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NOTE ON FLEX CURVES AND THEIR APPLICATIONS

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1. Introduction

1.1. Let $f(x,y)\in \mathbf{C}[x,y]$ be a polynomial. We consider an affine curve $C^a(f):=\{(x,y)\in \mathbf{C}^2; f(x,y)=0\}$ and the projective curve $C(f)$ defined by the closure of $C^a(f)$ in $\mathbf{P}^2$.

In this talk, we introduce the notion of a flex curve of degree ℓ, denoted by $\mathcal{F}^{(\ell)}(f;P)$, at a smooth point P of a given curve $C^a(f)$.

As an application, we construct, in Section 5, another projective curve C_3 of degree 12 with 27 cusps for which $\pi_1(\mathbf{P}^2-C_3)$ is abelian. The triple $\{C_1,C_2,C_3\}$ gives the first example of a triple of projective curves such that $\deg C_i=12$, $i=1,2,3$ and they have 27 cusps but their complements are topologically not homeomorphic. This implies that the moduli space of curves of degree 12 with 27 cusps has at least 3 irreducible components. The pair $\{C_1,C_3\}$ is of particular interest to us as the cyclic covering argument (see for example [2]) cannot distinguish C_1 and C_3 as their Alexander polynomials are trivial.

2. Cyclic coverings and discriminant polynomials

We say that f is a polynomial of type $(a,b;d)$ where a,b are coprime positive integers if the degree of $f(x,y)$ with weight$(x)=a$ and weight$(y)=b$ is d. We denote it by $\deg_P(f)=d$ where $P:={}^t(a,b)$. We denote the set of such polynomials by $\mathcal{M}(a,b;d)$. For a given $f\in\mathcal{M}(a,b;d)$, *the P-principal part at infinity* of f is defined by the sum of the monomials in $f(x,y)$ which has weight d and we denote it by f^P. $f^P(x,y)$ is by definition a weighted homogeneous polynomial of type $(a,b;d)$ and we can factorize it as $f^P(x,y)=cx^ry^s\prod_{i=1}^k(y^a+\alpha_ix^b)^{\nu_i}$ where $c\in\mathbf{C}^*$ and $\alpha_1,\dots,\alpha_k$ are mutually distinct non-zero complex numbers. We say that $f(x,y)$ is *wt-convenient at infinity*, if $r=s=0$. We say that f is *wt-generic at infinity* if f is wt-convenient and $\nu_1=\cdots=\nu_k=1$.

If f is wt-convenient at infinity of type $(a,b;d)$, the Newton diagram $\Delta(f)$ has a unique outside face and $a,b|d$ and f is a monic polynomial of degree d/b in y (respectively of degree d/a in x) up to a multiplication of a constant.

2.1. **Cyclic covering branched along a line.** In this section, we assume that $f(x,y)$ is a wt-convenient polynomial of type $(a,b;abd)$. Let $D_\beta:=\{y=\beta\}$ be a fixed horizontal line. We consider a cyclic covering

$$\varphi_m:\mathbf{C}^2\to\mathbf{C}^2,\quad (x,y)\mapsto(x,(y-\beta)^m+\beta)$$

which is branched along D_β and put $f_m(x,y):=f(\varphi_m(x,y))$. We are mainly interested in constructing plane curves $C_m=\{f_m(x,y)=0\}$, which has as many cusps as possible, starting from a given curve C. For this purpose, we often take a tangent line of a flex point as the branching line D_β. We use the following lemma which can be proved by the exact same argument as in the proof of Theorem (3.4) of [28].

Lemma 2.1. *Let $f(x,y)$ be a wt-convenient polynomial of type $(a,b;abd)$. Let*

$$\iota_1 : \mathbf{C}^2 - C^a(f) \cup D_\beta \to \mathbf{C}^2 - C^a(f), \quad \iota_2 : \mathbf{C}^2 - C^a(f) \cup D_\beta \to \mathbf{C}^2 - D_\beta$$

be the inclusion maps. Assume that the canonical homomorphism

$$\iota_{1\sharp} \times \iota_{2\sharp} : \pi_1(\mathbf{C}^2 - C^a(f) \cup D_\beta) \to \pi_1(\mathbf{C}^2 - C^a(f)) \times \pi_1(\mathbf{C}^2 - D_\beta)$$

is an isomorphism. Then $\varphi_{m\sharp} : \pi_1(\mathbf{C}^2 - C^a(f_m)) \to \pi_1(\mathbf{C}^2 - C^a(f))$ is an isomorphism. Moreover
(1) if $\pi_1(\mathbf{C}^2 - C^a(f) \cup D_\beta)$ is abelian, $\pi_1(\mathbf{C}^2 - C^a(f_m))$ and $\pi_1(\mathbf{P}^2 - C(f_m))$ are also abelian.
(2) If $\deg_y f(x,y) = \deg f(x,y)$, $\pi_1(\mathbf{P}^2 - C(f_m))$ is a central extension of $\pi_1(\mathbf{P}^2 - C(f))$ by the cyclic group $\mathbf{Z}/m\mathbf{Z}$. Namely we have a central extension:

$$1 \to \mathbf{Z}/m\mathbf{Z} \to \pi_1(\mathbf{P}^2 - C(f_m)) \to \pi_1(\mathbf{P}^2 - C(f)) \to 1$$

