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Covering dimension and nonlinear equations

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For a set S in a Banach space, we denote by $\dim(S)$ its covering dimension ([1],p.42). Recall that, when S is a convex set, the covering dimension of S coincides with the algebraic dimension of S, this latter being understood as ∞ if it is not finite ([1], p.57). Also, $\overline{S}$ and $\mathrm{conv}(S)$ will denote the closure and the convex hull of S, respectively.

In [3], we proved what follows.

THEOREM A ([3], Theorem 1). - *Let X, Y be two Banach spaces, $\Phi : X \to Y$ a continuous, linear, surjective operator, and $\Psi : X \to Y$ a continuous operator with relatively compact range.*

Then, one has

$$\dim(\{x \in X : \Phi(x) = \Psi(x)\}) \geq \dim(\Phi^{-1}(0)).$$

In the present paper, we improve Theorem A by establishing the following result.

THEOREM 1. - *Let X, Y be two Banach spaces, $\Phi : X \to Y$ a continuous, linear, surjective operator, and $\Psi : X \to Y$ a completely continuous operator with bounded range.*

Then, one has

$$\dim(\{x \in X : \Phi(x) = \Psi(x)\}) \geq \dim(\Phi^{-1}(0)).$$

PROOF. First, assume that Φ is not injective. For each $x \in X$, $y \in Y$, $r > 0$, we denote by $B_X(x, r)$ (resp. $B_Y(y, r)$) the closed ball in X (resp. Y) of radius r centered at x (resp. y). By the open mapping theorem, there is $\delta > 0$ such that

$$B_Y(0, \delta) \subseteq \Phi(B_X(0, 1)).$$

Since $\Psi(X)$ is bounded, there is $\rho > 0$ such that

$$\overline{\Psi(X)} \subseteq B_Y(0, \rho).$$

Consequently, one has

$$\overline{\Psi(X)} \subseteq \Phi\left(B_X\left(0, \frac{\rho}{\delta}\right)\right).$$

Now, fix any bounded open convex set A in X such that

$$B_X\left(0, \frac{\rho}{\delta}\right) \subseteq A.$$

Put

$$K = \overline{\Psi(A)}.$$

Since Ψ is completely continuous, K is compact. Fix any positive integer n such that $n \leq \dim(\Phi^{-1}(0))$. Also, fix $z \in K$. Thus, $\Phi^{-1}(z) \cap A$ is a convex set of dimension at least n. Choose $n+1$ affinely independent points $u_{z,1}, ..., u_{z,n+1}$ in $\Phi^{-1}(z) \cap A$. By the open mapping theorem again, the operator Φ is open, and so, successively, the multifunctions $y \to \Phi^{-1}(y)$, $y \to \Phi^{-1}(y) \cap A$, and $y \to \overline{\Phi^{-1}(y) \cap A}$ are lower semicontinuous. Then, applying the classical Michael theorem ([2], p.98) to the restriction to K of the latter multifunction, we get $n+1$ continuous functions $f_{z,1}, ..., f_{z,n+1}$, from K into $\overline{A}$, such that, for all $y \in K, i = 1, ..., n+1$, one has

$$\Phi(f_{z,i}(y)) = y$$

and

$$f_{z,i}(z) = u_{z,i}.$$

Now, for each $i = 1, ..., n+1$, fix a neighbourhood $U_{z,i}$ of $u_{z,i}$ in A in such a way that, for any choice of w_i in $U_{z,i}$, the points $w_1, ..., w_{n+1}$ are affinely independent. Now, put

$$V_z = \bigcap_{i=1}^{n+1} f_{z,i}^{-1}(U_{z,i}).$$

Thus, V_z is a neighbourhood of z in K. Since K is compact, there are finitely many $z_1, ..., z_p \in K$ such that $K = \cup_{j=1}^{p} V_{z_j}$. For each $y \in K$, put

