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Compact and reductive subgroups of the group of holomorphic automorphisms of $\mathbb{C}^n$

FRANK KUTZSCHEBAUCH
Mathematisches Institut, Rheinsprung 21
4051 Basel, Switzerland
kutzsche@math.unibas.ch

1. INTRODUCTION

This article is centered around the following question about complex reductive subgroups of the group $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ of holomorphic automorphisms of $\mathbb{C}^n$:

Holomorphic Linearization Problem. *Let $G \hookrightarrow \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ be a complex reductive subgroup of the group $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ of holomorphic automorphisms of $\mathbb{C}^n$. Can one conjugate this subgroup by a single automorphism into the general linear group $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$, i.e., is every action of a complex reductive group on $\mathbb{C}^n$ linearizable?*

It is natural to ask the same question for compact groups. These two questions are strongly related since complex reductive groups are exactly those groups G which are the universal complexifications of its maximal compact subgroup K, i.e., $G = K^{\mathbb{C}}$. Recall that K is a totally real submanifold of half real dimension in $G = K^{\mathbb{C}}$ hence K is an identity set in G for holomorphic maps. So if we have a non-linearizable holomorphic action $\phi : G \times \mathbb{C}^n \to \mathbb{C}^n$ of a complex reductive group $G = K^{\mathbb{C}}$ on $\mathbb{C}^n$, then by restricting that action $\phi|_K : K \times \mathbb{C}^n \to \mathbb{C}^n$ to the maximal compact subgroup K of G we get a non-linearizable K-action. Indeed if $\phi|_K$ would be conjugate by an automorphism to a linear K-representation, then the same automorphism conjugates α to the corresponding linear representation of $K^{\mathbb{C}}$. The aim of the first section is to explain the equivalence of the holomorphic linearization questions for compact and complex reductive groups. In fact we prove that the group $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ is a complex group in the following sense:

Let H be a real Lie group acting by holomorphic automorphisms on $\mathbb{C}^n$. Then the action extends to a holomorphic action of the universal complexification $H^{\mathbb{C}}$ on $\mathbb{C}^n$.

Also we give two interesting consequences of this extension result. The first is that some real Lie groups can not act effectively on $\mathbb{C}^n$ by holomorphic transformations and the second is concerning Fatou-Bieberbach domains in $\mathbb{C}^2$ invariant under a complete holomorphic vectorfield.

In the second section we give a short overview over the positive results known about holomorphic linearization and in the third section we explain the method to construct counterexamples to the holomorphic linearization problem found by DERKSEN and the author.

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2. Complexifying an action of a real Lie group on $\mathbb{C}^n$

We start with some notation.

We say that a real Lie group G acts on a complex space X by holomorphic transformations if the action map $\phi : G \times X \to X$ is real analytic and for all $g \in G$ the map $\phi(g, \cdot) : X \to X$ is holomorphic (in fact a holomorphic automorphism). If G is a complex Lie group and the map ϕ is holomorphic we say that G acts holomorphically on X.

Remark 2.1. Every continous action of a real Lie group on a complex space by holomorphic automorphisms is already real analytic ([2] 1.6.). So it would be sufficient to require only continouity of the action map $\phi : G \times X \to X$ in the first part of the above definition.

If X is a complex manifold the compact-open topology makes the group of holomorphic automorphisms $\mathrm{Aut_{hol}}(X)$ of X a topological group. The action map $\phi : G \times X \to X$ is continous iff the corresponding group homomorphism of G into $\mathrm{Aut_{hol}}(X)$ is continous. By the above remark we see that an action of a real Lie group G on $\mathbb{C}^n$ by holomorphic transformations is the same as a continous group homomorphism $\alpha : G \to \mathrm{Aut_{hol}}(\mathbb{C}^n)$. An effective action of G on $\mathbb{C}^n$ is the same as a subgroup of $\mathrm{Aut_{hol}}(\mathbb{C}^n)$ which is as a topological group isomorphic to G.

Before we formulate the extension result we recall the definition of the universal complexification of a real Lie group.

Definition. A complex Lie group $G^{\mathbb{C}}$ together with a continous (hence real-analytic) homomorphism $i : G \to G^{\mathbb{C}}$ is called the universal complexification of a real Lie group G if it has the following property: For any continous group homomorphism $\psi : G \to H$ of G into a complex Lie group H there exists a unique complex Lie group homomorphism $\psi^{\mathbb{C}} : G^{\mathbb{C}} \to H$ such that the diagram:

$$\begin{tikzcd} G \arrow[rr, "\psi"] \arrow[dr, "i"'] & & H \\ & G^{\mathbb{C}} \arrow[ur, "\psi^{\mathbb{C}}"'] & \end{tikzcd}$$

commutes.

The construction for $G^{\mathbb{C}}$ can be found in [13] in the case where G is connected. If the group G is not connected the universal complexification can be easily constructed from the universal complexification of the component of the identity of G (see the proof of proposition 2.3). We remark that in general the map $i : G \to G^{\mathbb{C}}$ is not injective. For the universal covering group of $SL_2(\mathbb{R})$ the universal complexification equals $SL_2(\mathbb{C})$ and the kernel of $i : \widetilde{SL_2}(\mathbb{R}) \to SL_2(\mathbb{C})$ is isomorphic to $\mathbb{Z}$. There are also examples where the kernel is of positive dimension (see [14]). We call a real Lie group *extendable*, if the homomorphism $i : G \to G^{\mathbb{C}}$ is injective. Linear groups and in particular compact groups are extendable.

Definition. We say that a holomorphic action $\phi^{\mathbb{C}} : G^{\mathbb{C}} \times X \to X$ of the universal complexification of a real Lie group G on a complex space X is the extension of an action $\phi : G \times X \to X$ of G on X by holomorphic transformations if

$$\phi^{\mathbb{C}}(i(g), x) = \phi(g, x) \quad \forall\, g \in G, x \in X.$$

2. Complexifying an action of a real Lie group on $\mathbb{C}^n$

The main point for our extension result is that one can complexify every one parameter subgroup of G, i.e. an action of the additive group $(\mathbb{R}, +)$. This result is due to FORSTNERIC [6]. Recall that a complex space X is called Liouville if every bounded from above plurisubharmonic function on X is constant.

