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# Fixed Points of Multivalued Nonexpansive Mappings in Certain Convex Metric Spaces

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1. Introduction. The investigation concerning convexity in metric spaces was initiated by Menger [11] in 1928. This investigation was developed by several authors [1]. The terms "metrically convex" and "convex metric space" are due to Blumenthal[1]. Throughout this report, let $X$ be a metric space with metric $d$.

**Definition 1** *$z \in X$ is said to be a between-point of $x, y$ if*

$$z \neq x \quad z \neq y, \quad \text{and} \quad d(x,y) = d(x,z) + d(z,y).$$

**Definition 2** *$X$ is metrically convex if for each pair $x, y \in X$ such that $x \neq y$, there exists $z \in X$ that is a between-point of $x, y$. Then $X$ is said to be a convex metric space.*

Let $T$ be a mapping of $X$ into itself. $T$ is said to be nonexpansive [2], if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x,y).$$

In 1970, W.Takahashi [14] introduced a notion of convexity into metric spaces, studied properties of such spaces and proved several fixed point theorems for nonexpansive mappings.

**Definition 3** *Put $I = [0,1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y).$$

*$X$ is called a convex metric space, if it has a convex structure.*

Such kind of convex metric space seems to be often called w-convex metric space.

In 1981/82, Kirk [7] introduced a notion of a metric space of hyperbolic type and showed that it is a w-convex metric space. As a consequence of the proof of theorem 1 [7], we have the following result.

**Theorem 1** *Let $X$ be a bounded w-convex metric space that has a unique convex structure and $T$ be a nonexpansive mapping of $X$ into itself. Then* $\inf_{x\in X} d(x,Tx) = 0$. *( i.e., $X$ has the almost fixed point property for nonexpansive mappings )*

On the other hand, in 1987, Kijima [5] generalized, in certain sense [cf. 15], the notion of w-convex metric spaces.

**Definition 4** *$X$ is said to be a convex metric space if for each pair $x, y \in X$ there exists $z \in X$ such that*

$$d(z,u) \leq \frac{d(x,u) + d(y,u)}{2} \quad \text{for all} \quad u \in X. \tag{$*$}$$

We shall call such $X$ a metric space with property (S).

**Example 1** *A dyadic cube in $R^n$.*

$$X = \left\{ \left(\tfrac{k_1}{2^{m_1}}, \cdots, \tfrac{k_n}{2^{m_n}}\right) \in R^n \ : \ k_i = 0, 1, 2, \cdots, 2^{m_i}\ ,\ m_i = 1, 2, \cdots\ ,\ i = 1, \cdots, n \right\}.$$

Recently, Kijima[6] proved the following result and generalized theorem 1.

*Let $X$ be a bounded metric space with property (S). Then* $\inf_{x\in X} d(x,Tx) = 0$. *( i.e., $X$ has the almost fixed point property for nonexpansive mappings of $X$ into itself )*

This result is proved for the case of Banach space, using the Banach contraction principle; for instance, see [2]. However, the proof dose not carry over to the case of metric space with property (S). Kijima [6] proved the result by introducing an $(\epsilon, n)$-sequence without using the Banach contraction principle.

Let $K(X)$ be the class of all nonempty compact subsets of $X$. A mapping $T$ of $X$ into $K(X)$ is said to be nonexpansive, if for each pair $x, y \in X$,

$$\mathcal{H}\,(Tx, Ty) \leq d(x,y)\,.$$

where $\mathcal{H}$ is the Hausdorff metric on $K(X)$.

In 1992, Shimizu and Takahashi[12] generalized Kijima's result in the case of multivalued nonexpansive mappings with nonempty compact-values.

**Theorem 2** *Let $X$ be a bounded metric space with property (S) and $T$ be a multivalued nonexpansive mapping of $X$ into $K(X)$. Then* $\inf_{x\in X} d(x,Tx) = 0$, *where* $d(x,Tx) = \inf_{y\in Tx} d(x,y)$. *( i.e., $X$ has the almost fixed point property for multivalued nonexpansive mappings of $X$ into $K(X)$ )*

We sketch the outline of the proof.
Suppose that $\inf_{x\in X} d(x,Tx) = 2\delta > 0$. $\forall\epsilon > 0$, $\exists x_0 \in X$ s.t.

$$d(x_0, Tx_0) \leq 2\delta + \epsilon.$$

Since $Tx_0$ is nonempty compact, $\exists y_0 \in X$ s.t.

