\mu$ *and* $\widetilde{\mathcal{H}} =$ $\{\nu \in \mathcal{H}; \int \nu\,d\mu = 0\}$.

Lemma 2 *Let* $\eta > 0$ *and* $\nu \in \widetilde{\mathcal{H}}$. *Then there exists a constant* $C = C(\eta)$ *such that*

$$\|\nu\|^2_{L^2_{-1-\eta}} \le C(\eta)\|\nabla\nu\|_{L^2}.$$

4 Sketch of The Proof of Theorem 1

Let $\alpha < 0$ and take $u_0(x) = \log(1+|x|^2)^{\alpha/2}$. Then

$$f(x) \equiv -\Delta u_0(x) = \frac{-2\alpha}{(1+r^2)^2} (\geq 0).$$

Consider the measure $d\mu = f(x)\,dx$. Then we have

$$\int d\mu = \int f(x)\,dx = -2\pi\alpha.$$

Let $\mathcal{H}$ be the closure of C_0^∞ w.r.t. the norm $\|\nu\|_{\mathcal{H}}^2 = \{\int |\nabla w|^2\,dx + \int w^2\,d\mu < +\infty\}$ and $\widetilde{\mathcal{H}} = \{w \in \mathcal{H}; \int w\,d\mu = 0\}$. Now $u = w + u_0$ is a solution to (4) iff w satisfies

$$\Delta w + \lambda k(x) e^{u_0+w}(1 - e^{u_0+w}) = f.$$

We will find a solution w in the class $\mathcal{H}$. Decompose $w \in \mathcal{H}$ into $w = \nu + c, \nu \in \widetilde{\mathcal{H}}$ with a constant c. Assume

$$\int \Delta w\,dx = 0.$$

ν and c should satisfy

$$e^{2c}\int k(x)e^{2(u_0+\nu)}\,dx - e^c \int k(x)e^{u_0+\nu}\,dx + (-2\pi\alpha)/\lambda = 0.$$

Thus the following condition is necessary:

$$(\int k(x)e^{u_0+\nu}\,dx)^2 + \frac{8\pi\alpha}{\lambda}\int k(x)e^{2(u_0+\nu)}\,dx \geq 0. \tag{6}$$

Let $\mathcal{H}_* = \{\nu \in \widetilde{\mathcal{H}}; \nu$ satisfies the condition above$\}$. Define the constant $c = c(\nu)$ as follows:

$$e^c = \frac{\int k(x)e^{u_0+\nu}\,dx + \sqrt{(\int k(x)e^{u_0+\nu}\,dx)^2 + \frac{8\pi\alpha}{\lambda}\int k(x)e^{2(u_0+\nu)}\,dx}}{2\int k(x)e^{2(u_0+\nu)}\,dx}. \tag{7}$$

Then consider the following minimizing problem:

$$\sigma = \inf_{\nu \in \mathcal{H}_*} I(\nu), \tag{8}$$

where

$$I(\nu) = \int \frac{1}{2}|\nabla \nu|^2 + \frac{\lambda}{2}k(x)e^{2(u_0+\nu+c)} - \lambda k(x)e^{u_0+\nu+c}\,dx - 2\pi\alpha c. \tag{9}$$

Lemma 3 *There exists a constant* $c = c(\alpha)$ *s.t.*

$$I(\nu) \geq -(-2\pi\alpha)\log\lambda - c(\alpha), \nu \in \partial\mathcal{H}_*.$$

Next take λ_0 sufficiently large s.t.

$$(\int k(x)e^{u_0}\,dx)^2 + \frac{8\pi\alpha}{\lambda_0}\int k(x)e^{2u_0}\,dx > 0.$$

Then 0 belongs to the interior of $\mathcal{H}_*$ for every $\lambda \geq \lambda_0$. On the other hand,

Lemma 4 *Taking* λ_0 *sufficiently large if necessary, there exist positive constants* $C_j = C_j(\alpha), j = 1, 2$ *s.t.*

$$I(0) \leq -C_1\lambda + C_2$$

for $\lambda \geq \lambda_0$.

Therefore, we have, taking λ_0 sufficiently large if necessary again,

$$I(0) < -1 + I(\nu), \nu \in \partial\mathcal{H}_*$$

for $\lambda \geq \lambda_0$. Hence, if there exists a minimizer ν_0 to the minimizing problem, ν_0 belongs to the interior of $\mathcal{H}_*$.

Lemma 5 *There exist positive constants* δ, C *s.t.*

$$I(\nu) \geq \delta\|\nabla\nu\|_{L^2}^2 - C, \nu \in \mathcal{H}_*.$$

Our assuption $l > 2$ need here! This lemma and the compactness of the embedding $\widetilde{\mathcal{H}} \hookrightarrow L^2(d\mu)$, we can conclude the existence of the minumizer ν_0, which belongs to the interior of $\mathcal{H}_*$. Note $\mathcal{H} \hookrightarrow W^2_{1,-1-\epsilon}$ for every $\epsilon < 1$.

Since ν_0 belongs to the interior of $\mathcal{H}_*$, we have

$$\langle I'(\nu_0), \phi\rangle = 0, \phi \in \widetilde{\mathcal{H}}.$$

Using $\int \phi f\,dx = 0$ and the definition of $c = c(\nu_0)$, this implies

$$\int \nabla\nu_0 \cdot \nabla\psi\,dx + \int\{\lambda k(x)(e^{2(u_0+\nu_0+c)} - e^{u_0+\nu_0+c}) + f\}\psi\,dx = 0$$

for every $\psi \in \mathcal{H}$. Thus $U(x) = u_0 + \nu_0 + c(\nu_0)$ satisfies $-\Delta U + \lambda k(x)e^U(e^U - 1) = 0$. Finally, since $\widetilde{\mathcal{H}} \hookrightarrow C_0$, we have $\nu_0(x) \to 0$ as $|x| \to \infty$. This conclude the desired result. □

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