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Weierstrass type expression of curves of genus two and modular forms.

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Abstract

The parameter space of versal deformation of curve singularity of type A_4 has a structure of $\mathbf{C}^*$-bundle in a wider sense. The aim of this paper is to clarify the structure of the bundle by a period mapping.

1 Introduction.

For any positive integer n, zeros of a polynomial $-y^2+x^{n+1}$ in $\mathbf{C}^2$ gives a curve singularity of type A_n. And the following polynomial

$$F_{A_n}(x,y,t) := -y^2+x^{n+1}+t_2x^{n-1}+\cdots+t_nx+t_{n+1}$$

expresses versal deformation of the above singularity. That is,

$$\Xi_{A_n} := \{(x,y,t)\in \mathbf{C}^2\times\mathbf{C}^n | F_{A_n}(x,y,t)=0\}$$

is called versal deformation of curve singularity of tipe A_n. The parameter space $\mathbf{C}^n$ is denoted by S_{A_n}:

$$S_{A_n} := \mathbf{C}^n(\ni t=(t_2,\ldots,t_{n+1}))\,.$$

On Ξ_{A_n} and S_{A_n} we define a $\mathbf{C}^*$-action as

$$\lambda\cdot(x,y,t) := \begin{cases} (\lambda x, \quad \lambda^{\frac{n+1}{2}}y, \quad \lambda\cdot t) & (n:\ \text{odd}) \\ (\lambda^2 x, \quad \lambda^{n+1}y, \quad \lambda\cdot t) & (n:\ \text{even}) \end{cases}$$

where

$$\lambda\cdot t := \begin{cases} (\lambda^2 t_2, \ \ldots\ , \lambda^{n+1}t_{n+1}) & (n:\ \text{odd}) \\ (\lambda^4 t_2, \ \ldots\ , \lambda^{2n+2}t_{n+1}) & (n:\ \text{even}). \end{cases}$$

By the action, the parameter space S_{A_n} is regarded as the total space of a $\mathbf{C}^*$-bundle in a wider sense. here we think of the following problem.

Problem 1.1 *Clarify the structure of the above* $\mathbf{C}^*$*-bundle* S_{A_n}.

At the present time, only for $n=1$ and $n=2$, answer to the problem is already known. As for the case $n=1$, the problem is trivial, because the base space $\mathbf{C}^*\backslash S_{A_1}$ consists of one point. As for the case $n=2$, answer to the problem is a classical result, which we will see later (in subsection 2.4). In the present paper we think of the problem for $n=4$. Here we avoid the problem for $n=3$. If n is an odd integer and $n\geq 3$, there are some circumstances, in which, the case of A_n is rather different from that of A_2. Therefore we cannot apply the way used in solving the problem of the case $n=2$, simply, to the problem for the n. As for the problem for the n, we have no idea now. In the case $n=2$, using a period mapping and applying a well-known frame of automorphic forms, we can see that the transition functions of the bundle S_{A_2} are given as a factor of automorphy. In the following section we review the frame of automorphic forms.

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2 A frame of automorphic forms.

In this section we review a well-known frame of automorphic forms.

2.1 Equivariant group action on a trivial bundle and a factor of automorphy.

Suppose X be a complex manifold, and G be a group acting on X discontinuously. Then the following (2-1-1), (2-1-2) are equivarent.

(2-1-1) To give a factor of automorphy $j : G \times X \to \mathbf{C}^*$.

(2-1-2) To give a G-action on $\mathbf{C}^* \times X$ which satisfies the following (i),(ii).

- **(i)** The G-action is commutative to the natural $\mathbf{C}^*$-action on $\mathbf{C}^* \times X$.
- **(ii)** The G-action is equivariant to the natural projection $\mathbf{C}^* \times X \to X$.