$$d(x_0, y_0) \leq 2(\delta + \epsilon).$$

Define $\{x_n\}$ and $\{y_0\}$ inductivery. Assume that $x_k$ and $y_0$ s.t. $y_k \in Tx_k$ are known. Choose $x_{k+1} \in X$ form $(*)$ such that

$$d(x_{k+1}, u) \leq \frac{d(x_k, u) + d(y_k, u)}{2}$$

for all $u \in X$.
Since $Tx_{k+1}$ is nonempty compact, we can choose $y_{k+1} \in X$ such that

$$y_{k+1} \in Tx_{k+1} \ \ \textit{and} \ \ d(y_k, y_{k+1}) = d(y_k, Tx_{k+1}).$$

$$\begin{aligned} d(y_k, y_{k+1}) &= d(y_k, Tx_{k+1}) \\ &\leq \sup_{y\in Tx_k} d(y, Tx_{k+1}) \\ &\leq \mathcal{H}(Tx_k, Tx_{k+1}) \\ &\leq d(x_k, x_{k+1}). \end{aligned}$$

By this inequality and induction using $(\epsilon, n)$-sequences, we have

$$d(x_k, x_{k+1}) \leq \delta + \epsilon$$

and

$$d(x_k, y_k) \leq 2(\delta + \epsilon)$$

for all nonnegative integer $k$. And by these inequalities and induction using $(\epsilon, n)$-sequences, we have

$$d(x_k, y_{k+n}) \geq (n+2)(\delta + \epsilon) - 2^{n+1}\epsilon$$

for all nonnegative integer $k$ and $n$.
By this inequality, we can choose $\{x_n^m\}$ , $\{y_n^m\} \subseteq X$ such that

$$d(x_0^m, Tx_0^m) \leq 2\delta + \frac{\delta}{2^m}$$

and

$$d(x_0^m, y_m^m) \geq (m+2)(\delta + \epsilon) - 2^{m+1}\frac{\delta}{2^m} > m\delta.$$

Hence we have

$$\lim_{m\to\infty} d(x_0^m, y_m^m) = \infty.$$

This contradicts the boundedness of $X$. Therefore we have

$$\inf_{x\in X} d(x, Tx) = 0.$$

By theorem 2, we have

**Theorem 3** *Let $X$ be a nonempty compact metric space with property (S) and $T$ be a multivalued nonexpansive mapping of $X$ into $K(X)$. Then $T$ has a fixed point, i.e., there exists $x_0 \in X$ such that $x_0 \in Tx_0$.*

Concerning fixed point theorems for multivalued nonexpansive mappings, in 1968, Markin [10] proved the first fixed point theorem.

**Theorem 4** *Let $H$ be a Hilbert space and $C$ be a nonemty bounded closed convex subset of $H$ and $T$ be a multivalued nonexpansive mapping of $C$ into $K(C)$ such that $Tx$ is convex for each $x \in C$. Then $T$ has a fixed point.*

He proved this theorem by proving that $(I - T)(C)$ is a closed subset of $C$. This theorem was generalized by several authors[3,16].

In 1974, Lim[8] generalized Markin's result to uniformly convex Banach spaces by transfinite induction as follows.

**Theorem 5** *Let $C$ be a nonempty bounded closed convex subset of uniformly convex Banach space $E$ and $T$ be a multivalued nonexpansive mapping of $C$ into $K(C)$. Then $T$ has a fixed point.*

We introduce a notion of uniformly convexity into convex metric spaces and prove a fixed point theorem for multivalued nonexpansive mappings in such spaces. Our theorem generalizes Lim's result and we can prove the theorem smartly by virture the filter theory.

2. Main results [13]. Let $X$ be a w-convex metric space and $W$ be its convex structure.

**Definition 5** *$X$ is said to be uniformly convex if for any $\epsilon > 0$, there exists $\alpha = \alpha(\epsilon)$ such that, for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$ , $d(z, y) \leq r$ and $d(x, y) \geq r\epsilon$,*

$$d(z, W(x, y, 1/2)) \leq r(1 - \alpha) < r.$$

**Example 2** *Uniformly convex Banach spaces.*

**Example 3** *Let $H$ be a Hilbert space and $X$ be a nonempty closed subset of $\{x \in X : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ then $(\alpha x + \beta y) / \|\alpha x + \beta y\| \in X$ $\delta(X) \leq \sqrt{2}/2$. Let $d(x, y) = \cos^{-1}\{(x, y)\}$ for all $x, y \in X$, where $(\cdot, \cdot)$ is the inner product of $H$. When we define a convex structure $W$ for $(X, d)$ adiquately, it is easily seen that $(X, d)$ becomes a complete and uniformly convex metric space[9].*

A convex metric space $X$ is said to have a property(C) if every decreasing sequence of nonempty bounded closed convex subsets of $X$ has a nonempty intersection. The authors proved the following results.