(3) If $m \times \deg_y f(x,y) \le \deg f(x,y)$, $\pi_1(\mathbf{P}^2 - C(f_m)) \cong \pi_1(\mathbf{P}^2 - C(f))$.

3. Flex curves and their limits

3.1. Flex and flex curves. Let $f(x,y)$ be a reduced polynomial. Recall that a smooth point $P \in C^a(f)$ is called *a flex* of order ℓ, $\ell \ge 1$ if the intersection multiplicity of $C^a(f)$ and the tangent line T_P at P is $\ell + 2$. There exists only finite number of flex points on $C^a(f)$ if $C^a(f)$ does not have any line component.

Take a smooth point $P = (\alpha, \beta) \in C^a(f)$ with $\partial f/\partial y(\alpha,\beta) \neq 0$. A smooth curve

$$D := \{(x,y) \in \mathbf{C}^2; y - (a_0 + a_1 x + \cdots + a_\ell x^\ell) = 0\}$$

is called a *flex curve of degree ℓ (of the first kind)* for $C^a(f)$ at P if the intersection multiplicity of $C(f)$ and D at P, denoted by $I(C(f), D; P)$, is strictly greater than ℓ. Consider the analytic function $y = \varphi(x)$ which is the solution of $f(x,y) = 0$ at P. Then *the flex curve of degree ℓ of the first kind is unique and it is defined by the polynomial*

$$y = \beta + \varphi'(\alpha)(x - \alpha) + \cdots + \frac{\varphi^{(\ell)}}{\ell!}(\alpha)(x - \alpha)^\ell$$

and we denote this affine curve by $\mathcal{F}^{(\ell)}(f;P)$. Note that the tangent line $y - \beta = \varphi'(\alpha)(x - \alpha)$ is the flex curve of degree 1 and P is a flex of order 1 if and only if $\varphi''(\alpha) = 0$. More generally *P is a flex point of order ℓ if and only if $\varphi^{(j)}(\alpha) = 0$ for $j = 2, \dots, \ell + 1$.*

3.2. Limit of flex curves. Let $P(t)$, $|t| \le \varepsilon$ be a parametrization of a branch γ at infinity such that $P(0) \in L_\infty \cap C(f)$ and $P(t) = (u(t), v(t)) \in C^a(f)$ for $t \neq 0$ and $|u(t)| \to \infty$. Let $Q = {}^t(a,b)$ be the associated covector to γ and $\lambda = b/a \in \mathbf{Q}$, the ratio at infinity. As $\gcd(a,b) = 1$ and $a > 0$, we observe that λ *is an integer if and only if $a = 1$.* Changing the parametrization if necessary, we can assume that

$$u(t) = t^{-am}, \quad v(t) = t^{-bm} c(t), \quad \xi := c(0) \neq 0, \ \gcd(a,b) = 1, \ a, m > 0. \tag{3.1}$$

We are interested in the behavior of the flex curve $\mathcal{F}^{(\ell)}(f; P(t))$ when $P(t)$ goes to infinity along a branch γ at infinity. The following gives a description of this limit in the case $\lambda \neq 0, 1, \dots, \ell$.

Theorem 3.2. 1. *Assume that Q is positive and $\lambda \neq 0, 1, \dots, \ell$. Then the limit of flex curves $\lim_{t\to 0} \mathcal{F}^{(\ell)}(f; P(t))$ in the space of projective curves of degree ℓ is given by $Z^\ell = 0$ $(= \ell L_\infty)$.*
2. *Assume that Q is mixed and $b < 0$. Then $\lim_{t\to 0} \mathcal{F}^{(\ell)}(f; P(t))$ is equal to $YZ^{\ell-1} = 0$.*

Here X, Y, Z are homogeneous coordinates of $\mathbf{P}^2$ and $Z = 0$ is the line at infinity L_∞.