Theorem 2.2. *If $\phi : \mathbb{R} \times X \to X$ is an action of $(\mathbb{R}, +)$ on a complex manifold X which is Stein and Liouville, then ϕ extends to a holomorphic action of $(\mathbb{C}, +)$ on X.*

PROOF. We sketch only the main ideas of the proof. For a detailed proof we refer to [6]. By ξ_ϕ we denote the globally integrable (real) vectorfield induced by ϕ, i.e.,

$$\xi_\phi(x) = \frac{d}{dt}|_{t=0}\phi(t, x)$$

and by J we denote the almost complex structure induced from the complex structure on X.

The global $\mathbb{R}$ action induces a local $\mathbb{C}$-action on X which can be constructed as follows: Fix a point $x \in X$ and extend the real-analytic map $\phi_x : \mathbb{R} \to X, \quad t \mapsto \phi(t, x)$ by analytic continuation to a holomorphic map $\tilde{\phi}_x : R_x \to X$ from some maximal Riemann domain R_x over $\mathbb{C}$ into X. One shows that R_x is schlicht, i.e., a domain in $\mathbb{C}$. Since the vectorfields ξ_ϕ and $J\xi_\phi$ commute R_x is an $\mathbb{R}$-invariant strip in $\mathbb{C}$ containing $\mathbb{R}$. So it is of the form $R_x = \{\lambda \in \mathbb{C} \ : \ -b(x) < Im(\lambda) < a(x)\}$, where $a(x), b(x) > 0$. Clearly $(-b(x), a(x))$ is the maximal interval for which the vectorfield $J\xi_\phi$ can be integrated starting from the point x. The subset $\Omega := \{(\lambda, x) \in \mathbb{C} \times X \ : \ \lambda \in R_x\}$ is just the saturation $p^{-1}(p(0 \times X))$ of X under the map $p : \mathbb{C} \times X \to X^*$ to the leaf space X^* of the local $\mathbb{C}$-action (for definition see [22], [10]). Hence it is open in X. This implies that $x \mapsto -a(x)$ and $x \mapsto -b(x)$ are lower semicontinous functions on X.

The most important point is to show that if X is Stein, then the functions $-a$ and $-b$ are plurisubharmonic. If one of the functions $-a$ or $-b$ is not plurisubharmonic one can find a Hartogs figure $H : \Delta \times \Delta_n \hookrightarrow \mathbb{C} \times X$ not entirely contained in Ω such that

$$H(\{(w_1, \tilde{w}) \in \Delta \times \Delta_n \ : \ |w_1| < \frac{1}{2} \ \text{ or } \ |\tilde{w}| > \frac{1}{2}\})$$

is contained in Ω thus proving that every holomorphic function defined on Ω can be extended to the bigger domain $\Omega \cup H(\Delta \times \Delta_n)$. Since a Stein manifold admits a closed holomorphic embedding into some linear space $\mathbb{C}^N$ the extension also holds for maps into Stein manifolds. In particular the local $\mathbb{C}$-action $\tilde{\phi} : \Omega \to X \quad (\lambda, x) \mapsto \tilde{\phi}_x(\lambda)$ could be extended contradicting the maximality in the definition of Ω.

Since in addition X is Liouville the plurisubharmonic functions $-a$ and $-b$ which are bounded from above by 0 are constant on X. This means that the vectorfield $J\xi_\phi$ can be integrated in both negative and positive directions for a fixed time independend on the starting point $x \in X$. This implies that it is globally integrable hence $-a$ and $-b$ are identically $-\infty$. $\square$

Now we are able to prove the extension result

Proposition 2.3. *Let G be a real Lie group acting by holomorphic transformations on a Stein manifold X which is Liouville, then the action extends to a holomorphic action of the universal complexification $G^{\mathbb{C}}$ on X.*

PROOF. Let $\mathfrak{g}$ denote the Lie algebra of G. First suppose G is connected. We shortly recall the construction of $G^{\mathbb{C}}$ in that case:

Take the universal covering group $\pi : \tilde{G} \to G$ and denote the kernel of π by Γ. Let $\tilde{G}^{\mathbb{C}}$ be the unique simply connected complex Lie group corresponding to the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $i_1 : \tilde{G} \to \tilde{G}^{\mathbb{C}}$ be the Lie group homomorphism corresponding to the inclusion homomorphism $\mathfrak{g} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$. Let H in $\tilde{G}^{\mathbb{C}}$ be the minimal complex normal subgroup containing $i_1(\Gamma)$. Then $G^{\mathbb{C}} = \tilde{G}^{\mathbb{C}}/H$ and $i : G \to G^{\mathbb{C}}$ is the group homomorphism determined by i_1. For details we refer to [13]. In the rest of the proof we will use the notation introduced above.

From the G-action we get an infinitesimal G-transformation group, i.e., a Lie algebra homomorphism $\alpha : \mathfrak{g} \to V(X)$ from $\mathfrak{g}$ into the Lie algebra of holomorphic vector fields $V(X)$ on X by differentiating the action:

$$\alpha(v)(x) := \frac{d}{dt}|_{t=0}\phi(\exp tv, x) \quad v \in \mathfrak{g}, x \in X.$$

Here holomorphic vectorfield means a real vectorfield inducing holomorphic transformations on X ($V \mapsto V - iJV$ gives an isomorphism between the holomorphic vectorfields and the holomorphic sections of the complex tangent bundle $T^{(1,0)}(X)$ of X). Clearly all the vectorfields $\alpha(v)$, $v \in \mathfrak{g}$ are globally integrable.

