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The Natural Criteria in Set-Valued Optimization*

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Abstract

We introduce some criteria of a minimization programming problem whose objective function is a set-valued map. For such criteria, we define some semicontinuities and prove certain theorems with respect to existence of solutions of the problem.

1. Introduction

Recently, set-valued analysis has been developed and many concepts and properties for set-valued maps are produced, see [2, 3, 4, 5]. Such a number of these concepts and properties are simple generalizations of the concepts in vector-valued optimization, however, such concepts are often not suitable for set-valued optimization, because they are only depend on some element of values of set-valued maps and not based on comparisons among values of set-valued maps. It is necessary and important to define concepts which are suitable for set-valued optimization.

In this paper, we consider what notions of set-valued maps are suitable for set-valued optimization, and then, we propose certain criteria, which are called by 'natural criteria', of solutions for set-valued optimization. Also, we investigate some properties for such solutions with such criteria.

2. The Natural Criteria and Minimal Solutions

First, we give some preliminary terminorogy in the paper. Let X be a topological space, S a nonempty subset of X, Y an ordered topological vector space with an ordering convex cone K, and F a map from X to 2^Y.

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Our set-valued optimization problem is written by

$$\text{(P)} \qquad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in S \end{array}$$

Above 'Minimize' is often interpreted like this way [3, 4, 5]:

$$x_0 \in S \text{ is a solution if } \mathrm{cl}F(x_0) \cap \mathrm{Min} \bigcup_{x \in S} F(x) \neq \emptyset.$$

However, the above solution x_0 only depends on some element of $F(x_0)$ and it does not depend on comparisons between the value $F(x_0)$ and another value of F, therefore the above interpretation is not suitable for set-valued optimization.

In this point of view, we assert that

some of criteria for set-valued optimization should be obtained by comparisons of values of the (set-valued) objective function.

We call such criteria based on the philosophy above 'Natural Criteria'.

To define such natural criteria, we introduce some relations between two sets like the order relation in topological vector spaces; though the number types of such relations is six, we treat two important relations of them, see, [6].

Definition 2.1. (SET RELATIONS)
For nonempty subsets A, B of Y,

- $A \leq^l B \overset{\text{def}}{\Longleftrightarrow} A + K \supset B$
- $A \leq^u B \overset{\text{def}}{\Longleftrightarrow} A \subset B - K$

We can see that $A \leq^l B$ and $B \leq^l A$ imply $\mathrm{Min}\, A = \mathrm{Min}\, B$, and $A \leq^u B$ and $B \leq^u A$ imply $\mathrm{Max}\, A = \mathrm{Max}\, B$, where $\mathrm{Min}\, A = \{x \in A | A \cap (x - K) = \{x\}\}$ and $\mathrm{Max} = -\mathrm{Min}(-A)$.

By using the set relations above, we define two types criteria of minimal solutions. In this paper, we assume that $F(x) + K$(resp. $F(x) - K$) is closed for each $x \in X$ when we consider l-type(resp. u-type) minimal solution.

Definition 2.2. (Minimal Solutions)

- $x_0 \in S$ is *l-type minimal solution* of (P) if
  $F(x) \leq^l F(x_0)$ and $x \in S$ imply $F(x_0) \leq^l F(x)$
- $x_0 \in S$ is *u-type minimal solution* of (P) if
  $F(x) \leq^u F(x_0)$ and $x \in S$ imply $F(x_0) \leq^u F(x)$

3. l-Type Semicontinuity of Set-Valued Maps and Existence Theorems

In the rest of the paper, we prove some existence theorems for our solutions defined by previous section. In this section, we investigate l-type solution and u-type in the next.

First, remember classical results with respect to existence of solution of some minimization problems:

(i) Let Z be a topological space, D a compact set in Z, and f a lower semicontinuous real-valued function on D. Then, f attains its minimum on D.