$$F(y) = \mathrm{conv}(\{f_{z_j,i}(y) : j = 1, ..., p,\ i = 1, ..., n+1\}).$$

Observe that, for some j, one has $y \in V_{z_j}$, and so $f_{z_j,i}(y) \in U_{z_j,i}$ for all $i = 1, ..., n+1$. Hence, $F(y)$ is a compact convex subset of $\Phi^{-1}(y) \cap \overline{A}$, with $\dim(F(y)) \geq n$. Observe also that the multifunction is F is continuous ([2], p.86 and p.89) and that the set $F(K)$ is compact ([2], p.90). Put

$$C = \overline{\mathrm{conv}(F(K))}.$$

Furthermore, note that, by continuity, one has $\Psi(\overline{A}) \subseteq K$. Finally, consider the multifunction $G : C \to 2^C$ defined by putting

$$G(x) = F(\Psi(x))$$

for all $x \in C$. Hence, G is a continuous multifunction, from the compact convex set C into itself, whose values are compact convex sets of dimension at least n. Consequently, by the result of [4], one has

$$\dim(\{x \in C : x \in G(x)\}) \geq n.$$

But, since

$$\{x \in C : x \in F(\Psi(x))\} \subseteq \{x \in C : x \in \Phi^{-1}(\Psi(x))\}$$

the conclusion follows ([1], p.220). Finally, if Φ is injective, the conclusion means simply that the set $\{x \in X : \Phi(x) = \Psi(x)\}$ is non-empty, and this is got readily proceeding as before. $\triangle$ In [3], we indicated some examples of application of Theorem A. We now point out an application of Theorem 1 which cannot be obtained from Theorem A. For a Banach space E, we denote by $\mathcal{L}(E)$ the space of all continuous linear operators from E into E, with the usual norm. Also, I will denote a (non-degenerate) compact real interval. THEOREM 2. - *Let E be an infinite-dimensional Banach space, $A : I \to \mathcal{L}(E)$ a continuous function and $f : I \times E \to E$ a uniformly continuous function with relatively compact range.*

Then, one has

$$\dim(\{u \in C^1(I, E) : u'(t) = A(t)(u(t)) + f(t, u(t)) \ \forall t \in I\}) = \infty.$$

PROOF. Take $X = C^1(I, E)$, $Y = C^0(I, E)$ and $\Phi(u) = u'(\cdot) - A(\cdot)(u(\cdot))$ for all $u \in X$. So, by a classical result, Φ is a continuous linear operator from X onto Y such that $\dim(\Phi^{-1}(0)) = \infty$. Next, put $\Psi(u) = f(\cdot, u(\cdot))$ for all $u \in X$. So, Ψ is an operator from X into Y with bounded range. From our assumptions, thanks to the Ascoli-Arzelà theorem, it also follows that Ψ is completely continuous. Then, the conclusion follows directly from Theorem 1. $\triangle$

Analogously, one gets from Theorem 1 the following

THEOREM 3. - *Let $A : I \to \mathcal{L}(\mathbf{R}^n)$ be a continuous function and $f : I \times \mathbf{R}^n \to \mathbf{R}^n$ a continuous and bounded function.*

Then, one has

$$\dim(\{u \in C^1(I, \mathbf{R}^n) : u'(t) = A(t)(u(t)) + f(t, u(t)) \ \forall t \in I\}) \geq n.$$

THEOREM 4. - *Let $a_1, ..., a_k$ be k continuous real functions on I. Further, let $f : I \times \mathbf{R}^k \to \mathbf{R}$ be a continuous and bounded function.*

Then, one has

$$\dim\left(\left\{u \in C^k(I) : u^{(k)}(t) + \sum_{i=1}^{k} a_i(t) u^{(k-i)}(t) = f(t, u(t), u'(t), ..., u^{(k-1)}(t)) \ \forall t \in I\right\}\right) \geq k.$$

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