In fact, if a factor of automorphy j is given, we can give a G-action on $\mathbf{C}^* \times X$ using j as follows:

$$\mathbf{C}^* \times X \ni (\lambda, x) \overset{\sigma}{\longmapsto} (j(\sigma, x)^{-1}\lambda, \sigma(x)) \in \mathbf{C}^* \times X \qquad (\sigma \in G)\,. \tag{1}$$

It can be easily seen that this G-action satisfy the above (i) and (ii). On the other hand, suppose that a G-action on $\mathbf{C}^* \times X$ satisfying (i) and (ii) is given. Then we define a map $j : G \times X \to \mathbf{C}^*$ by the following relation:

$$(1, x) \overset{\sigma}{\longmapsto} (j(\sigma, x)^{-1}, \sigma(x)) \qquad (\sigma \in G, x \in X)\,. \tag{2}$$

Then this j is a factor of automorphy. Those two procedures now explained are inverse to each other.

2.2 Invariant ring and ring of automorphic forms.

In general, when a group G is acting on a ring R, we denote by R^G the G-invariant subring of R. And for any complex analytic space Y, we denote by $\mathcal{O}(Y)$ the ring of all of holomorphic functions on Y.

When (2-1-1) (or, equivarently, (2-1-2)) is satisfied, there are well known relations of four rings as follows:

$$\begin{aligned} &[\text{the ring of } (G, j)\text{-automorphic forms on } X] \\ &\cong \mathcal{O}(X)[\lambda, \lambda^{-1}]^G \subseteq \mathcal{O}(\mathbf{C}^* \times X)^G \cong \mathcal{O}((\mathbf{C}^* \times X)/G)\,. \end{aligned} \tag{3}$$

It seems that in (3), only the first isomorphism is non-trivial (at least, to me). Here we explain the correspondence which gives the first isomorphism of (3). Suppose f be an element of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$. We express f as Laurent polynomial in λ:

$$f(\lambda, x) = \sum_{k \in \mathbf{Z}} \lambda^k f_k(x) \qquad (\text{ finite sum}) \tag{4}$$

where $f_k \in \mathcal{O}(X)$. From the expansion, f satisfies the equality

$$f(j(\sigma, x)^{-1}\lambda, \sigma(x)) = \sum_k j(\sigma, x)^{-k} \lambda^k f_k(\sigma(x)) \tag{5}$$

for any $\sigma \in G$. Because f is G-invariant, (1), (4) and (5) imply that

$$f_k(\sigma(x)) = j(\sigma,x)^k f_k(x) \quad (\forall \sigma \in G,\ \forall x \in X,\ \forall k \in \mathbf{Z})\,.$$

That is, f_k is a (G,j)-automorphic form of weight k. On the oter hand, for given finite set $\{f_k\}$ (where each f_k is a (G,j)-automorphic form of weight k), if we define f by (4), we can easily see that f is an element of $\mathcal{O}(X)[\lambda,\lambda^{-1}]^G$.

2.3 Our plan.

We denote by D_{A_n} the discriminant set of S_{A_n}:

$$D_{A_n} := \{t \in S_{A_n} | F_{A_n}(x,0,t) \text{ has multiple roots.}\} \tag{6}$$

We treat $S_{A_n}-D_{A_n}$ rather than S_{A_n} itself. Suppose that there exist X and G which make the left hand side of the following diagram

$$\begin{tikzcd}
S_{A_n}-D_{A_n} \arrow[d] & (\mathbf{C}^*\times X)/G \arrow[l, "\sim"'] \arrow[d] & \mathbf{C}^*\times X \arrow[l, "u"'] \arrow[d, shift right] & \ni & (1,x) \\
\mathbf{C}^*\backslash(S_{A_n}-D_{A_n}) & X/G \arrow[l, "\sim"'] & X \arrow[l] \arrow[ul, "u\circ s"] \arrow[u, shift right, "s"'] & \ni & x \arrow[u, mapsto, "s"']
\end{tikzcd}$$