Reduction Operation. Put $\xi := \lim_{t\to 0} y(t)/x(t)^b$. Take the normalized automorphism of degree b:

$$\varphi_1 : \mathbf{C}^2 \to \mathbf{C}^2, \quad (x, y) \mapsto (x, y - \xi x^b)$$

and take $x_1 = x, \quad y_1 = y - \xi x^b$ as the coordinates of the target space. With respect to these coordinates, the parametrization of γ is given by

$$x_1 = u(t), \quad \text{and} \quad y_1 = v_1(t), \quad \text{where} \quad v_1(t) := v(t) - \xi u(t)^b$$

Note that $\varphi_1 \in G_{II}(\ell)'$. By the choice of coordinates, we have

$$\text{(3.3)} \qquad \mathrm{ord}_{t=0} x_1(t) = \mathrm{ord}_{t=0} x(t), \quad \mathrm{ord}_{t=0} v_1(t) > \mathrm{ord}_{t=0} v(t)$$

We call this operation *the reduction operation for* $P(t)$. The coordinates (x_1, y_1) are called *the primitive limit tangential coordinates* and φ_1 is called *the primitive limit tangential automorphism.* Let $Q_1 = {}^t(a_1, b_1)$ be the associated covector and let λ_1 be the ratio at infinity with respect to the coordinates (x_1, y_1). Note that λ_1 (and thus Q_1) is characterized by the equality $\lambda_1 = \lim_{t\to 0} \log|y_1(t)|/\log|x_1(t)|$. By (3.3), we have that $\lambda_1 < \lambda$. If λ_1 is still positive integer (so $a_1 = 1$) and $P(t)$ is still not reduced with respect to the coordinates (x_1, y_1), we put $\xi_1 := \lim_{t\to 0} y_1(t)/x_1(t)^{\lambda_1}$ and we do another reduction operation, putting $x_2 = x_1$, $y_2 := y_1 - \xi_1 x_1^{\lambda_1}$. Note that reduction operations stops at a finite step. In fact, each operation strictly decrease the ratio at infinity. Assume that the reduction operation stops at β-th step and let (x_β, y_β) be the last coordinates and let $Q_\beta = {}^t(a_\beta, b_\beta)$ be the associated covector with respect to this coordinates. By the assumption, either (a) $Q_\beta = {}^t(a_\beta, b_\beta)$ is a positive covector with $a_\beta > 1$, or (b) Q_β is mixed.

Definition 3.4. We say that $P(t)$ is *of a positive type* or *of a mixed type* depending on whether Q_β is positive or mixed respectively. In any case, we can write $x_\beta = x, \quad y_\beta = y + \sum_{i=0}^{b} c_i x^i$ for $c_0, \ldots, c_b \in \mathbf{C}$. Define $\psi(x, y) = (x, y + \sum_{i=0}^{b} c_i x^i)$. Then $\psi \in G_{II}(b)' \subset G_{II}(\ell)'$. We call ψ *the limit tangential automorphism* of the branch $P(t)$. Clearly ψ is the composition of the primitive limit tangential automorphisms.

Theorem 3.5. *Let $P(t)$ be a branch γ at infinity and let $\psi(x, y) = (x, y + \sum_{i=0}^{\ell} c_i x^i)$ be the limit tangential automorphism of $P(t)$. If the type of $P(t)$ is positive, $\lim_{t\to 0} \mathcal{F}^{(\ell)}(f; P(t))$ is given by ℓL_∞. If the type of $P(t)$ is mixed, $\lim_{t\to 0} \mathcal{F}^{(\ell)}(f; P(t))$ is given by $YZ^{\ell-1} + \sum_{i=0}^{\ell} c_i X^i Z^{\ell-i}) = 0$. Thus $\lim_{t\to 0} \mathcal{F}^{(\ell)}(f; P(t)) \cap \mathbf{C}^2 = \{y + \sum_{i=0}^{b} c_i x^i = 0\}$.*

In the case that $P(t)$ is of a positive type, we simply say that the flex curves $\mathcal{F}^{(\ell)}(f; P(t))$ *disappear* at infinity, as $\lim_{t\to 0} \mathcal{F}^{(\ell)}(f; P(t)) \cap \mathbf{C}^2 = \emptyset$.

4. Flex covering

4.1. **Flex covering.** In this section, we introduce the notion of *flex coverings*.