**Theorem 6** *Let $X$ be a complete and uniformly convex metric space. Then $X$ has the property(C).*

We sckech the outline of the proof. Let $\{K_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty bounded closed convex subsets of $X$. Suppose that for each $n \geq 1$, $\delta(K_n) > 0$. Then for each $n \geq 1$, there exists $x, y \in K_n$ such that

$$d(x, y) \geq \frac{\delta(K_n)}{2} \text{ and } d(z, x) \leq \delta(K_n) \ , \ d(z, y) \leq \delta(K_n) \ \text{ for all } z \in K_n.$$

Since $X$ is uniformly convex, for each $n \geq 1$, there exists $u_n^1 \in K_n$ such that

$$d\left(z, u_n^1\right) \leq \delta(K_n)(1 - \alpha) \quad \text{for all } \ z \in K_n.$$

Put

$$K_n^1 = \left\{u_n^1, u_{n+1}^1, \cdots\right\}.$$

Then we have for each $n \geq 1$,

$$K_n^1 \neq \phi, K_n^1 \subseteq K_n, \ and \ K_{n+1}^1 \supseteq K_n^1.$$

Suppose that for each $n \geq 1$, $\delta\left(K_n^1\right) > 0$. Put for each $n \geq 1$,

$$B_n^1 = \bigcap\nolimits_{k=0}^{\infty} B\left[u_{n+k}^1, \delta\left(K_n^1\right)\right].$$

Note that for each $n \geq 1$, $\overline{co}K_n^1 \subseteq B_n^1$, $\delta\left(K_n^1\right) \leq \delta\left(K_n\right)(1-\alpha)$ and

$$\delta\left(\overline{co}K_n^1\right) \leq \delta\left(B_n^1\right) \leq \delta\left(B\left[u_n^1, \delta\left(K_n^1\right)\right]\right) \leq 2\delta\left(K_n\right)(1-\alpha).$$

And we have for each $n \geq 1$, there exist $x, y \in K_n^1$ such that

$$d(x,y) \geq \frac{\delta\left(K_n^1\right)}{2}, d(z,x) \leq \delta\left(K_n^1\right) \ and \ d(z,y) \leq \delta\left(K_n^1\right) \ for \ all \ z \in \overline{co}K_n^1.$$

Since $X$ is uniformly convex, there exists $u_n^2 \in \overline{co}K_n^1$ such that

$$d\left(z, u_n^2\right) \leq \delta\left(K_n\right)(1-\alpha)^2$$

for all $z \in \overline{co}K_n^1$. Put $K_n^2 = \left\{u_n^2, u_{n+1}^2, \cdots\right\}$. Then we have for each $n \geq 1$,

$$\delta\left(\overline{co}K_n^2\right) \leq 2\delta\left(K_n\right)(1-\alpha)^2.$$

By the same method as above, we obtain for each $n \geq 1$,

$$\overline{co}K_n^3, \overline{co}K_n^4, \cdots, \ and \ u_n^3, u_n^4, \cdots.$$

And we have for each $n \geq 1$,

$$K_n \supseteq \overline{co}K_n^1 \supseteq \overline{co}K_n^2 \supseteq \cdots$$

and

$$\delta\left(\overline{co}K_n^m\right) \leq 2\delta\left(K_n\right)(1-\alpha)^m \to 0, \ as \ m \to \infty.$$

Since $X$ is complete, for each $n \geq 1$, there exists $u_n \in K_n$ such that

$$\bigcap\nolimits_{m=1}^{\infty} \overline{co}K_n^m = \{u_n\}.$$

Since for each $n \geq 1$, $\bigcap_{m=1}^{\infty} \overline{co} K_n^m \supseteq \bigcap_{m=1}^{\infty} \overline{co} K_{n+1}^m$, we have

$$u_1 = u_2 = \cdots \ .$$

Hence we have, for each $n \geq 1$, there exists $u \in X$ such that

$$u \in \bigcap_{m=1}^{\infty} \overline{co} K_n^m \subseteq K_n.$$

So we have

$$\bigcap_{n=1}^{\infty} K_n \neq \phi.$$

To prove our main theorem, we need a lemma about filters on $X$. Concerning the filter theory, for instance, see[4]. Let $\mathcal{B}$ be a filterbase on $X$ that contains at least one nonempty bounded subset in $\mathcal{B}$. Put for each $x \in X$

$$r(x, \mathcal{B}) \underset{def.}{=} \inf_{A \in \mathcal{B}} \sup_{y \in A} d(x, y).$$