We define a complex Lie algebra homomorphism from the complexified Lie algebra $\alpha^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to V(X)$ into the holomorphic vectorfields on X by $\alpha^{\mathbb{C}}(v+iw) = \alpha(v) + J\alpha(w)$. By theorem 2.2 all vectorfields $J\alpha(v)$, $v \in \mathfrak{g}$ are globally integrable. Together with the globally integrable vectorfields $\alpha(v)$, $v \in \mathfrak{g}$ they form a spanning set (as a vector space) of the real Lie algebra underlying $\mathfrak{g}^{\mathbb{C}}$. By a theorem of PALAIS (theorem III of ch. 4 in [22]) all vectorfields in the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ are globally integrable and they integrate to a (real analytic, not necessarily effective) action $\phi_1^{\mathbb{C}} : \tilde{G}^{\mathbb{C}} \times X \to X$ of the simply connected complex Lie group $\tilde{G}^{\mathbb{C}}$ corresponding to $\mathfrak{g}^{\mathbb{C}}$. Clearly this action is holomorphic ([2] p.25). We have

$$\phi_1^{\mathbb{C}}(i_1(g), x) = \phi_1(g, x), \quad x \in X, g \in G \tag{$*$}$$

since these two G-actions have the same infinitesimal generator α. This implies that $i_1(\Gamma)$ is contained in the ineffectivity I of the action $\phi_1^{\mathbb{C}}$. The ineffecticity is a normal subgroup of $\tilde{G}^{\mathbb{C}}$ and it is also a complex Lie subgroup ([2] 1.7. Prop.1). Hence $I < H$ and the $\tilde{G}^{\mathbb{C}}$-action $\phi_1^{\mathbb{C}}$ factors to a $G^{\mathbb{C}}$-action $\phi^{\mathbb{C}}$. This is the desired extension, the property $\phi^{\mathbb{C}}(i(g), x) = \phi(g, x)$, $\quad g \in G, x \in X$ follows from $(*)$ and the definition of i. This proves the theorem for connected groups.

If the Lie group G is not connected one easily constructs the extension $\phi^{\mathbb{C}}$ from an extension $\phi_0^{\mathbb{C}} : G_0^{\mathbb{C}} \times X \to X$ of the restricted action $\phi_1 = \phi|_{G_0 \times X} : G_0 \times X \to X$. Here G_0 is the component of the identity of G. For that write G in the form $G \cong G \times_{G_0} G_0$. The multiplication on the right hand side is given by $[g, h] \cdot [g_1, h_1] = [gg_1, g_1^{-1}hg_1h_1]$. Now $G^{\mathbb{C}} := G \times_{G_0} G_0^{\mathbb{C}}$ is the universal complexification of G where the group multiplication is given by $[g, h] \cdot [g_1, h_1] = [gg_1, \beta_{g_1}(h)h_1]$. Here β_{g_1} is the unique Lie group automorphism of $G_0^{\mathbb{C}}$ determined by the Lie group automorphism $h \mapsto g_1^{-1}hg_1$ of G_0. It is straightforward to check that $\phi^{\mathbb{C}}([g, h], x) := \phi(g, \phi_0^{\mathbb{C}}(h, x))$ defines an extension of ϕ. □

Corollary 2.4. *If a real Lie group G is not extendable, then it can not act effectively on $\mathbb{C}^n$ by holomorphic transformations. For instance there is no subgroup of* $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ *isomorphic as a topological group to* $\widetilde{SL_2}(\mathbb{R})$.

PROOF. First of all note that $\mathbb{C}^n$ is Stein and Liouville (see for example [21] 1.38). If G acts on $\mathbb{C}^n$ by holomorphic transformations by proposition 2.3 the action extends to a holomorphic action of $G^{\mathbb{C}}$. Hence the kernel of $i: G \to G^{\mathbb{C}}$ is contained in the ineffectivity of the G-action. The second statement is only a reformulation of the first part and uses the fact that $\widetilde{SL_2(\mathbb{R})}$ is not extendable. □

REMARK 2.5. In the case where the Lie group G is compact HEINZNER proved the existence of a universal complexification $X^{\mathbb{C}}$ of a Stein space X, i.e., one embedds X holomorphically and G-equivariantly as an open subset into some $G^{\mathbb{C}}$ space $X^{\mathbb{C}}$ so that every holomorphic G-map from X to some complex $G^{\mathbb{C}}$-space Y extends uniquely to a holomorphic $G^{\mathbb{C}}$-map from $X^{\mathbb{C}}$ into Y (see [9], [10]). Proposition 2.3 implies that if X in addition is Liouville, then one doesn't have to enlarge X, i.e. $X = X^{\mathbb{C}}$.

Corollary 2.6. *The holomorphic linearization questions for compact and complex reductive groups are equivalent.*

PROOF. Suppose every holomorphic action of a complex reductive group on $\mathbb{C}^n$ is linearizable. Given an action of a compact group K by holomorphic transformations on $\mathbb{C}^n$ we can extend this action by proposition 2.3 to a holomorphic action of the complex reductive group $K^{\mathbb{C}}$ on $\mathbb{C}$. By assumption this action is conjugate by an automorphism to a linear action. The same automorphism clearly conjugates the K-action to the restriction to K of the linear $K^{\mathbb{C}}$-action. The other direction of the equivalence is clear (see Introduction). □

For the last corollary of our extension result recall that a Fatou-Bieberbach domain is a proper open subset $\Omega \subset \mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$. It seems to be unknown whether there exists a Fatou-Bieberbach domain in $\mathbb{C}^2$ which is invariant under the flow of a globally integrable holomorphic vectorfield, i.e., under a $(\mathbb{R},+)$-action by holomorphic transformations on $\mathbb{C}^2$ ([7]). We prove that it doesn't exist in the special case of a holomorphic vectorfield with periodic flow, i.e., of an action of the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$:

Corollary 2.7. *Let Ω be a domain in $\mathbb{C}^2$ which is biholomorphic to $\mathbb{C}^2$ and invariant under an S^1-action by holomorphic transformations on $\mathbb{C}^2$. Then $\Omega = \mathbb{C}^2$.*