Proposition 4.1. *Assume that f is an irreducible polynomial and let $P \in C^a(f)$ be a smooth point with a generic flex curves of degree $\ell \geq 2$ and $\mathcal{F}^{(\ell)}(f; P) \neq C^a(f)$. Then the topology of the pair $(\mathbf{P}^2, C(f) \cup \mathcal{F}^{(\ell)}(f; P) \cup L_\infty)$ does not depend on a generic P.*

Definition 4.2. Let $P = (\alpha, \beta)$ be a smooth point of $C^a(f)$ and let $h(x, y) := y - (a_0 + a_1 x + \cdots + a_\ell x^\ell)$ be the defining polynomial of $\mathcal{F}^{(\ell)}(f; P)$. We consider the automorphism $\psi \in G_{II}(\ell)'$ and the admissible change of coordinates (x_1, y_1) defined by

$$\psi(x, y) = (x_1, y_1), \quad x_1 = x, \qquad y_1 = y - (a_0 + a_1 x + \cdots + a_\ell x^\ell)$$

As $f^{\psi}(x_1, y_1) = f(x_1, y_1 + (a_0 + a_1 x_1 + \cdots + a_\ell x_1^\ell))$, $f^{\psi}(\alpha, 0) = f(\alpha, \beta) = 0$ and $(\alpha, 0)$ is a flex point of order $\geq \ell + 1$ of $C^a(f^{\psi})$ with the tangent line $y_1 = 0$ by Proposition ?? and $h^{\psi} = y_1$.

Now we take the cyclic covering transform of $C^a(f^{\psi})$ of degree ℓ branched along $y = 0$ and we define

$$\widetilde{f}(x, y) := f^{\psi}(x, y^\ell) = f(x, y^\ell + (a_0 + a_1 x + \cdots + a_\ell x^\ell))$$

and put $\mathcal{C}^{(\ell)}(f; P) := C^a(\widetilde{f})$. We call $\mathcal{C}^{(\ell)}(f; P)$ *the flex covering transform of degree ℓ of $C^a(f)$ at P*. Put $\varphi_\ell(x, y) = (x, y^\ell)$ and $\varphi'_\ell := \psi^{-1} \circ \varphi_\ell$. Then $\varphi'_\ell : (\mathbf{C}^2, \mathcal{C}^{(\ell)}(f; P)) \to (\mathbf{C}^2, C^a(f))$ can be considered as a cyclic covering branched along $\mathcal{F}^{(\ell)}(f; P)$. We call $\mathcal{C}^{(\ell)}(f; P)$ *the generic flex covering transform of degree ℓ at P*, if $\mathcal{F}^{(\ell)}(f; P)$ is a generic flex curve of degree ℓ.

Recall that $S_\ell = {}^t(1, \ell)$. The following is immediate from the definition.

Proposition 4.3. *Assume that f is a polynomial of type $(1, \ell; d)$, $\ell \geq 2$, and $\mathcal{C}^{(\ell)}(f; P)$ is the generic flex covering transform of degree ℓ at P. We assume also that $d > \ell$ and thus $\mathcal{F}^{(\ell)}(f; P) \neq C^a(f)$. Then $\widetilde{f}(x, y)$ is a polynomial of type $(1, 1; d)$. If f^{S_ℓ} is given by $cx^r y^s \prod_{i=1}^k (y + \alpha_i x^\ell)^{\nu_i}$, we have $\widetilde{f}^{S_1}(x, y) = cx^r (y^\ell + a_\ell x^\ell)^s \prod_{i=1}^k (y^\ell + (\alpha_i + a_\ell) x^\ell)^{\nu_i}$.*

4.2. **Flex curves and limit flex curves of polynomials of type $(a, b; d)$.** In this section we assume that $f(x, y)$ is a wt-convenient irreducible polynomial of type $(a, b; d)$ and we consider flex curves of degree ℓ with $\ell \leq [b/a]$. First we write f as

$$f(x, y) = c \prod_{j=1}^{k} (y^a - \xi_j x^b)^{\nu_j} + (\text{lower terms}), \quad c \neq 0 \tag{4.4}$$

Let $P(t)$, $|t| \leq \varepsilon$, be the parametrization of a branch at infinity. If $a > 1$, $b/a \notin \mathbf{Z}$ and by Theorem 3.2, $\mathcal{F}^{(\ell)}(f; P)$ disappears. Assume that $a = 1$ and $\ell = b$. The following describes the limit of flex curves in the simplest case.

Proposition 4.5. *Assume that $a = 1$ and suppose that $P(t)$ corresponds to the factor $y - \xi_p x^\ell$ with $\nu_p = 1$. Then $\lim_{t \to 0} \mathcal{F}^{(\ell)}(f; P(t))$ is a (unique) smooth curve C defined by $y - \sum_{i=0}^\ell a_i x^i = 0$ where $a_0, \ldots, a_\ell$ are characterized by $a_\ell = \xi_p$ and $\deg_x f(x, \sum_{i=0}^\ell a_i x^i) < d - \ell$.*

It turns out that flex covering transforms often give interesting new curves starting from a simpler curve $C^a(f)$. In [26], we have constructed Zariski's three cuspidal quartic and a non-conical six cuspidal sextics, using flex covering. The following criterion is useful to construct a curve with an abelian fundamental group.

