数理解析研究所講究録
1034 巻 1998 年 185-198

中立型微分方程式のある終局的正値解が存在するための必要十分条件

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1. Introduction

In this paper we consider the first order neutral differential equation

$$\frac{d}{dt}[x(t)+h(t)x(\tau(t))]+\sigma f(t,x(g(t)))=0, \tag{1.1}$$

where $\sigma=+1$ or -1. It is assumed throughout this paper that:

(a) $\tau:[t_0,\infty)\longrightarrow\mathbb{R}$ is continuous and strictly increasing, $\tau(t)<t$ for $t\ge t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$;
(b) $h:[\tau(t_0),\infty)\longrightarrow\mathbb{R}$ is continuous;
(c) $g:[t_0,\infty)\longrightarrow\mathbb{R}$ is continuous and $\lim_{t\to\infty}g(t)=\infty$;
(d) $f:[t_0,\infty)\times(0,\infty)\longrightarrow[0,\infty)$ is continuous and $f(t,u)$ is nondecreasing in $u\in(0,\infty)$ for any fixed $t\in[t_0,\infty)$.

By a solution of (1.1), we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x,\infty)$ for some $t_x\ge t_0$.

Recently there has been considerable investigation of the existence of positive solutions of first order neutral differential equations. We refer the reader to [1–20]. In particular, it is known that (1.1) has a solution x satisfying

$$0<\liminf_{t\to\infty}x(t)\le\limsup_{t\to\infty}x(t)<\infty \tag{1.2}$$

if and only if

$$\int_{t_0}^{\infty}f(t,a)dt<\infty\quad\text{for some } a>0 \tag{1.3}$$

when one of the following cases holds:

(i) $|h(t)|\le\lambda<1$ and $h(t)h(\tau(t))\ge0$ ([1, 5, 6, 13, 14, 16]);
(ii) $h(t)\equiv1$ and $\tau(t)=t-\tau$ $(\tau>0)$ ([1, 17]);
(iii) $1<\mu\le h(t)\le\lambda<\infty$ ([1, 16]).

Here, λ, μ and τ are constants. However, very little is known about the existence of solution x of (1.1) satisfying (1.2) in a different case, such as

$$\liminf_{t\to\infty} h(t) < 1 < \limsup_{t\to\infty} h(t). \tag{1.4}$$

In this paper, we consider the case

$$h(t) > -1 \quad \text{and} \quad h(\tau(t)) = h(t), \quad t \geq t_0. \tag{1.5}$$

A pair of the functions $h(t) = 1 + (1/2)\sin t$ and $\tau(t) = t - 2\pi$ gives a typical example satisfying (1.5). We easily see that if (1.5) holds, then

$$x(t) = \frac{b}{1+h(t)} \quad (b > 0)$$

is a positive solution of the unperturbed equation $\frac{d}{dt}[x(t) + h(t)x(\tau(t))] = 0$, and so it is natural to expect that, if f is small enough in some sense, (1.1) possesses a solution x which behaves like the function $b/[1 + h(t)]$ as $t \to \infty$. In fact, the following theorem will be shown.

Theorem. *Suppose that* (1.5) *holds. Then* (1.1) *has a positive solution* x *satisfying*

$$x(t) = \frac{b}{1+h(t)} + o(1) \quad (t \to \infty) \quad \textit{for some } b > 0 \tag{1.6}$$

if and only if (1.3) *holds.*

If (1.5) holds, then there are constants μ and λ such that $-1 < \mu \leq h(t) \leq \lambda < \infty$ for $t \geq t_0$. Then it is worthwhile to note that a positive solution x with the asymptotic property (1.6) satisfies (1.2)

2. Proof of Theorem

First we prove the "only if" part of Theorem.

Proof of the "only if" part. Let x be a solution of (1.1) which satisfies (1.6). Put $y(t) = x(t) + h(t)x(\tau(t))$. Then (1.5) implies that $y(t) = b + o(1)$ as $t \to \infty$. Integration of (1.1) over $[T, \infty)$ yields

$$b - y(T) + \sigma \int_T^\infty f(s, x(g(s)))ds = 0,$$

where $T \geq t_0$. Hence we obtain

$$\int_T^\infty f(s, x(g(s)))ds < \infty.$$

Noting that x satisfies (1.2) and using the monotonicity of f, we conclude that (1.3) holds.