Diagram-1

commutative, where u is a natural projection, and s is a global section of the trivial bundle $\mathbf{C}^* \times X \to X$ defined as in the above diagram. Then by (3), the ring $\mathbf{C}[t_2,\ldots,t_{n+1}]$ is regarded as a subring of $\mathcal{O}(X)[\lambda,\lambda^{-1}]^G$, and hence it is regarded as a subring of the ring of (G,j)-automorphic forms. Moreover, transition functions of the bundle $S_{A_n}-D_{A_n}$ is given as a factor of automorphy j. By the way, the G-actions on the total space and on the base space of the bundle $\mathbf{C}^* \times X \to X$ are equivariant to the projection. Hence, by the relation (2) the section s satisfies

$$s(\sigma(x)) = j(\sigma,x)\cdot\sigma(s(x)) \qquad (\forall\sigma \in G,\ \forall x \in X)\,.$$

Moreover, the $\mathbf{C}^*$-actions on $(\mathbf{C}^* \times X)/G$ and on $\mathbf{C}^* \times X$ are equivariant to the map u. And, in addition, u is G-invariant. Therefore, we have

$$(u\circ s)(\sigma(x)) = j(\sigma,x)\cdot(u\circ s)(x) \qquad (\forall\sigma \in G,\ \forall x \in X)\,.$$

Keeping the above frame in mind, we consider the problem 1.1 for $n=4$ according to the following plan.

(2-3-1) First we find X and G which give **Diagram-1**.

(2-3-2) Next we investigate the effect of G-action on the map $u \circ s$ to obtain a factor of automorphy j explicitly.

2.4 Example. (A_2-type curve singularity.)

As an example, we review the answer to the problem 1.1 for $n = 2$. In order to adapt the problem to the theory of Weierstrass' elliptic function, we modify the definition of F_{A_2} as follows:

$$F_{A_2} := -y^2 + 4x^3 - g_2 x - g_3 .$$

Then $S_{A_2} = \mathbf{C}^2$ and $D_{A_2} = \{g \in S_{A_2} | g_2^3 - 27g_3^2 = 0\}$. In this case, using the following multi-valued holomorphic mapping:

$$S_{A_2} - D_{A_2} \ni g \mapsto \left(\int_{A(g)} \frac{dx}{y}, \int_{B(g)} \frac{dx}{y} \Big/ \int_{A(g)} \frac{dx}{y} \right) \in \mathbf{C}^* \times \mathbf{H} , \tag{7}$$

we can apply the above frame to $S_{A_2} - D_{A_2}$, where $G = SL(2, \mathbf{Z})$ and $X = \mathbf{H}(:= \{\tau \in \mathbf{C} | \Im\tau > 0\})$. As a consequence, we obtain that $S_{A_2} - D_{A_2} \cong \mathbf{C}^* \times \mathbf{H}/SL(2, \mathbf{Z})$. Moreover, we have $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d$, and obtain the expression of g_i $(i = 2, 3)$ as (G, j)-automorphic forms, which coincide to the well-known expressions as Eisenstein series.

3 Definition of period mapping.

We denote that $S := S_{A_4}$, $\Xi := \Xi_{A_4}$, and $D := D_{A_4}$. Discriminant of the polynomial $F_{A_4}(x, 0, t) \in (\mathbf{C}[t])[x]$ is as follows:

$$\begin{aligned} \Delta(t) : = \; & 3125t_5^4 - 3750t_2t_3t_5^3 + 2000t_2t_4^2t_5^2 + 2250t_3^2t_4t_5^2 - 900t_2^3t_4t_5^2 + 825t_2^2t_3^2t_5^2 \\ & +108t_2^5t_5^2 - 1600t_3t_4^3t_5 + 560t_2^2t_3t_4^2t_5 - 630t_2t_3^3t_4t_5 - 72t_2^4t_3t_4t_5 + 108t_3^5t_5 \\ & +16t_2^3t_3^3t_5 + 256t_4^5 - 128t_2^2t_4^4 + 144t_2t_3^2t_4^3 + 16t_2^4t_4^3 - 27t_3^4t_4^2 - 4t_2^3t_3^2t_4^2 . \end{aligned}$$