We denote by $\lim_{A \in \mathcal{B}} \sup_{y \in A} d(x, y)$ the righthand side of above definition.

**Lemma 1** *Let $X$ be a complete and uniformly convex metric space. Let $K$ be a nonempty closed convex subset of $X$ and $\mathcal{F}$ be a filter on $X$ that contains at least one nonempty bounded set of $\mathcal{F}$. Then, there exists a unique $u_0 \in K$ such that*

$$r(u_0, \mathcal{F}) = \inf_{x \in K} r(x, \mathcal{F}).$$

We sketch the outline of the proof. Put $r = \inf_{x \in K} r(x, \mathcal{F})$ and

$$K_n = \left\{ z \in K \ : \ r(z, \mathcal{F}) \leq r + \frac{1}{n} \right\}.$$

Then $\{K_n\}$ is a decreasing sequence of bounded closed convex subsets of $K$. By the previous theorem, we have

$$\bigcap K_n \neq \phi.$$

So there exists $u_0 \in K$ such that

$$r(u_0, \mathcal{F}) = \inf_{x \in K} r(x, \mathcal{F}).$$

The uniqueness of $u_0$ follows from uniformly convexity of $X$.

Our main theorem is as follows.

**Theorem 7** *Let $X$ be a bounded, complete and uniformly convex metric space. If $T$ is a multivalued nonexpansive mapping of $X$ into $K(X)$. Then $T$ has a fixed point.*

We sketch the outline of the proof. By theorem 2, there exists $\{x_n\}$ such that

$$\lim_n d(x_n, Tx_n) = 0.$$

Put $A_n = \{x_n, x_{n+1}, \cdots\}$ for every $n \geq 1$. Since $\{A_n\}$ is a filterbase on $X$, it generates the filter $\mathcal{F}$ on $X$. Hence there exists an ultrafilter $\mathcal{U}$ on $X$ and $\inf_{A\in\mathcal{U}}\sup_{x\in A} d(x, Tx) = 0$. On the other hand, by lemma 1, there exists a unique $u_0 \in X$ such that

$$r(u_0, \mathcal{U}) = \inf_{x\in X} r(x, \mathcal{U}).$$

Since $Tx$ is nonempty compact for all $x \in X$, there exist $Sx \in Tx$ and $Px \in Tu_0$ such that

$$d(x, Sx) = d(x, Tx) \;\; and \;\; d(Sx, Px) = d(Sx, Tu_0).$$

Since $P$ is a mapping of $X$ into $Tu_0$, $P(\mathcal{U})$ is a filterbase on $Tu_0$ and generates an ultrafilter on $Tu_0$. Since $Tu_0$ is compact, $P(\mathcal{U})$ converges to a point $p_0 \in Tu_0$.

$$\begin{aligned} r(p_0, \mathcal{U}) &= \inf_{A\in\mathcal{U}}\sup_{x\in A} d(p_0, x) \\ &\leq \inf_{A\in\mathcal{U}}\sup_{xinA} \{d(p_0, Px) + d(Px, Sx) + d(Sx, x)\} \\ &= \inf_{A\in\mathcal{U}}\sup_{x\in A} \{d(p_0, Px) + d(Sx, Tu_0) + d(x, Tx)\} \\ &\leq \inf_{A\in\mathcal{U}}\sup_{x\in A} \{d(p_0, Px) + \mathcal{H}(Tx, Tu_0) + d(x, Tx)\} \\ &\leq \inf_{A\in\mathcal{U}}\sup_{x\in A} \{d(p_0, Px) + d(x, u_0) + d(x, Tx)\} \\ &= \inf_{A\in\mathcal{U}}\sup_{x\in A} d(x, u_0) \\ &= r(u_0, \mathcal{U}). \end{aligned}$$

By lemma 1, we have

$$u_0 = p_0 \in Tu_0.$$

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