PROOF. By proposition 2.3 the S^1-action extends to a holomorphic action of $S^{1\mathbb{C}} = \mathbb{C}^\star$. A result of SUZUKI [25] states that $\mathbb{C}^\star$-actions on $\mathbb{C}^2$ are linearizable. So we can assume after a holomorphic change of coordinates that the $\mathbb{C}^\star$-action on $\mathbb{C}^2$ is of the form

$$\lambda, (z, w) \mapsto \lambda^a \cdot z, \lambda^b \cdot w, \quad \lambda \in \mathbb{C}^\star, \ (z, w) \in \mathbb{C}^2$$

for some integers a and b. Applying proposition 2.3 to $\Omega \cong \mathbb{C}^2$ we see that Ω is not only S^1-invariant but also $\mathbb{C}^\star$-invariant. We consider the following three cases:

Case 1: a and b are both nonzero and have the same sign

The action on $\mathbb{C}^2$ has exactly one fixed point, the point $(0,0)$. Since a differentiable S^1-action on R^n always admits a fixed point (see for instance [4] IV 1.5) we conclude that $(0,0)$ is contained in Ω. Since Ω is open it contains an open neighborhood U of $(0,0)$. Since every $\mathbb{C}^\star$-orbit in $\mathbb{C}^2$ has $(0,0)$ as limit point and Ω is $\mathbb{C}^\star$ invariant we have $\mathbb{C}^2 = \mathbb{C}^\star \cdot U \subset \Omega$.

Case 2: one of the numbers a or b is zero (say $a = 0$, hence $b \neq 0$)

Like in case 1 we conclude that Ω containes a neighborhood of some fixed point $(z_0, 0)$. Linearization at a fixed point shows that the actions on Ω and $\mathbb{C}^2$ are locally the same. Since the $\mathbb{C}^\star$-action on $\Omega \cong \mathbb{C}^2$ is linearizable too, in some coordinates on $\Omega \cong \mathbb{C}^2$ it is of the same form like that on $\mathbb{C}^2$ (see also remark 3.2 below). In particular the fixed point set is biholomorphic to $\mathbb{C}$. Since an injective holomorphic map from $\mathbb{C}$ to $\mathbb{C}$ is an biholomorphism we conclude that Ω contains the fixed point set $\{(z,0) \in \mathbb{C}^2 : z \in \mathbb{C}\}$ together with some open neighborhood U. Again we have $\mathbb{C}^\star \cdot U = \mathbb{C}^2$ and the desired conclusion like in case 1.

Case 3: a and b are both nonzero and have different signs

Consider the $\mathbb{C}^\star$-invariant map $\pi : \mathbb{C}^2 \to \mathbb{C}$ $(z,w) \mapsto z^{|b|} \cdot w^{|a|}$. The fibres of this map are the closed $\mathbb{C}^\star$-orbits and the cross of axis $\{zw = 0\}$. The composition of the injection $i : \Omega \hookrightarrow \mathbb{C}^2$ with π is by the big Picard theorem surjective or leaves out one point. Like in the other cases we know that Ω contains an neighborhood of the fixed point $(0,0)$ hence the cross of axes. If $\pi \circ i$ is surjective Ω contains points from each $\mathbb{C}^\star$-orbit in $\mathbb{C}^2$ so we are done. If the map $\pi \circ i$ leaves out one point we conclude $\Omega = \mathbb{C}^2 \setminus \{\text{one closed } \mathbb{C}^\star - \text{orbit}\}$. But this contradicts the simply-connectedness of Ω. $\square$

REMARK 2.8. For $n \geq 3$ there are clearly Fatou-Bieberbach domains in $\mathbb{C}^n$ invariant under an action of $(\mathbb{R}, +)$. Just take any Fatou-Bieberbach domain Ω_1 in $\mathbb{C}^2$ and consider the domain $\Omega := \Omega_1 \times \mathbb{C}^{n-2}$ in $\mathbb{C}^n$. Clearly Ω is a Fatou-Bieberbach domain in $\mathbb{C}^n$ and it is invariant under any $(\mathbb{R}, +)$-action of the form $t, (z, w) \mapsto z, \phi(t, w)$, $t \in \mathbb{R}, z \in \mathbb{C}^2, w \in \mathbb{C}^{n-2}$ where ϕ is some $(\mathbb{R}, +)$-action on $\mathbb{C}^{n-2}$.

3. POSITIVE RESULTS ON HOLOMORPHIC LINEARIZATION

For the convenience of the reader we first recall the notion of categorical quotient for an action $G \times X \to X$ of a Lie group G on a complex space X.

Definition. A complex space $X//G$ together with a G-invariant holomorphic map $\pi_X : X \to X//G$ is called categorical quotient for the action $G \times X \to X$ if it satisfies the following universality property:

For every holomorphic G-invariant map $\psi : X \to Y$ from X to some complex G-space Y there exists a unique holomorphic G-invariant map $\tilde{\psi} : X//G \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\pi_X"'] & & Y \\ & X//G \arrow[ur, "\tilde{\psi}"'] & \end{tikzcd}$$

commutes.

The existence of the categorical quotient in the case where G is a complex reductive group acting holomorphically on a Stein space X was proved by SNOW [24] and if G is a compact group acting by holomorphic transformations on a Stein space X by HEINZNER [9]. As a topological space $X//G$ is just the topological quotient of X with respect to the equivalence relation R associated to the algebra $\mathcal{O}^G(X)$ of G-invariant holomorphic functions on X:

$$R = \{(x,y) \in X \times Y, f(x) = f(y) \quad \forall\, f \in \mathcal{O}^G(x)$$

3. Positive results on holomorphic linearization

If G is reductive and X Stein, the map $\pi_X : X \to X//G$ parametrices the closed G-orbits in X, i.e., every π_X-fibre containes exactly one closed G-orbit O and moreover every G-orbit in the fibre contains O in its closure. If G is a finite group the categorical quotient is the same as the orbit quotient X/G. The following easy example will be used in chapter 3.