(ii) Let Z be a complete metric space, $f : Z \to \mathbf{R} \cup \{\infty\}$ a lower semicontinuous and proper function which is bounded from below. Then there exists $z_0 \in Z$ such that $f(z) \geq f(z_0) - \epsilon d(z, z_0)$ for all $z \in Z$. (Ekeland's variational theorem, [1])

(iii) Let Z be a Banach space, C a closed convex cone in Z, $C \subset \{z \in Z | \langle z, z^* \rangle + \epsilon ||z|| \geq 0\}$ for some $z^* \in Z$, $\epsilon > 0$, and D a nonempty closed subset of Z such that z^* is bounded from below on D. Then, $\operatorname{Min} D \neq \emptyset$. (Phelps' extreme theorem, [1])

We can find that some of theorems are concerned with concept of lower-semicontinuity of real-valued functions. Remember the lower-semicontinuity of set-valued maps: A set-valued function $F : X \to 2^Y$ said to be lower semicontinuous at $\bar{x}$ if for any $y \in F(\bar{x})$ and for any net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$, there exists a net of elements $y_\lambda \in F(x_\lambda)$ converging to y. However, the notion is a generalization of the continuity of real-valued functions, it is not a generalization of the lower-semicontinuity. Then, we define some lower-semicontinuities of set-valued maps which are generalizations of the lower-semicontinuities of real-valued functions. To this end, we define the *upper limit* and the *lower limit* of $\{A_\lambda\}$, see [2].

Definition 3.1. ($\operatorname{Lim\,inf}_\lambda A_\lambda$)
For $\{A_\lambda\} \subset 2^Y$, $(\Lambda, <)$: a directed set,

$$\operatorname*{Lim\,inf}_{\lambda} A_\lambda = \text{the set of limit points of } \{a_\lambda\}, a_\lambda \in A_\lambda;$$

$$\operatorname*{Lim\,sup}_{\lambda} A_\lambda = \text{the set of cluster points of } \{a_\lambda\}, a_\lambda \in A_\lambda.$$

In general,

$$\operatorname*{Lim\,inf}_{\lambda} A_\lambda \subset \operatorname*{Lim\,sup}_{\lambda} A_\lambda$$

By using this, we define four kinds of l-type lower semicontinuity of set-valued maps.

Definition 3.2. (**l-type Lower Semicontinuity**)
A set-valued map F is said to be

- l-type **(A)** lower semicontinuous if
  for each net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$ and for each open set U with $U \leq^l F(\bar{x})$, there exists $\hat{\lambda}$ such that $\hat{\lambda} < \lambda$ implies $U \leq^l F(x_\lambda)$.

- l-type **(B)** lower semicontinuous if
  for each net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$, $F(\bar{x}) \leq^l \mathrm{Lim\,inf}_\lambda(F(x_\lambda) + K)$.

- l-type **(C)** lower semicontinuous if
  for each $\bar{x}$, l-$\mathcal{L}(F(\bar{x})) = \{x \in S | F(x) \leq^l F(\bar{x})\}$ is closed.

- l-type **(D)** lower semicontinuous if
  for each net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$ and $\lambda < \lambda'$ implies $F(x_{\lambda'}) \leq^l F(x_\lambda)$, $F(\bar{x}) \leq^l \mathrm{Lim\,inf}_\lambda(F(x_\lambda) + K)$.

Note that, if a set-valued map F is presented by $F(x) = \{f(x)\}$ for each $x \in X$, $f : X \to \mathbf{R}$, l-type (A), l-type (B), or l-type (C) lower-semicontinuous are equivalent to the ordinary lower-semicontinuous of real-valued functions.

Proposition 3.1. We have the following:

- l-type **(A)** l.s.c. $\Rightarrow$ l-type **(B)** l.s.c.
- l-type **(B)** l.s.c. $\Rightarrow$ l-type **(C)** l.s.c.
- l-type **(C)** l.s.c. $\Rightarrow$ l-type **(D)** l.s.c.

