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A REMARK ON NAIVE HEIGHT OF A POLARIZED ABELIAN VARIETY AND ITS APPLICATIONS

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In the talk we have presented a partial generalization of a theorem due to Masser and David concerning the canonical height on a polarized abelian variety over a number field. The essential part of the proof was to show an inequality between naive heights of isogenous polarized abelian varieties.

Let k be a number field, A a g-dimensional abelian variety over k, and $\mathcal{M}$ a very ample line bundle over A. The *theta group* $\mathcal{G}(\mathcal{M})$ of $\mathcal{M}$ is defined as

$$\mathcal{G}(\mathcal{M}) := \left\{ \phi \in \mathrm{GL}(\mathcal{M}/A); \begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{M} \\ \downarrow & & \downarrow \\ A & \xrightarrow{\text{transl.}} & A \end{array} \right\}.$$

Via the multiplication on the fibers the multiplicative group $\mathbb{G}_\mathrm{m}$ is a subgroup of $\mathcal{G}(\mathcal{M})$. Moreover we know that there exists a finite sequence $\delta = (d_1, \dots, d_g)$ of rational integers such that d_{i+1} divides d_i and such that the theta group $\mathcal{G}(\mathcal{M})$ is isomorphic over an algebraic closure $\bar{k}$ of k to the group

$$\mathcal{G}(\delta) := \mathbb{G}_\mathrm{m} \times \prod_{i=1}^{g} \boldsymbol{\mu}(d_i) \times \bigoplus_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z},$$

where $\boldsymbol{\mu}(d_i)$ is the group of d_i-th roots of unity in $\bar{k}$. We do not mention here the group structure of $\mathcal{G}(\delta)$. The sequence δ is the *type* of $\mathcal{M}$ and the integer $d := \prod_{i=1}^{g} d_i$ is the *degree* of $\mathcal{M}$. An isomorphism s of $\mathcal{G}(\mathcal{M})$ onto $\mathcal{G}(\delta)$ which is restricted to the identity on $\mathbb{G}_\mathrm{m}$ is called a *theta-structure on* $(A, \mathcal{M})$. For the detail we refer the reader to [4], [5], or [2].

What is good about a theta-structure s is that a basis $(\theta_j)_{j=1}^{d}$ of the finite dimensional vector space of global sections of $\mathcal{M}$ is determined up to a constant. The *naive*

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height h of the triple $(A, \mathcal{M}, s)$ is by definition the logarithmic height of the $\bar{k}$-valued point $(\theta_j(0))_j$ of the $(d-1)$-dimensional projective space $\mathbb{P}^{d-1}$, i.e.,

$$h(A, \mathcal{M}, s) := \frac{1}{[k : \mathbb{Q}]} \sum_v [k_v : \mathbb{Q}_v] \log \max_j |\theta_j(0)|_v.$$

Here the sum is over the set of normalized absolute values on k.

With some conditions on $\mathcal{M}$ and s, the naive height is in fact a height on a moduli scheme. Namely, we have a quasi-projective scheme $M_\delta \hookrightarrow \mathbb{P}^{d-1}$ over $\operatorname{Spec} \mathbb{Z}[d^{-1}]$ such that the set of triples $(A, \mathcal{M}, s)$ of type δ is in one-to-one correspondence with the set of $\bar{k}$-valued points of M_δ. The correspondence is given by associating $(A, \mathcal{M}, s)$ with the point $(\theta_j(0))_{j=1}^d$ as described above. For the precise statement, see [5, Section 6].

Next we consider the situation that the line bundle $\mathcal{M}$ over A is an inverse image by an isogeny. That is, let B be another abelian variety over k, $\mathcal{N}$ a very ample line bundle over B, and $f\colon A \to B$ an isogeny such that $f^*\mathcal{N} \simeq \mathcal{M}$. Then we obtain naturally the theta group $\mathcal{G}(\mathcal{N})$ of $\mathcal{N}$ from the theta group $\mathcal{G}(\mathcal{M})$ of $\mathcal{M}$ by taking a subquotient. Hence if the theta-structure s on $(A, \mathcal{M})$ is nice with respect to the formation of the subquotient, it induces a theta-structure t on $(B, \mathcal{N})$. In this case we say s and t are *compatible* [2, Definition 1.4].

The following is the key result mentioned at the beginning.

Theorem 0.1. *Notation is the same as above. If s and t are compatible, then we have*

$$h(A, \mathcal{M}, s) \geq h(B, \mathcal{N}, t). \quad \square$$

As an application we obtain the next theorem.

Let $\mathcal{L}$ be an arbitrary ample line bundle over A and set $\mathcal{M} := (\mathcal{L} \otimes (-1)^*\mathcal{L})^{\otimes 4}$. The line bundle $\mathcal{M}$ is very ample. Let $q_{\mathcal{L}}$ be the quadratic part of the canonical height attached to $\mathcal{L}$. For a finite extension field F of k, the function $q_{\mathcal{L}}$ is positive definite on the finite dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} A(F)$. The question is that what is the non-zero minimum value of $q_{\mathcal{L}}$ on the lattice $A(F)/\text{torsion}$ in $\mathbb{R} \otimes_{\mathbb{Z}} A(F)$. We do not go now into the Lang-Silverman conjecture or the Lehmer conjecture as to lower bounds of the minimum. We only note that a lower bound as below together with the rank of $A(F)$ provides us with an explicit estimate for the number of F-valued points with bounded heights.

Theorem 0.2. *If the abelian variety A is* simple *and a theta-structure on $(A, \mathcal{M})$ is defined* over k, *then there exists a positive constant $C = C(g)$ such that*

$$\min_{A(F)\ni P:\ \text{non-torsion}} q_{\mathcal{L}}(P) > \frac{C}{h(A,\mathcal{M})^{3g}\Delta^{3g+1}(1+\log\Delta)^{2g}D^{2g+1}(1+\log D)^{2g}},$$

where $h(A,\mathcal{M}) := \min_s h(A,\mathcal{M},s)$, $\Delta := [k:\mathbb{Q}]$, *and* $D := [F:k]$. □

Remark 0.3. When $\deg \mathcal{L} = 1$, the theorem is due to Masser [3] and David [1]. □

Remark 0.4. The constant C in the theorem is effective if a set of generators of the ring of automorphic forms and a set of generators of the ideal of cusp forms with respect to a certain congruence subgroup is well understood. Precisely their Fourier coefficients and their algebraic equations over the subring of theta functions. □

Handling as examples Jacobians of curves, we often encounter non-simple abelian varieties. His starting point is that the speaker would like to solve the following.

Problem 0.5. Remove the assumption *simple* in the theorem of Masser and David.

This seems possible if we are able to settle the next question in addition to our result in the talk.

Question 0.6. Can we replace the naive height h with the Faltings stable height modulo a moderate error term?

References

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