By (6), we have $D = \{t \in S \mid \Delta(t) = 0\}$. In addition, π denotes the natural projection $\Xi \ni (x, y, t) \longmapsto t \in S$, and X_t denotes $\pi^{-1}(t)$. We take a point $t_0 \in S - D$. t_0 is used as a base point of the fundamental group of $S - D$. Projection $\pi : \Xi - \pi^{-1}(D) \to S - D$ has the property of local triviality. Hence $\pi_1(S - D, t_0)$ acts on $H_1(X_{t_0}, \mathbf{Z})$. Moreover, this action preserves the intersection form $\langle\ ,\ \rangle$ on $H_1(X_{t_0}, \mathbf{Z})$. Therefore we have the following representation:

$$\rho : \pi_1(S - D, t_0) \longrightarrow Aut(H_1(X_{t_0}, \mathbf{Z}), \langle\ ,\ \rangle)$$

(monodromy representation), where $Aut(H_1(X_{t_0}, \mathbf{Z}), \langle\ ,\ \rangle)$ denotes all of automorphisms of $H_1(X_{t_0}, \mathbf{Z})$ which preserve the intersection form $\langle\ ,\ \rangle$. $\Gamma := \rho(H_1(X_{t_0}, \mathbf{Z}))$ is called as monodromy group. We take a symplectic basis of $H_1(X_{t_0}, \mathbf{Z})$ as in **Figure-1**. Then by the basis, the following group isomorphism holds:

$$Aut(H_1(X_{t_0}, \mathbf{Z}), \langle\ ,\ \rangle) \cong Sp(4, \mathbf{Z}) , \tag{8}$$

and by the isomorphism, Γ is regarded as a subgroup of $Sp(4, \mathbf{Z})$. Now we define a covering space of $S - D$ as follows:

$$\widehat{S-D} := (\text{universal covering space of } S - D)/(\text{kernel of } \rho).$$

$\widehat{S-D}$ is called as monodromy covering. Natural projection $\widehat{S-D} \to S-D$ is denoted by σ. Here we can define a period mapping.

$$P : \widehat{S-D} \ni h \longmapsto \begin{pmatrix} \omega_{11}(h) & \omega_{12}(h) & \omega_{13}(h) & \omega_{14}(h) \\ \omega_{21}(h) & \omega_{22}(h) & \omega_{23}(h) & \omega_{24}(h) \end{pmatrix} \in \mathbf{C}^{2\times 4},$$

$$\omega_{ij}(h) := \int_{A_j(h)} \frac{x^{i-1}dx}{y},$$

where $A_1(h)$, $A_2(h)$, $A_3(h) = B_1(h)$, $A_4(h) = B_2(h)$ are symplectic basis of $H_1(X_{\sigma(h)}, \mathbf{Z})$ and depend on h "continuously". That is, each $A_j(h)$ is a local system. Note that, On some $h_0 \in \sigma^{-1}(t_0)$, we take $A_j(h_0)$ $(j = 1, 2, 3, 4)$ as in the **Figure-1**.