EXAMPLE 3.1. Let S^1 act on $\mathbb{C}^2$ by the rule

$$\lambda, (z, w) \mapsto \lambda^a \cdot z, \lambda^b \cdot w, \quad \lambda \in \mathbb{C}^\star,\ (z, w) \in \mathbb{C}^2.$$

Since $S^{1\mathbb{C}} = \mathbb{C}^\star$ the S^1-invariant holomorphic functions on $\mathbb{C}^2$ are $\mathbb{C}^\star$-invariant where $\mathbb{C}^\star$ acts by the same rule. Hence they are constant on $\mathbb{C}^\star$-orbits. If a and b have the same sign all $\mathbb{C}^\star$-orbits contain the point $(0,0)$ in its closure. So the value of a S^1-invariant holomorphic function at some point (z, w) is the same as the value at $(0,0)$. So the invariant functions are constant and $C^2//S^1 = \mathbb{C}^2//C^\star = \{\text{one point}\}$.

If a and b have different signs (and no common divisor to make the action effective) all invariant holomorphic functions on $\mathbb{C}^2$ are functions in one variable of $z^{|b|} \cdot w^{|a|}$. So $\pi_{\mathbb{C}^2} : \mathbb{C}^2 \to \mathbb{C}^2//S^1 = \mathbb{C}^2//\mathbb{C}^\star \cong \mathbb{C}$ is given by $(z, w) \mapsto z^{|b|} \cdot w^{|a|}$.

The categorical quotient $X//G$ is a normal complex space if X is normal and contractible if X is contractible [11]. So if $\mathbb{C}^n//G$ has dimension one it is necessarily smooth and contractible and admits no bounded holomorphic functions, hence is biholomorphically to $\mathbb{C}$.

Now we formulate some positive results about holomorphic linearization.

First of all we remark that a local linearization of an action of a compact group K on a manifold X in a K-invariant neighborhood of a fixed point x is easyly achieved by averaging over the group K (with respect to the Haar measure) a suitable local biholomorpism from a neighborhood U of x in X to a neighborhood of 0 in the tangent space T_xX (see for instance [2] 2.2). For this note that differentiating the action map $\phi : K \times X \to X$ with respect to the first variable one gets a K-action on the tangent bundle TX hence a linear representation of K on the tangent space T_xX at the fixed point x. This implies the following useful observation

REMARK 3.2. If an action of a compact group on $\mathbb{C}^n$ by holomorphic transformations is linearizable, it is conjugate to the tangent representation at some fixed point.

Moreover if the reductive group $K^{\mathbb{C}}$ acts and X is Stein one can extend this local K-equivariant biholomorphism to a $K^{\mathbb{C}}$-equivariant biholomorphism between $K^{\mathbb{C}}$-invariant neighborhoods, which are saturated with respect to the categorical quotient maps. This is a special case of the holomorphic version of LUNA's slice theorem ([9] 5.5).

From this one easy deduces the classical fact that holomorphic actions of a reductive group G on $\mathbb{C}^n$ with zero dimensional categorical quotient are linearizable, i.e., we are speaking about the case that $\mathbb{C}^n//G$ is just one point or equivalently all G-invariant holomorphic functions on $\mathbb{C}^n$ are constant.

The first remarkable result is due to SUZUKI [25] who gave a classification of the proper holomorphic $(\mathbb{C}, +)$-actions on $\mathbb{C}^2$. Here proper means that the limit set of each $\mathbb{C}$-orbit is finite. $\mathbb{C}^\star$-actions on a Stein space are proper in this sense.

Theorem 3.3. *Holomorphic actions of $\mathbb{C}^\star$ on $\mathbb{C}^2$ are linearizable.*

By proposition 2.3 the same holds for S^1-actions by holomorphic transformations on $\mathbb{C}^2$. In chapter 3 we will give examples of nonlinearizable $\mathbb{C}^*$-actions on $\mathbb{C}^n$ for all $n \geq 4$. The linearization question for $\mathbb{C}^*$ on $\mathbb{C}^3$ is open.

The complex algebraic analogue of the linearization question for reductive groups is studied quite well. We refer the interested reader to the papers by KRAFT [18], [17]. We only remark that algebraic linearization of $\mathbb{C}^*$-actions on $\mathbb{C}^n$ is open in general, but has a positive solution for $n \leq 3$ [16]. The first non-linearizable algebraic actions were found by SCHWARZ [23]. They come from non G-trivial G-vectorbundles over representations. More concretely he proved that for some reductive groups G (e.g. $O_2(\mathbb{C})$) there exist algebraic actions on $\mathbb{C}^n$ of the form

$$g,(z,w) \mapsto \alpha(g)\cdot z, \varphi(g,z)\cdot w, \quad g \in G, (z,w) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} = \mathbb{C}^n, \tag{1}$$

which are not algebraically linearizable. Here $\alpha : G \to GL_{n_1}(\mathbb{C})$ is a linear representation and $\phi : G \times \mathbb{C}^{n_1} \to GL_{n_2}(\mathbb{C})$ is an algebraic map satisfying tho two conditions $\phi(e,z) = id$ and $\phi(g,\alpha(h)\cdot z)\cdot \phi(h,z) = \phi(gh,z)$, $\quad z \in \mathbb{C}^{n_1}, g, h \in G$ to make (1) an action. All examples of SCHWARZ have one-dimensional categorical quotient. His original aim together with KRAFT was to prove that algebraic actions with one dimensional categorical quotient are algebraically linearizable. This turned out to be false, the first counterexamples of the form (1) found by SCHWARZ have one dimensional quotient. But they proved that algebraic actions with one dimensional categorical quotient are holomorphically linearizable [20]. This result was generalized to the natural setting by JIANG [15].

Theorem 3.4. *Holomorphic actions of reductive groups on $\mathbb{C}^n$ with one dimensional categorical quotient are linearizable.*

Note that this result generalizes theorem 3.3.