Theorem 4.6. *Assume that f is an irreducible polynomial and $\pi_1(\mathbf{C}^2 - C^a(f)) \cong \mathbf{Z}$. Assume further that for a smooth point $P = (\alpha, \beta)$ of $C^a(f)$ with $\partial f / \partial y(\alpha, \beta) \neq 0$ $\pi_1(\mathbf{C}^2 - C^a(f) \cup \mathcal{F}^{(\ell)}(f; P))$ is abelian. Then the fundamental groups $\pi_1(\mathbf{C}^2 - \mathcal{C}^{(\ell)}(f; P))$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}^{(\ell)}(f; P))$ are abelian.*

Corollary 4.7. *Assume that $f(x, y)$ is a convenient irreducible polynomial of type $(a, b; abd)$ and $\pi_1(\mathbf{C}^2 - C^a(f)) \cong \mathbf{Z}$ and $\deg_{S_\ell} f > \ell + 1$. Assume also that $C^a(f)$ has a disappearing generic flex curve of degree ℓ. Then $\pi_1(\mathbf{C}^2 - \mathcal{C}^{(\ell)}(f; P))$ is abelian for any generic P.*

5. Construction of Zariski's triple

5.1. **Zariski's non-conical six cuspidal sextic.** In our previous paper [28], we have constructed a Zariski pair of projective curves $\{C_1, C_2\}$ of degree 12 with 27 (2,3)-cusps such that $\pi_1(\mathbf{P}^2 - C_1)$ is a finite non-abelian (meta-cyclic) group of order 36 and $\pi_1(\mathbf{P}^2 - C_2)$ is isomorphic to $G(2, 3; 4)$ which is a $\mathbf{Z}_2$-extension of the free product $\mathbf{Z}_2 * \mathbf{Z}_3$.

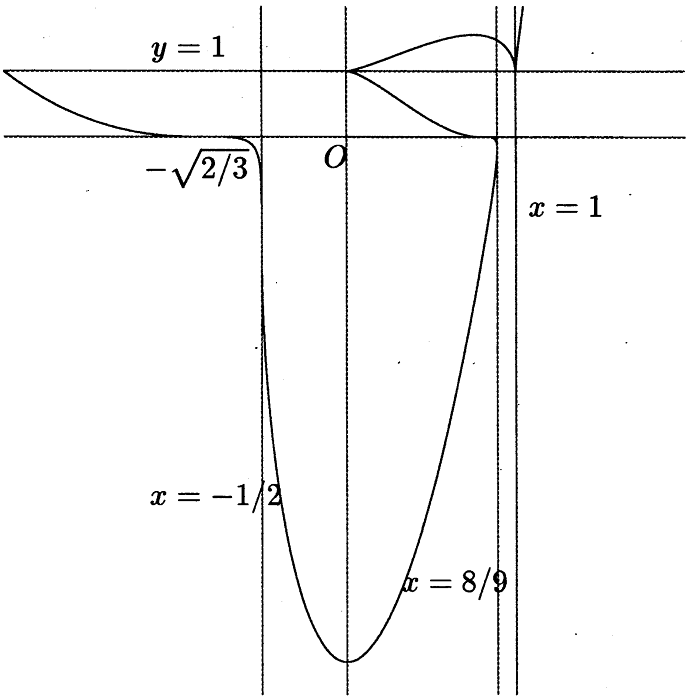

FIGURE 1. Graph of B_1

The purpose of this section is to construct another projective curve C_3 of degree 12 with 27 cusps such that $\pi_1(\mathbf{P}^2 - C_3)$ is abelian. The triple $\{C_1, C_2, C_3\}$ gives the first example of a triple of projective curves whose complements are topologically different. Therefore the moduli space of curves of degree 12 with 27 cusps has at least 3 irreducible components.

We start with the following family of affine curves $B_1(t)$ of type (1,2;6) which is defined by $B_1(t) := \{(x,y) \in \mathbf{C}^2; f_t(x,y) = 0\}$, $t \in \mathbf{R}$. In [26], we have used this family to construct a non-conical six cuspidal sextic where:

$$(5.1) \qquad \begin{aligned} f_t(x,y) :=& x^2(x-1)^2(x^2+2x+a_{00}) + x(x-1)(a_{12}x^2+a_{11}x+a_{10})(y-1) \\ &+ (a_{22}x^2+a_{21}x+a_{20})(y-1)^2 + a_{30}(y-1)^3 \end{aligned}$$

where $h = 1 - 3t + 3t^2$ and

$$\left\{ \begin{array}{llll} a_{00} = -(3t-2)^2/h, & a_{12} = (-7+24t-27t^2+9t^3)/h, & a_{11} = 3(1-2t-t^2+3t^3)/h, & \\ a_{10} = 4-6t, \quad a_{22} = 3t^2-9t+5, & a_{21} = -4+6t, & a_{20} = -h, & a_{30} = (t-1)^3 \end{array} \right.$$