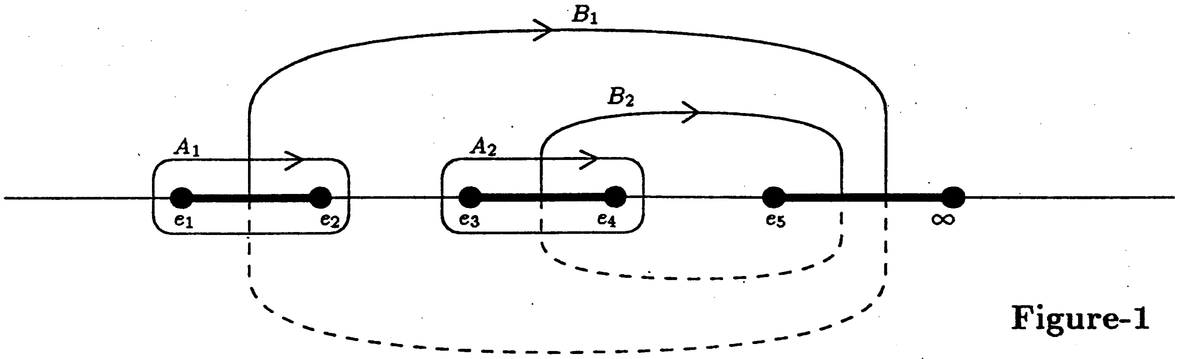

Figure-1

Remark. Each $A_j(t)$ is multi-valued on $S - D$. But, on $\widehat{S-D}$, each $A_j(t)$ is single-valued. In fact $\widehat{S-D}$ is the minimal covering on which each $A_j(t)$ is single-valued. Therefore the above period map P is single-valued.

By the definition of P, each $P(h)$ $(h \in \widehat{S-D})$ is a 2×4 matrix. We define a map φ as

$$\varphi : Image(P) \ni (\Omega_A \ \Omega_B) \longmapsto (\Omega_A^{-1}\Omega_B) \in \mathbf{H}_2$$

where Ω_A, Ω_B denote the left 2×2 part, the right 2×2 part of the 2×4 matrix $P(h)$, respectively, and $\mathbf{H}_2$ denotes the Siegel upper half space of genus two.

4 Results(1) —— towards (2-3-1).

The $\mathbf{C}^*$-action on $S - D$ can be lifted to an action on $\widehat{S-D}$, where the $\mathbf{C}^*$-action is fixed point free. Hence it can be easily seen that $\widehat{S-D}$ is regarded as the total space of a $\mathbf{C}^*$-bundle in the strict sense. By the way, in the Problem for n=2, we can see that $\widehat{S_{A_2}-D_{A_2}}$ is isomorphic to the trivial bundle $\mathbf{C}^* \times \mathbf{H}$ via the period mapping (7). As for the Problem for n=4, we obtained the following theorem.

Theorem 4.1 ([4]) *The above period mapping P gives an isomorphism*

$$\widehat{S-D} \cong \mathbf{C}^* \times \mathbf{H}_2^0 \tag{9}$$

as $\mathbf{C}^*$-bundles, where

$$\mathbf{H}_2^0 := \mathbf{H}_2 - A, \qquad A := \{M \circ \tau | M \in Sp(4, \mathbf{Z}), \quad \tau \in \mathbf{H}_2, \ \tau \text{ is a diagonal matrix}\}. \tag{10}$$

(Definition of $M \circ \tau$ is given in subsection 5.2.)

(Outline of the proof.) To prove the theorem we give a global section of the bundle $\widehat{S-D} \to \mathbf{H}_2^0$ by using Rosenhain's formula [7]. The formula says that, in our situation, the following equalities

$$\frac{e_3-e_1}{e_2-e_1}=\frac{\vartheta_{0000}^2\vartheta_{0100}^2}{\vartheta_{1000}^2\vartheta_{1100}^2},\qquad \frac{e_4-e_1}{e_2-e_1}=\frac{\vartheta_{0100}^2\vartheta_{0001}^2}{\vartheta_{1100}^2\vartheta_{1001}^2},\qquad \frac{e_5-e_1}{e_2-e_1}=\frac{\vartheta_{0000}^2\vartheta_{0001}^2}{\vartheta_{1000}^2\vartheta_{1001}^2} \tag{11}$$

hold, where

- $e_1,\ldots,e_5$ are five roots of $F(x,0,\sigma(h))$,
- τ is a period matrix, which is obtained from $X_{\sigma(h)}$ with a basis $A_j(\sigma(h))$ $\{j = 1,2,3,4\}$ of $H_1(X_{\sigma(h)},\mathbf{Z})$,