An application of an equivariant version of Grauert's Oka principle implies that all holomorphic actions of reductive groups G of the above form (1) are holomorphically linearizable. So the only up to now known method to construct counterexamples to complex algebraic linearization does not work in the holomorphic setting. To explain this we introduce the following setting and notations:

Let X be a complex space equipped with a holomorphic action of a reductive group G. A holomorphic fibre bundle $\pi : E \to X$ over X with structure group H and typical fibre F is called a holomorphic G-bundle over X if G acts holomorphically on the total space E such that the projection π is equivariant and the automorphisms of the fibres $E_x \to E_{g\cdot x}$ induced from the G-action are given by biholomorphic maps which belong to H. We call such bundles holomorphically G-isomorphic if there exists a G-equivariant holomorphic bundle isomorphism between them. Let K be a maximal compact subgroup of G, i.e., $G = K^{\mathbb{C}}$. The following theorem and corollary are due to HEINZNER and the author [12]

Theorem 3.5. *Two holomorphic G bundles with a complex Lie group as structure group are holomorphically G-isomorphic iff they are topologically K-isomorphic.*

Corollary 3.6. *Holomorphic actions of the form* (1) *are linearizable.*

PROOF. The action (1) is an G-action on the vector bundle (structure group is the complex Lie group $GL_n(\mathbb{C})$) $E = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \to C^{n_1} = X$ over the Stein manifold

$X = \mathbb{C}^{n_1}$ with linear action α. Since the base is K-equivariantly retractible to the fixed point 0, an equivariant version of the covering homotopy theorem for compact group actions implies that the bundle E is topologically K-isomorphic to the bundle $F = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \to C^{n_1} = X$ equipped with the action

$$g, (z, w) \mapsto \alpha(g) \cdot z, \varphi(g, 0) \cdot w, \quad g \in G, (z, w) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} = \mathbb{C}^n \qquad (2).$$

Note that $g \mapsto \varphi(g, 0)$ is the linear representation of G on the fibre over the fixed point $0 \in \mathbb{C}^{n_1}$. By theorem 3.5 there exists a holomorphic G-equivarant biholomorphism between the two bundles E and F. This is the automorphism of the total space $\mathbb{C}^n$ which conjugates the action (1) to the linear action (2). □

For the last result recall that the group of overshears $\mathrm{Sh_n}$ is the subgroup of $\mathrm{Aut_{hol}}(\mathbb{C}^n)$ generated by affine automorphisms and automorphisms of the form

$$(z_1, \dots, z_n) \mapsto (a(z_2, \dots, z_n)z_1 + b(z_2, \dots, z_n), z_2, \dots, z_n)$$

where a, b are arbitrary holomorphic functions on $\mathbb{C}^{n-1}$ and a is invertible. By a theorem of ANDERSÉN and LEMPERT this is a dense (but proper) subgroup of $\mathrm{Aut_{hol}}(\mathbb{C}^n)$ $n \geq 2$ [3]. The following theorem was proved by AHERN and RUDIN in the case of finite cyclic groups [1] and generalized by KRAFT and the author to the present form [19].

Theorem 3.7. *Every holomorphic action of a compact goup K on $\mathbb{C}^2$ by elements of Sh_2 is linearizable.*

In view of theorem 3.4 this result is only interesting for finite groups. By the methods explained in chapter 3 one can for instance prove that there are non-linearizable holomorphic actions of finite cyclic groups $\mathbb{Z}/\mathbb{Z}_n$ on C^m for $m \geq n+2$ ([5]). The smallest dimension in which non-linearizable holomorphic actions are known is 4. So the linearization question for finite groups on $\mathbb{C}^2$ is open.

4. Non-linearizable actions

The first counterexamples to the holomorphic linearization problem were constructed by DERKSEN and the author in [5]. There the following theorem is proved:

Theorem 4.1. *For every complex reductive Lie group G (except the trivial group) there exists a natural number N_G such that for all $n \geq N_G$ there exists an effective non-linearizable holomorphic action of G on $\mathbb{C}^n$.*

We will not prove this theorem here in full generality. Instead we explain the main idea restricting ourselves to the interesting case $G = \mathbb{C}^\star$. The construction uses the existence of holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ which are not equivalent to the standard embedding proved by FORSTNERIC, GLOBEVNIK and ROSAY in [8],

Definition. A proper holomorphic embedding $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ is called *straightenable* if there exists a $\beta \in \mathrm{Aut_{hol}}(\mathbb{C}^n)$ such that

$$\beta \circ \varphi(z_1, \dots, z_k) = (z_1, \dots, z_k, 0, \dots, 0).$$

Since every automorphism of $\mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$ extends to an automorphism of $\mathbb{C}^n$ this is equivalent to the existence of $\beta \in \mathrm{Aut_{hol}}(\mathbb{C}^n)$ with $\beta(\varphi(\mathbb{C}^k)) = \mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$.

We will use the following special version of non-straightenable embeddings (see [5] Cor.2.4)

Theorem 4.2. *There exists a non-straightenable proper holomorphic embedding $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^2$ such that in addition all embeddings $\varphi_k : \mathbb{C} \times \mathbb{C}^k \hookrightarrow \mathbb{C}^2 \times \mathbb{C}^k$ defined by $\varphi_k(z, y) = (\varphi(z), y)$ are non-straightenable too for all integer $k > 0$.*

In the proof of proposition 4.4 we will need the fact that if the image of an embedding $\varphi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is contained in an linear subspace of high enough codimension this embedding is straightenable which is made precise in the following lemma. For a proof we refer to [5] (Lemma 2.5).