Note that $B_1(t)$ has two cusps at (1,1), (0,1) and two flexes of order 1 at $(\pm\sqrt{t}, 0)$ with the same tangent line $y = 0$. The pull-back by the flex double covering $\varphi : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi(x,y) = (x, y^2)$ gives a family of six cuspidal sextics $B_2(t)$ which is defined by the polynomial $\hat{f}_t(x,y) := f_t(x, y^2)$. We will study the fiber $B_1(2/3)$ and $B_2(2/3)$ in detail, as they are of special interest. They are defined by the polynomials:

$$(5.2) \qquad f_{2/3}(x,y) = x^2(x-1)^2((x+1)^2 - y) + (x^2-1)(y-1)^2/3 - (y-1)^3/27$$

$$(5.3) \qquad \hat{f}_{2/3}(x,y) = x^2(x-1)^2((x+1)^2 - y^2) + (x^2-1)(y^2-1)^2/3 - (y^2-1)^3/27$$

This family of curves enjoys the following properties.

(a) *The polynomials f_t and $\hat{f}_t$ are generic at infinity for $t \neq 2/3$ but $f_{2/3}$ and $\hat{f}_{2/3}$ are degenerated and their principal parts at infinity are given by $(y - 3x^2)^3/27$ and $(y^2 - 3x^2)^3/27$ respectively.*

However the curve $B_2(2/3)$ *has no singularity at infinity but it has two flex points at infinity* $[1;\pm\sqrt{3};0]$ *with their common tangent line* $L_\infty := \{Z=0\}$.

(b) *For each* t, $B_1(t)$ *has two cusps at* $S_1 := (1,1)$ *and* $S_2 := (0,1)$. *The tangent cone at* S_1 *is constant and it is defined by* $x^2 = 0$. *The tangent cone at* S_2 *is not constant. For* $B_1(2/3)$, *it is given by* $y^2 = 0$.

(c) *The discriminant* $\Delta_y(f_{2/3})(x)$ *is given by*

$$\Delta_y(f)(x) = x^3(-8+9x)(2x+1)^2(x-1)^4 \tag{5.4}$$

and $\deg_x(\Delta_y(f_{2/3})) = 10$ *and* $\deg_x(\Delta_y(f_t)) = 12$ *for* $t \neq 2/3$. *(Two roots of* $\Delta_y(f_t)(x) = 0$ *disappear at infinity when* $t \to 2/3$.*)* $x=0$ *and* $x=1$ *are roots of* $\Delta_y(f_t)(x) = 0$ *of multiplicity* 3 *and* 4 *respectively and*

($\star$) : *other roots of* $\Delta_y(f_t)(x) = 0$ *are simple for a generic* t.

For the last assertion ($\star$), by the algebraicity of the condition, it is enough to give one generic t such that this is the case. For example, $\Delta_y(f_{3/4})(x)$ is given by

$$-x^3(5504x^5 - 14704x^4 - 22048x^3 + 81816x^2 - 5712x - 40131)(x-1)^4/87808$$

(d) *The discriminant of* $\hat{f}$ *is given by* ?? *and the equality* $f_{2/3}(x,0) = (3x^2-2)^3/27$, *as*

$$\Delta_y(\hat{f}_{2/3})(x) = cx^6(-8+9x)^2(2x+1)^4(x-1)^8(3x^2-2)^3 \tag{5.5}$$

(e) $(-1/2,-5/4)$ *is a flex of order 1 of* $B_1(2/3)$ *whose tangent line is the vertical line* $x=-1/2$. *Thus* $C^a(\hat{f}_{2/3})$ *has two flexes at* $(-1/2,\pm\sqrt{5}i/2)$ *with the common tangent line* $x=-1/2$.

We have seen in section 3 that a smooth point $P \in C^a(f)$ is a flex point of order 1 if and only if d^2y/dx^2 vanishes at P. As $dy/dx = -\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y}$, this is equivalent to $\mathcal{F}(f)(P) = 0$ where

$$\mathcal{F}(f) := \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2$$

Using the algebraicity of this condition and the information from the real graph of $B_1(t)$ and $B_2(t)$ for $|t-2/3| \le \varepsilon$ with ε sufficiently small, we can show that

(f) *there exists a family of flexes* $(\alpha(t),\beta(t))$ *of* $B_1(t)$ *for* $|t-2/3| \le \varepsilon$ *(*ε*: sufficiently small) such that* $(\alpha(2/3),\beta(2/3)) = (-1/2,-5/4)$. *Similarly there exists two families of flexes* $(\gamma(t),\pm\delta(t)i)$ *of* $B_2(t)$ *such that* $(\gamma(2/3),\delta(2/3)i) = (-1/2,\sqrt{5}i/2)$.