and for any $\varepsilon = (\varepsilon'\varepsilon'') = (\varepsilon_1'\ldots\varepsilon_g'\varepsilon_1''\ldots\varepsilon_g'') \in \mathbf{Z}^{2g}$ and $\tau \in \mathbf{H}_g$,

$$\vartheta_\varepsilon = \vartheta_\varepsilon(\tau) := \sum_{n\in\mathbf{Z}^g} \exp\left[\pi i(n+\frac{\varepsilon'}{2})\tau{}^t(n+\frac{\varepsilon'}{2}) + 2\pi i(n+\frac{\varepsilon'}{2})\frac{{}^t\varepsilon''}{2}\right] \tag{12}$$

are theta constants of genus g, where $\mathbf{H}_g$ denotes Siegel upper half space of genus g. Here we use only theta constants of genus two.

In our situation, the equality $e_1+\cdots+e_5 = 0$ holds. Therefore, by using some formulas of theta constants, (11) imply that the following equality holds,

$$(e_1,\ldots,e_5) = (\lambda^2\alpha_1,\ldots,\lambda^2\alpha_5)$$

for some $\lambda \in \mathbf{C}^*$, where

$$\begin{aligned}
\alpha_1 &:= \frac{1}{5}(-\vartheta_{1000}^2\vartheta_{1100}^2\vartheta_{1001}^2 - \vartheta_{0000}^2\vartheta_{0100}^2\vartheta_{1001}^2 - \vartheta_{1000}^2\vartheta_{0100}^2\vartheta_{0001}^2 - \vartheta_{0000}^2\vartheta_{1100}^2\vartheta_{0001}^2)\,,\\
\alpha_2 &:= \frac{1}{5}(+\vartheta_{1000}^2\vartheta_{1100}^2\vartheta_{1001}^2 - \vartheta_{0010}^2\vartheta_{0110}^2\vartheta_{1001}^2 - \vartheta_{0011}^2\vartheta_{0110}^2\vartheta_{1000}^2 - \vartheta_{0011}^2\vartheta_{0010}^2\vartheta_{1100}^2)\,,\\
\alpha_3 &:= \frac{1}{5}(+\vartheta_{0000}^2\vartheta_{0100}^2\vartheta_{1001}^2 + \vartheta_{0010}^2\vartheta_{0110}^2\vartheta_{1001}^2 - \vartheta_{0110}^2\vartheta_{1111}^2\vartheta_{0100}^2 - \vartheta_{0010}^2\vartheta_{1111}^2\vartheta_{0000}^2)\,,\\
\alpha_4 &:= \frac{1}{5}(+\vartheta_{1000}^2\vartheta_{0100}^2\vartheta_{0001}^2 + \vartheta_{0011}^2\vartheta_{0110}^2\vartheta_{1000}^2 + \vartheta_{0110}^2\vartheta_{1111}^2\vartheta_{0100}^2 - \vartheta_{0011}^2\vartheta_{1111}^2\vartheta_{0001}^2)\,,\\
\alpha_5 &:= \frac{1}{5}(+\vartheta_{0000}^2\vartheta_{1100}^2\vartheta_{0001}^2 + \vartheta_{0011}^2\vartheta_{0010}^2\vartheta_{1100}^2 + \vartheta_{0010}^2\vartheta_{1111}^2\vartheta_{0000}^2 + \vartheta_{0011}^2\vartheta_{1111}^2\vartheta_{0001}^2)\,.
\end{aligned}$$

Using those functions, we define a map $F : \mathbf{H}_2^0 \ni \tau \mapsto (t_2,\ldots,t_5) \in S-D$ by

$$t_i = (-1)^i \sum_{1\le\nu_1<\cdots<\nu_i\le 5} \alpha_{\nu_1}\cdots\alpha_{\nu_5} \qquad (i=2,3,4,5)\,. \tag{13}$$