The following notation will be used:

$$\tau^0(t) = t; \quad \tau^i(t) = \tau(\tau^{i-1}(t)), \quad i = 1, 2, \dots ;$$
$$\tau^{-i}(t) = \tau^{-1}(\tau^{-(i-1)}(t)), \quad i = 2, 3, \dots ,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. We note here that $\tau^{-p}(t) \to \infty$ as $p \to \infty$ for each fixed $t \geq t_0$. Otherwise, there is a constant $c \geq t_0$ such that $\lim_{p\to\infty} \tau^{-p}(t) = c$, because of $\tau^{-p}(t) < \tau^{-(p+1)}(t)$. Letting $p \to \infty$ in $\tau^{-p}(t) = \tau^{-1}(\tau^{-(p-1)}(t))$, we have $c = \tau^{-1}(c)$ which contradicts $\tau(t) < t$ for $t \geq t_0$.

Note that $[t_0, \infty) = \cup_{p=0}^{\infty} [\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)]$ and that the range of $h(t)$ for $t \in [t_0, \tau^{-1}(t_0)]$ is identical to the range of $h(t)$ $(= h(\tau^p(t)))$ for $t \in [\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)]$, $p = 0, 1, 2, \dots$. Thus it is possible to take a sufficiently large number $T \geq t_0$ such that

$$h(T) = \max\{h(t) : t \in [t_0, \infty)\}$$

and

$$T_* \equiv \min\{\tau(T), \inf\{g(t) : t \geq T\}\} \geq t_0.$$

Let $C[T_*, \infty)$ denote the Fréchet space of all continuous functions on $[T_*, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*, \infty)$. Let $\eta \in C[T, \infty)$ be fixed such that $\eta(t) \geq 0$ for $t \geq T$ and $\lim_{t\to\infty} \eta(t) = 0$. We consider the set Y of all functions $y \in C[T_*, \infty)$ which is nondecreasing on $[T, \infty)$ and satisfies

$$y(t) = y(T) \quad \text{for } t \in [T_*, T], \quad 0 \leq y(t) \leq \eta(t) \quad \text{for } t \geq T.$$

It is easy to see that Y is a closed convex subset of $C[T_*, \infty)$.

To prove the "if" part of Theorem, the following Proposition is used.

Proposition. *Suppose that* (1.5) *holds. Let* $\eta \in C[T, \infty)$ *with* $\eta(t) \geq 0$ *for* $t \geq T$ *and* $\lim_{t\to\infty} \eta(t) = 0$. *For this* η, *define* Y *as above. Then there exists a mapping* $\Phi : Y \longrightarrow C[T_*, \infty)$ *which possesses the following properties*:

(a) *For each* $y \in Y$, $\Phi[y]$ *satisfies*

$$\Phi[y](t) + h(t)\Phi[y](\tau(t)) = y(t), \quad t \geq T \quad \textit{and} \quad \lim_{t\to\infty} \Phi[y](t) = 0;$$

(b) Φ *is continuous on* Y *in the* $C[T_*, \infty)$*-topology, i.e., if* $\{y_j\}_{j=1}^{\infty}$ *is a sequence in* Y *converging to* $y \in Y$ *uniformly on every compact subinterval of* $[T_*, \infty)$, *then* $\Phi[y_j]$ *converges to* $\Phi[y]$ *uniformly on every compact subinterval of* $[T_*, \infty)$.

Let us first show the "if" part of Theorem. The proof of Proposition is deferred to the next section.

Proof of the "if" part. Put

$$\eta(t) = \int_t^\infty f(s,a)ds, \quad t \geq T.$$

We use Proposition for this η. We can take constants $b > 0$, $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < \delta + \varepsilon \leq \frac{b}{1 + h(t)} \leq a - \varepsilon, \quad t \geq T_*.$$

Define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_t^\infty F\left(s, \frac{b}{1 + h(g(s))} + \sigma\Phi[y](g(s))\right) ds, & t \geq T, \\ (\mathcal{F}y)(T), & t \in [T_*, T], \end{cases}$$

where

$$F(t,u) = \begin{cases} f(t,a), & u \geq a, \\ f(t,u), & \delta \leq u \leq a, \\ f(t,\delta), & u \leq \delta. \end{cases}$$

It is easy to see that $\mathcal{F}$ is well defined on Y and maps Y into itself.