Lemma 4.3. *If $\varphi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is a proper holomorphic embedding, then the embedding $\varphi_1 : \mathbb{C}^k \hookrightarrow \mathbb{C}^{n+k} = \mathbb{C}^n \times \mathbb{C}^k$ defined by $\varphi_1(z) = (\varphi(z), 0)$ is straightenable.*

From now on we fix a proper holomorphic embedding $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^2$ like in the preceeding theorem. Also we fix a holomorphic function $f \in \mathcal{O}(\mathbb{C}^2)$ which vanishes precisely on the closed submanifold $\varphi(\mathbb{C})$ in $\mathbb{C}^2$ and the gradient of f does not vanish when f vanishes, in other words f generates the ideal $I_{\varphi(\mathbb{C})}(\mathbb{C}^2)$ of $\varphi(\mathbb{C})$. Such a function exists by the solution of the multiplicative Cousin problem.

Define
$$X = \{(x, y, z, w) \in \mathbb{C}^4 : f(x, y) = w \cdot z\}.$$

Since the gradient of f does not vanish on $\{f(x, y) = 0\} = \varphi(\mathbb{C})$ the set X is a smooth submanifold of $\mathbb{C}^4$. The crucial point of our construction is the following

Proposition 4.4. *If f is any holomorphic function on $\mathbb{C}^2$, whose zero set is biholomorphic to $\mathbb{C}$, and with non-vanishing gradient on its zero set, then the manifold $X \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^4$.*

PROOF. We have

$$X \times \mathbb{C} = \{(x, y, z, w, u) \in \mathbb{C}^5 : f(x, y) = w \cdot z\}.$$

The map $x, y, z, w, u \mapsto x, y, w \cdot u, z, w, u$ gives a biholomorphism $\tau_1 : X \times \mathbb{C} \to A_1$ where A_1 is the submanifold of $\mathbb{C}^6$ defined by

$$A_1 := \{(x, y, v, z, w, u) \in \mathbb{C}^6 : f(x, y) = w \cdot z, v = u \cdot z\}$$

The inverse mapping τ_1^{-1} is given by $(x, y, v, z, w, u) \mapsto (x, y, z, w, u)$.

By Lemma 4.3 there exists an automorphism $\alpha \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^3)$ with

$$\alpha(\varphi(\xi), 0) = (\xi, 0, 0). \qquad (\star)$$

The automorphism of $\mathbb{C}^6$ defined by $(x, y, v, z, w, u) \mapsto \alpha(x, y, v), z, w, u$, i.e., in the first 3 coordinates α and identity in the other 3 coordinates gives if restricted to A_1 a biholomorphism τ_2 from A_1 to the submanifold A_2 of $\mathbb{C}^6$ defined by:

$$A_2 := \{(a, b, c, z, w, u) \in \mathbb{C}^6 : (\alpha^{-1})^\star f(a, b, c) = w \cdot z, (\alpha^{-1})^\star v(a, b, c) = u \cdot z\}$$

Because of $(\star)$ the Ideal in $\mathcal{O}(\mathbb{C}^3)$ generated by the two functions b and c is the same as the Ideal generated by the functions $(\alpha^{-1})^\star v$ and $(\alpha^{-1})^\star f$, namely the Ideal

$I_{\mathbb{C}\times\{0\}}$ of the first coordinate axis. We can express one set of generators through the other, that means we find holomorphic functions $a_{ij}, b_{ij} \in \mathcal{O}(\mathbb{C}^3)$ $\quad i,j = 1,2$ with

$$b = a_{11} \cdot (\alpha^{-1})^\star f + a_{12} \cdot (\alpha^{-1})^\star v, \quad c = a_{21} \cdot (\alpha^{-1})^\star f + a_{22} \cdot (\alpha^{-1})^\star v \tag{1}$$

and

$$(\alpha^{-1})^\star f = b_{11} \cdot b + b_{12} \cdot c, \quad (\alpha^{-1})^\star v = b_{21} \cdot b + b_{22} \cdot c. \tag{2}$$

We claim that the restriction τ_3 to A_3 of the holomorphic map $S_1 : \mathbb{C}^6 \to \mathbb{C}^6$ defined by

$$S_1(a,b,c,z,w,u) = a,b,c,z,a_{11}(a,b,c)\cdot w + a_{12}(a,b,c)\cdot u, a_{21}(a,b,c)\cdot w + a_{22}(a,b,c)\cdot u$$

gives a biholomorphism τ_3 from A_2 to the submanifold A_3 of $\mathbb{C}^6$ defined by

$$A_3 := \{(a,b,c,z,w,u) \in \mathbb{C}^6 : b = w \cdot z, c = u \cdot z\}.$$

We will prove that the inverse $\tau_3^{-1} : A_3 \to A_2$ is given by the restriction of $S_2 : \mathbb{C}^6 \to \mathbb{C}^6$ defined by

$$S_2(a,b,c,z,w,u) = a,b,c,z,b_{11}(a,b,c)\cdot w + b_{12}(a,b,c)\cdot u, b_{21}(a,b,c)\cdot w + b_{22}(a,b,c)\cdot u$$

to A_3. For that consider the holomorphic maps

$$\psi_1 : \mathbb{C}^3 \times \mathbb{C}^\star \to \mathbb{C}^6, \quad a,b,c,z \mapsto a,b,c,z,\frac{(\alpha^{-1})^\star f}{z},\frac{(\alpha^{-1})^\star v}{z}$$

and

$$\psi_2 : \mathbb{C}^3 \times \mathbb{C}^\star \to \mathbb{C}^6, \quad a,b,c,z \mapsto a,b,c,z,\frac{b}{z},\frac{c}{z}.$$

The submanifold A_2 is the topological closure (and therefore the holomorphic closure, i.e.,the smallest analytic set containing) of the image of ψ_1 in $\mathbb{C}^6$. Also A_3 is the closure of the image of ψ_2 in $\mathbb{C}^6$.

From (1) follows $\psi_2 = S_1 \circ \psi_1$ and (2) implies $\psi_1 = S_2 \circ \psi_2$. So we have $\psi_1 = S_2 \circ S_1 \circ \psi_1$. This means that $S_1 \circ S_2$ is the identity on the image of ψ_1 hence it is the identity on the closure of the image of ψ_1, on A_2. Analogously follows that $S_2 \circ S_1$ is the identity on A_3.