Then, the map F is "lifted" to a map $\hat{F} : \mathbf{H}_2^0 \to \widehat{S-D}$ and $\hat{F}$ is a global section of the bundle $\widehat{S-D} \to \mathbf{H}_2^0$. Therefore, the isomorphism (9) is obtained. □

By the theorem, we can take $\mathbf{H}_2^0$ as X and Γ as G in the **Diagram-1.**

5 Results(2) —— towards (2-3-2).

5.1 Preliminary.

In the previous section, we used $\hat{F}$ as a global section of the bundle $\widehat{S-D} \to \mathbf{H}_2^0$ to give isomorphism (9). As a result, in our trivialization, we obtain a factor of automorphy $j : \Gamma \times \mathbf{H}_2^0 \to \mathbf{C}^*$. From now on, we obtain j explicitly by clarifying factors of automorphy of t_2, t_3, t_4, t_5 (or $\alpha_1, ..., \alpha_5$) under the action of Γ, where we regard t_i as functions on $\mathbf{H}_2^0$ by the equality (13). Here we note the following three points.

(5-1-1) Each t_i (or α_i) is a homogeneous polynomial of theta constants.

(5-1-2) Γ is regarded as a subgroup of $Sp(4, \mathbf{Z})$ by the isomorphism (8).

(5-1-3) The transformation formula of theta constants under the action of the full modular group is well-known. (cf. [6])

So we investigate the effects of Γ-action on $\alpha_1, ..., \alpha_5$.

5.2 Transformation formula of theta constants.

Following to [6], here we give short review of the transformation formula of theta constants defined in (12). It is well-known that, for $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, \mathbf{Z})$, $\tau \in \mathbf{H}_g$, $\varepsilon = (\varepsilon' \varepsilon'') \in \mathbf{Z}^{2g}$, the following equality holds:

$$\vartheta_{M\circ\varepsilon}(M \circ \tau) = \kappa(M)\exp(\pi i \phi(M, \varepsilon))\sqrt{\det(C\tau + D)}\vartheta_\varepsilon(\tau) , \tag{14}$$

where

$$M \circ \tau := (A\tau + B)(C\tau + D)^{-1}, \qquad M \circ \varepsilon := \varepsilon M^{-1} + ((C^tD)_0(A^tB)_0),$$
$$\phi(M, \varepsilon) := \frac{1}{4}\{-\varepsilon'^tDB^t\varepsilon' + 2\varepsilon''^tCB^t\varepsilon' - \varepsilon''^tCA^t\varepsilon'' + 2(\varepsilon'^tD - \varepsilon''^tC)^t(A^tB)_0\}$$

where for $g \times g$ matrix $X = (x_{ij})$, we write $X_0 := (x_{11}, x_{22}, ..., x_{gg})$. In (14), $\kappa(M)$ is a constant, which depends on M and is independent to ε and τ. Moreover, It is well known that $\kappa(M)^8 = 1$ for any $M \in Sp(2g, \mathbf{Z})$. As for $\kappa(M)$, more various properties are known.

5.3 Definition of a group Γ'.

For any two positive integers g and n, the principal congruence subgroup of level n, genus g is defined as follows:

$$\Gamma_g(n) := \{M \in Sp(2g, \mathbf{Z}) | M \equiv I_{2g} \bmod n\} .$$

(Note that $\Gamma_g(1) = Sp(2g, \mathbf{Z})$.) It is well known that $\Gamma_2(1)/\Gamma_2(2)$ is isomorphic to the 6-th Symmetric group S_6. (See, for example, [3].) Usually, the isomorphism is given by the action of $\Gamma_2(1)$ over the following six odd theta characteristics of genus two:

$$\langle 1\rangle := (0101),\ \langle 2\rangle := (0111),\ \langle 3\rangle := (1011),\ \langle 4\rangle := (1010),\ \langle 5\rangle := (1110),\ \langle 6\rangle := (1101). \tag{15}$$

Here we give an isomorphism explicitly.