Since Φ is continuous on Y, the Lebesgue dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Let I be an arbitrary compact subinterval of $[T, \infty)$. We find that

$$|(\mathcal{F}y)'(t)| \leq \max\{f(s,a) \,:\, s \in I\}, \quad t \in I,$$

so that $\{(\mathcal{F}y)'(t)\}_{y \in Y}$ is uniformly bounded on I. The mean value theorem shows that $\mathcal{F}(Y)$ is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for t_1, $t_2 \in [T_*, T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*, \infty)$. Obviously, $\mathcal{F}(Y)$ is uniformly bounded on $[T_*, \infty)$. Hence, by the Ascoli-Arzela theorem, $\mathcal{F}(Y)$ is relatively compact. Consequently, we are able to apply the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$ and we conclude that there exists a $\tilde{y} \in Y$ such that $\tilde{y} = \mathcal{F}\tilde{y}$. Set

$$x(t) = \frac{b}{1 + h(t)} + \sigma\Phi[\tilde{y}](t).$$

Proposition implies that x satisfies (1.6) and that there exists a number $\tilde{T} \geq T$ such that $\delta \leq x(g(t)) \leq a$ for $t \geq \tilde{T}$. Then $F(t, x(g(t))) = f(t, x(g(t)))$ for $t \geq \tilde{T}$.

Observe that

$$
\begin{aligned}
(2.1)\qquad & x(t) + h(t)x(\tau(t)) \\
&= \frac{b}{1+h(t)} + h(t)\frac{b}{1+h(\tau(t))} + \sigma[\Phi[\tilde{y}](t) + h(t)\Phi[\tilde{y}](\tau(t))] \\
&= b + \sigma\tilde{y}(t) \\
&= b + \sigma\int_t^\infty f(s, x(g(s)))ds, \quad t \geq \tilde{T}.
\end{aligned}
$$

By differentiation of (2.1), we see that x is a solution of (1.1). The proof is complete.

3. Proof of Proposition

The purpose of this section is to prove Proposition. Throughout this section, we assume that (1.5) holds.

For each $y \in Y$, we define the function $\Psi[y]$ by

$$
\Psi[y](t) = \begin{cases} \sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t)), & t \geq \tau(T), \\ \Psi[y](\tau(T)), & t \in [T_*, \tau(T)], \end{cases}
$$

where $H(t) = \max\{1, h(t)\}$. We note that $H(\tau(t)) = H(t)$ and $H(t) \geq 1$ for $t \geq t_0$.

Lemma 1.

(i) *For each $y \in Y$, the series*

$$
\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))
$$

converges uniformly on $[\tau(T), \infty)$, hence $\Psi[y]$ is well defined and is continuous on $[T_*, \infty)$;

(ii) *For each $y \in Y$, $\Psi[y]$ satisfies*

$$
(3.1)\qquad 0 \leq \Psi[y](t) \leq \eta(\tau^{-1}(t)), \quad t \geq \tau(T),
$$

and

$$
(3.2)\qquad \Psi[y](t) + H(t)\Psi[y](\tau(t)) = y(t), \quad t \geq T;
$$

(iii) *Ψ is continuous on Y in the $C[T_*, \infty)$-topology.*

Proof. (i) Let $y \in Y$. We set

$$
\Psi_m[y](t) = \sum_{i=1}^{m}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t)), \quad t \geq \tau(T), \quad m = 1, 2, \ldots .
$$

Now we claim that

$$
(3.3)\qquad 0 \leq \Psi_m[y](t) \leq \eta(\tau^{-1}(t)), \quad t \geq \tau(T)
$$

for $m = 1, 2, \dots$. Since y is nondecreasing on $[T, \infty)$ and $H(t) \geq 1$, we have

$$y(\tau^{-1}(t)) - [H(t)]^{-1} y(\tau^{-2}(t)) \geq 0, \quad t \geq \tau(T), \tag{3.4}$$

and

$$[H(t)]^{-1} y(\tau^{-1}(t)) \leq \eta(\tau^{-1}(t)), \quad t \geq \tau(T). \tag{3.5}$$