Finally the map $\tau_4 : A_3 \to \mathbb{C}^4$, $\quad (a,b,c,z,w,u) \mapsto a,z,w,u$ is a biholomorphism. The composition $\tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1$ provides the desired biholomorphism from X to $\mathbb{C}^4$.

□

REMARK 4.5. We do not know whether the manifold X itself is biholomorphic to $\mathbb{C}^3$ (if $\{f = 0\} \cong \mathbb{C}$ is straightenable then this is clearly true). If for some non-straightenable $\{f = 0\} \cong \mathbb{C}$ the manifold X would be biholomorphic to $\mathbb{C}^3$ then we would have a non-linearizable $\mathbb{C}^\star$-action on $X \cong \mathbb{C}^3$ (see the proof of proposition 4.7 below), if it is not biholomorphic to $\mathbb{C}^3$ then this would be a counterexample to the following open problem.

Problem 4.6 (Holomorphic Zariski Cancellation Problem). *Let Z be a complex manifold such that $Z \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+1}$ $(n \geq 2)$. Does it follow that $Z \cong \mathbb{C}^n$?*

Now we come to the non-linearizable $\mathbb{C}^*$-actions. Since $X \times \mathbb{C} \cong \mathbb{C}^4$ we have for each $k \geq 1$ a biholomorphism $\alpha_k : X \times \mathbb{C}^k \cong \mathbb{C}^{k+3}$. The submanifold $X \times \mathbb{C}^k \subset \mathbb{C}^4 \times \mathbb{C}^k$ is stable under the linear $\mathbb{C}^*$-action on $\mathbb{C}^4 \times \mathbb{C}^k$ given by

$$\mathbb{C}^* \times \mathbb{C}^4 \times \mathbb{C}^k \to \mathbb{C}^4 \times \mathbb{C}^k$$

$$\lambda \cdot (x, y, z, w, t_1, \ldots, t_k) = (x, y, \lambda z, \lambda^{-1} w, t_1, \ldots, t_k).$$

The restriction of this action to $X \times \mathbb{C}^k$ induces via α_k a holomorphic $\mathbb{C}^*$-action $\sigma_k : \mathbb{C}^* \times \mathbb{C}^{k+3} \to \mathbb{C}^{k+3}$.

Proposition 4.7. *The action σ_k is not linearizable ($k \geq 1$). So for all $l \geq 4$ there exists a non-linearizable $\mathbb{C}^*$ action on $\mathbb{C}^l$.*

PROOF. Suppose $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{3+k}$ is a biholomorphic $\mathbb{C}^*$-equivariant map, where $\mathbb{C}^*$ acts linearly on $\mathbb{C}^{3+k}$. This representation of $\mathbb{C}^*$ on $\mathbb{C}^{k+3}$ must be isomorphic to the representation of $\mathbb{C}^*$ on the tangent space of some fixed point of $X \times \mathbb{C}^k$ (see remark 3.2). With respect to some coordinates, this action is given by

$$\lambda \cdot (z, w, u_1, \ldots, u_{k+1}) = (\lambda z, \lambda^{-1} w, u_1, \ldots, u_{k+1}).$$

The categorical quotient $\pi_Y : Y \to Y//G$ of a G-invariant closed subspace Y of a Stein G-space Z is the restriction of the categorical quotient $\pi_Z : Z \to Z//G$. This follows from the fact that all G-invariant functions on Y extend to G-invariant holomorphic functions on Z (first extend to some holomorphic function, then average over the maximal compact subgroup K of $G = K^{\mathbb{C}}$). Using this together with example 3.1 one sees that the categorical quotient of $X \times \mathbb{C}^k$ is given by $\pi_{X \times \mathbb{C}^k} : X \times \mathbb{C}^k \to \mathbb{C}^{2+k}$,

$$\pi_{X \times \mathbb{C}^k}(x, y, z, w, t_1, \ldots, t_k) = (x, y, t_1, \ldots, t_k)$$

and the categorical quotient of $\mathbb{C}^{3+k}$ is given by $\pi_{\mathbb{C}^{3+k}} : \mathbb{C}^{3+k} \to \mathbb{C}^{2+k}$,

$$\pi_{\mathbb{C}^{3+k}}(z, w, u_1, \ldots, u_{k+1}) = (zw, u_1, \ldots, u_{k+1}).$$

The fixed point set $(X \times \mathbb{C}^k)^{\mathbb{C}^*}$ is

$$\{(x, y, z, w, t_1, \ldots, t_k) \in X \times \mathbb{C}^k \mid f(x, y) = z = w = 0\}.$$

Its image under $\pi_{X \times \mathbb{C}^k}$ is $\varphi(\mathbb{C}) \times \mathbb{C}^k \subset \mathbb{C}^2 \times \mathbb{C}^k$. On the other hand $(\mathbb{C}^{3+k})^{\mathbb{C}^*}$ is

$$\{(z, w, u_1, \ldots, u_{k+1}) \in \mathbb{C}^{3+k} \mid z = w = 0\}$$

and its image under $\pi_{\mathbb{C}^{3+k}}$ is $\{0\} \times \mathbb{C}^{1+k} \subset \mathbb{C}^{2+k}$. Since equivariant maps map the fixed point set into the fixed point set, the $\mathbb{C}^*$-equivariant biholomorphism $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{3+k}$ induces a biholomorphism $\gamma : \mathbb{C}^{2+k} \xrightarrow{\cong} \mathbb{C}^{2+k}$ of the categorical quotients such that the image of the fixed point set under $\pi_{X \times \mathbb{C}^k}$ is mapped onto the image of the other fixed point set under $\pi_{\mathbb{C}^{3+k}}$, i.e., $\gamma(\varphi(\mathbb{C}) \times \mathbb{C}^k) = \{0\} \times \mathbb{C}^{k+1}$. This contradicts the choice of $\varphi(\mathbb{C})$ according to theorem 4.2. $\square$

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