For any $M \in \Gamma_2(1)$, the following map:

$$\varepsilon \bmod (2\mathbf{Z})^4 \longmapsto M \circ \varepsilon \bmod (2\mathbf{Z})^4 \tag{16}$$

gives rise to a permutation of the six elements in (15). Hence for any $M \in \Gamma_2(1)$ and for any $i \in \{1,\ldots,6\}$, there is a $M(i) \in \{1,\ldots,6\}$ such that

$$\langle M(i)\rangle \equiv M \circ \langle i\rangle \bmod (2\mathbf{Z})^4 .$$

Then the map $i \mapsto M(i)$ is a permutation of $\{1,\ldots,6\}$. Terefore we have a group homomorphism $\Gamma_2(1) \to S_6$. It can be easily seen that the homomorphism is surjective, and that its kernel is $\Gamma_2(2)$. So we obtain an isomorphism $\Gamma_2(1)/\Gamma_2(2) \cong S_6$.

Here we treat the following subgroup Γ':

Definition 5.1 $\Gamma' := \{M \in \Gamma_2(1) | M \circ (1101) \equiv (1101) \bmod (2\mathbf{Z})^4\}$.

This subgroup Γ' has the following property.

$$\Gamma_2(2) \subset \Gamma' \subset \Gamma_2(1) \qquad \text{and} \qquad \Gamma'/\Gamma_2(2) \cong S_5 .$$

The following lemma brings the above subgroup Γ' to our notice.

Lemma 5.2 (A'Campo [1]) $\Gamma' = \Gamma$.

5.4 Factors of automorphy of α_i.

We denote $\chi(M) := \kappa(M)^2 \exp[2\pi i\phi(M,(1101))]$ for any $M \in \Gamma_2(1)$. We can easily obtain the following obvious lemma by using transformation formula of theta constants.

Lemma 5.3 $\Gamma' \ni M \mapsto \chi(M) \in \mathbf{C}^*$ *is a group homomorphism.*

Moreover, the transformation formula (14) implies the following lemma.

Lemma 5.4 ([4]) *For any* $i \in \{1,2,3,4,5\}$, *for any* $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \Gamma$, *and for any* $\tau \in \mathbf{H}_2$, *the following equality holds.*

$$\alpha_{M(i)}(M \circ \tau) = \chi(M) \det(C\tau + D)^3 \alpha_i(\tau) .$$

By the lemma, consequently, we obtain the following theorem.

Theorem 5.5 ([4]) *Under the trivialization of* $\widehat{S-D}$ *by* $\hat{F}$, *we conclude that*

$$j(M,\tau)^2 = \chi(M) \det(C\tau + D)^3 \qquad (\forall M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \Gamma, \tau \in \mathbf{H}_2) .$$

6 Comparison with the ring of Siegel modular forms.

In the frame given in section 2, if we take $X = \mathbf{H}_2$, $G = \Gamma_2(1)$ and $j(\begin{pmatrix} D & C \\ B & A \end{pmatrix}, \tau) := \det(C\tau + D)$, then the ring of (G, j)-automorphic forms on X is the ring of ordinary Siegel modular forms of genus two. Igusa showed (in [2], [3]) that the ring is

$$\mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}] \qquad \text{(indices denote weights)} \tag{17}$$

where $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ are algebraically independent over $\mathbf{C}$, and $\chi_{35}^2 \in \mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}]$. On the other hand, the ring which we pay attention to is

$$\mathbf{C}[t_2, t_3, t_4, t_5] \qquad \text{(indices denote half of weights)}$$

where t_2, t_3, t_4, t_5 are algebraically independent over $\mathbf{C}$. Therefore, though we have not yet found conditions which determine the ring $\mathbf{C}[t_2, t_3, t_4, t_5]$ in the ring of (Γ, j)-automorphic forms, we can already see that algebraic structures of our ring $\mathbf{C}[t_2, t_3, t_4, t_5]$ is different from that of the ring (17).

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