For example, $\mathcal{F}(f_{2/3})(0,-8) > 0$ and $\mathcal{F}(f_{2/3})(-3/4,b) < 0$ where b is the real solution of $f_{2/3}(-3/4,b) = 0$. For the existence of flexes for $B_2(t)$, we consider the graph of the real curve $C^a(f')$ with $f'(x,y,t) := f(x,iy,t)$. Then $(\gamma(t),\pm\delta(t))$ are real flexes of $C^a(f')$. For later purpose, we put $A_\pm(t) = (\gamma(t),\pm\delta(t)i)$. Note that $\gamma(t) \neq \alpha(t)$ for $t \neq 2/3$. $\alpha(t),\beta(t),\gamma(t)$ and $\delta(t)$ are real numbers and we may assume $\delta(t) > 0$.

(g) *The tangent lines at* $A_+(t)$ *and* $A_-(t)$ *are not vertical for* $t \neq 2/3$.

Note that the inverse image of a flex point of $B_1(t)$ is not a flex of $B_2(t)$ if the tangent line of a flex is not vertical. Here "vertical" implies $\partial f/\partial y$ vanishes. In fact, assume that $\gamma(t) = \alpha(t)$ and the tangent line at $(\alpha(t),\beta(t))$ is vertical. This implies that $\Delta_y(f_t,y)(x)$ has the factor $(x-\gamma(t))^2$ by Example ??, 2 of Section 2. However this is impossible as we have seen in (c).

5.2. **Recipe of the construction of a curve C_3.** The generic (2,2)-fold cyclic covering $\mathcal{C}_{2,2}(B_2(t))$ of $B_2(t)$ gives a curve of degree 12 with 24 cusps with $\pi_1(\mathbf{P}^2 - \mathcal{C}_{2,2}(B_2(t))) \cong \mathbf{Z}_{12}$. See [28]. It is the purpose of this section to put three more cusps without breaking the commutativity of the fundamental group. We first take the double covering along the tangent line at the flex $A_-(t)$ and we denote the pull-back of $B_2(t)$ by $B_3(t)$. We have seen that the tangent lines at $A_\pm(t)$ degenerate at $t = 2/3$ into the same line $x = -1/2$. Thus $B_3(t)$ has 13 cusps for $t \neq 2/3$ and 14 cusps for $t = 2/3$. Let $g_t(x, y)$ be the defining polynomial of $B_3(t)$. Though $\pi_1(\mathbf{C}^2 - B_3(2/3))$ may be not abelian, we can show, using the geometry of this degeneration, that $\pi_1(\mathbf{C}^2 - B_3(t))$ is abelian for any t, provided $|t - 2/3|$ is sufficiently small and $t \neq 2/3$ (thus abelian for any generic t). We will see that flex curves of $B_3(2/3)$ disappear at infinity. Finally we take a *generic flex double covering* $\mathcal{F}^{(2)}(g_t; P)$ which is a curve of degree 12 with 27 cusps and we put $C_3 := \mathcal{F}^{(2)}(g_t; P)$. We will refer the detail of the proof to [30].

6. Alexander polynomial

6.1. **Alexander polynomial through Fox calculus.** We quickly recall the definition of Alexander polynomial of a finitely representable group G with a surjective homomorphism $\varphi : G \to \mathbf{Z}$. Let F_n be a free group with n-generators $X_1, \dots, X_n$ and let G be a finitely represented group with n-generators $x_1, \dots, x_n$ and assume that the kernel of the surjective map $\psi : F_n \to G$, defined by $\psi(X_i) = x_i$, is normally generated by $R_1, \dots, R_s \in F_n$. Let $C[F_n]$, $\mathbf{C}[G]$ and $\mathbf{C}[\mathbf{Z}]$ be the respective group rings of F_n, G and $\mathbf{Z}$ with $\mathbf{C}$ coefficients. We can identify $\mathbf{C}[\mathbf{Z}]$ with the Laurent polynomial ring $\mathbf{C}[t, t^{-1}]$, under a fixed generator $t \in \mathbf{Z}$. There are canonical homomorphisms $\psi_* : C[F_n] \to \mathbf{C}[G]$ and $\varphi_* : \mathbf{C}[G] \to \mathbf{C}[t, t^{-1}]$ which are induced by ψ and φ. The j-th Fox derivative $\partial/\partial X_j$ is a linear map $\mathbf{C}[F_n] \to \mathbf{C}[F_n]$, characterized by the following properties:

$$\frac{\partial X_i}{\partial X_j} = \delta_{ij}, \quad \frac{\partial uv}{\partial X_j} = \frac{\partial u}{\partial X_j} + u\frac{\partial v}{\partial X_j}, \quad u, v \in \mathbf{C}[F_n]$$