Theorem 3.1. (Existence of l-type Solutions 1)
Let X be a topological space and Y an ordered topological vector space. If S is a nonempty compact subset of X and $F : S \to 2^Y$ is a l-type **(D)** l.s.c. set-valued map, then there exists a l-type minimal solution of (P).

Theorem 3.2. (Existence of l-type Solutions 2)
(X, d) : a complete metric space
Y : an ordered locally convex space with the cone K
$F : X \to 2^Y$ satisfies the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta\}$ such that
  - $\inf \langle y^*, F(\cdot)\rangle : S \to \mathbf{R}$
  - $F(x_1) \leq^l F(x_2), x_1, x_2 \in S \Rightarrow \inf \langle y^*, F(x_2)\rangle - \inf \langle y^*, F(x_1)\rangle \geq d(x_2, x_1)$
- $F : S \to 2^Y$ is l-type **(C)** l.s.c.

Then, there exists a l-type minimal solution of (P).

4. u-Type Semicontinuity of Set-Valued Maps and Existence Theorems

In this section, we investigate set-valued optimization with the u-type relation in the same way as the last section. First we define lower-semicontinuities of set-valued maps.

Definition 4.1. (u-type Lower Semicontinuity) A set-valued map F is said to be

- u-type **(A)** lower semicontinuous if
  for each net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$ and for each open set U with $F(\bar{x}) \cap U \neq \emptyset$, for any λ, there exists $\lambda' > \lambda$ such that $(F(x_\lambda) - K) \cap U \neq \emptyset$.

- u-type **(B)** lower semicontinuous if
  for each net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$, $F(\bar{x}) \leq^u \operatorname{Lim\,sup}_\lambda (F(x_\lambda) - K)$.

- u-type **(C)** lower semicontinuous if
  for each $\bar{x}$, u-$\mathcal{L}(F(\bar{x})) = \{x | F(x) \leq^u F(\bar{x})\}$ is closed.

- u-type **(D)** lower semicontinuous if
  for each net $\{x_\lambda\}$ with $x_\lambda \to \bar{x}$ and $\lambda < \lambda'$ implies $F(x_{\lambda'}) \leq^u F(x_\lambda)$, $F(\bar{x}) \leq^u \operatorname{Lim\,sup}_\lambda (F(x_\lambda) - K)$.

We can see that, if a set-valued map F is written by $F(x) = \{f(x)\}$ for each $x \in X$, $f : X \to \mathbf{R}$, u-type (A), u-type (B), or u-type (C) lower-semicontinuous are equivalent to the ordinary lower-semicontinuous of real-valued functions.

Proposition 4.1. (u-type Lower Semicontinuity) We have the following:

- u-type **(B)** l.s.c. $\Rightarrow$ u-type **(A)** l.s.c.

- u-type **(B)** l.s.c. $\Rightarrow$ u-type **(C)** l.s.c.

- u-type **(C)** l.s.c. $\Rightarrow$ u-type **(D)** l.s.c.

Then, we have two theorems with respect to existence of u-type solutions:

Theorem 4.1. (Existence of u-type Solutions 1)
Let X be a topological space and Y an ordered topological vector space. If S is a nonempty compact subset of X and $F : S \to 2^Y$ is a u-type **(D)** l.s.c. set-valued map, then there exists a u-type minimal solution of (P).

Theorem 4.2. (Existence of u-type Solutions 2)
(X, d) : a complete metric space
Y : an ordered locally convex space with the cone K
$F : X \to 2^Y$ satisfies the following conditions:

- there exists $y^* \in K^+ \setminus \{\theta\}$ such that
  - $\sup \langle y^*, F(\cdot)\rangle : S \to \mathbf{R}$
  - $F(x_1) \leq^u F(x_2), x_1, x_2 \in S \Rightarrow \sup \langle y^*, F(x_2)\rangle - \sup \langle y^*, F(x_1)\rangle \geq d(x_2, x_1)$
- $F : S \to 2^Y$ is u-type **(C)** l.s.c.

Then, there exists a u-type minimal solution of (P).

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