To (G, ψ, φ), we associate $s \times n$-matrix M of $\mathbf{C}[t, t^{-1}]$ coefficients whose (i, j)-entry is given by $a_{ij} := \varphi_*(\psi_*(\frac{\partial R_i}{\partial X_j}))$. The *Alexander polynomial* of G with respect to $\varphi : G \to \mathbf{Z}$ is defined by the greatest common divisor of $(n-1) \times (n-1)$-minors of M. This does not depend on the choice of the representation ψ and the choice of generators $R_1, \dots, R_s$ of the kernel. See [8] and [21] for further detail.

Let C be an irreducible affine curve of degree d. Then we know that $H_1(\mathbf{C}^2 - C; \mathbf{Z}) \cong \mathbf{Z}$ and the Hurewicz homomorphism together with this identification gives a surjective homomorphism

$$\xi : \pi_1(\mathbf{C}^2 - C) \to H_1(\mathbf{C}^2 - C; \mathbf{Z}) \cong \mathbf{Z}$$

We fix a generator $t \in H_1(\mathbf{C}^2 - C; \mathbf{Z})$ so that any lasso $\tau \in \pi_1(\mathbf{C}^2 - C)$ for C corresponds to t through the Hurewicz homomorphism. The Alexander polynomial of C is defined by that of $\pi_1(\mathbf{C}^2 - C)$ with respect to the Hurewicz homomorphism ξ and we denote it by $\mathcal{A}_C(t)$.

Example 6.1. 1. Let C be an irreducible affine curve with an abelian fundamental group. Then we can use the trivial representation $\psi = \mathrm{id} : F_1 \to \mathbf{Z}$. Thus $\mathcal{A}_C(t) = 1$.

2. Let Z_6 be a six cuspidal conical sextic with respect to a generic line at infinity (see [Z1]). Then the fundamental group $\pi_1(\mathbf{C}^2 - Z_6)$ has the representation $\langle x, y; xyx = yxy \rangle$ (see for example [26]). As the generators x, y are lassos for Z_6, they corresponds to t via Hurewicz homomorphism. We can take $s = 1$, $R_1 = XYXY^{-1}X^{-1}Y^{-1}$. Then $\partial R_1/\partial X$ and $\partial R_1/\partial Y$ give $\pm(t^2 - t + 1)$ respectively. Thus $\mathcal{A}_{Z_6}(t) = t^2 - t + 1$.

3. Let Z_4 be the three cuspidal quartic with respect to a generic line at infinity. Then $\pi_1(\mathbf{C}^2 - Z_4)$ is generated by two generators x, y with two relations $R_1 = xyx(yxy)^{-1}$ and

$R_2 = x^2y^{-2}$. Then $\partial R_1/\partial X$ and $\partial R_1/\partial Y$ give the same polynomial $t^2 - t + 1$. $\partial R_2/\partial X$ and $\partial R_2/\partial Y$ give the linear form $\pm(t+1)$. Thus we have $\mathcal{A}_{Z_4}(t) = 1$.

The following is an immediate consequence of Lemma 2.1.

Lemma 6.2. *Let $f(x,y)$ be a wt-convenient polynomial of type $(a,b;d)$ and let $D = \{y = 0\}$ and $f_m(x,y) := f(x,y^m)$. Let $\iota_1 : \mathbf{C}^2 - C^a(f) \cup D \to \mathbf{C}^2 - C^a(f)$ and $\iota_2 : \mathbf{C}^2 - C^a(f) \cup D \to \mathbf{C}^2 - D$ be the respective inclusion map and assume that the canonical homomorphism*

$$\iota_{1\sharp} \times \iota_{2\sharp} : \pi_1(\mathbf{C}^2 - C^a(f) \cup D) \to \pi_1(\mathbf{C}^2 - C^a(f)) \times \pi_1(\mathbf{C}^2 - D)$$

is isomorphism. Then the Alexander polynomial of $C^a(f_m)$ is equal to that of $C^a(f)$.

Applying this lemma to the generic (3,3)-covering transform $C_1 := \mathcal{C}_{3,3}(Z_4)$ of the Zariski's three cuspidal quartic Z_4 and the curve C_3 which we have constructed in the section 5, we obtain:

Corollary 6.3. *The Alexander polynomials of C_1 and C_3 coincide.*

Such a pair of non-irreducible plane curves are known by [3]

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