Hence, we easily see that (3.3) holds for the cases $m = 1$ and 2. If $m \geq 3$ is odd, we can rewrite $\Psi_m[y](t)$ as

$$\begin{aligned}\Psi_m[y](t) = &\sum_{j=1}^{(m-1)/2} [H(t)]^{-(2j-1)} [y(\tau^{-(2j-1)}(t)) - [H(t)]^{-1} y(\tau^{-2j}(t))] \\ &+ [H(t)]^{-m} y(\tau^{-m}(t))\end{aligned}$$

and

$$\begin{aligned}\Psi_m[y](t) = &[H(t)]^{-1} y(\tau^{-1}(t)) \\ &- \sum_{j=1}^{(m-1)/2} [H(t)]^{-2j} [y(\tau^{-2j}(t)) - [H(t)]^{-1} y(\tau^{-(2j+1)}(t))].\end{aligned}$$

If $m \geq 4$ is even, we can rewrite $\Psi_m[y](t)$ as

$$\Psi_m[y](t) = \sum_{j=1}^{m/2} [H(t)]^{-(2j-1)} [y(\tau^{-(2j-1)}(t)) - [H(t)]^{-1} y(\tau^{-2j}(t))]$$

and

$$\begin{aligned}\Psi_m[y](t) = &[H(t)]^{-1} y(\tau^{-1}(t)) \\ &- \sum_{j=1}^{(m/2)-1} [H(t)]^{-2j} [y(\tau^{-2j}(t)) - [H(t)]^{-1} y(\tau^{-(2j+1)}(t))] \\ &- [H(t)]^{-m} y(\tau^{-m}(t)).\end{aligned}$$

From (3.4) and (3.5) we conclude that (3.3) holds for $m = 3, 4, \dots$.

Using (3.3), we find that if $m \geq p \geq 1$, then

$$\begin{aligned}&\left| \sum_{i=p}^{m} (-1)^{i+1} [H(t)]^{-i} y(\tau^{-i}(t)) \right| \\ &= \left| \sum_{i=1}^{m-p+1} (-1)^{(i+p-1)+1} [H(t)]^{-(i+p-1)} y(\tau^{-i}(\tau^{-p+1}(t))) \right| \\ &= |(-1)^{(p-1)} [H(t)]^{-(p-1)} \Psi_{m-p+1}[y](\tau^{-p+1}(t))| \\ &\leq \eta(\tau^{-p}(t)), \quad t \geq \tau(T).\end{aligned} \tag{3.6}$$

Here, we have used the equality $H(t) = H(\tau^{-p+1}(t))$, $p \geq 1$. Since $\eta(\tau^{-p}(t)) \to 0$ as $p \to \infty$, the series $\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))$ converges for each fixed $t \in [\tau(T), \infty)$. From (3.6) it follows that

$$\sup_{t\in[\tau(T),\infty)} \left|\sum_{i=p}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))\right|$$
$$\leq \sup_{t\in[\tau(T),\infty)} \eta(\tau^{-p}(t)) = \sup_{t\in[\tau^{-p+1}(T),\infty)} \eta(t) \to 0 \quad \text{as } p \to \infty,$$

which shows that the series $\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))$ converges uniformly on $[\tau(T), \infty)$.

(ii) Letting $m \to \infty$ in (3.3), we have (3.1). It is easy to check that (3.2) holds.

(iii) Let $\varepsilon > 0$. There is an integer $p \geq 1$ such that

$$\sup_{t\in[\tau(T),\infty)} \eta(\tau^{-(p+1)}(t)) = \sup_{t\in[\tau^{-p}(T),\infty)} \eta(t) < \frac{\varepsilon}{3}.$$

Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Take an arbitrary compact subinterval I of $[\tau(T), \infty)$. There exists an integer $j_0 \geq 1$ such that

$$\sum_{i=1}^{p} |y_j(\tau^{-i}(t)) - y(\tau^{-i}(t))| < \frac{\varepsilon}{3}, \quad t \in I, \quad j \geq j_0.$$

It follows from (3.6) that

$$\begin{aligned}
&|\Psi[y_j](t) - \Psi[y](t)| \\
&\leq \sum_{i=1}^{p}[H(t)]^{-i}|y_j(\tau^{-i}(t)) - y(\tau^{-i}(t))| \\
&\quad + \left|\sum_{i=p+1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y_j(\tau^{-i}(t))\right| + \left|\sum_{i=p+1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))\right| \\
&\leq \sum_{i=1}^{p}|y_j(\tau^{-i}(t)) - y(\tau^{-i}(t))| + 2\eta(\tau^{-(p+1)}(t)) < \varepsilon, \quad t \in I, \quad j \geq j_0,
\end{aligned}$$

which implies that $\Psi[y_j]$ converges $\Psi[y]$ uniformly on I. It is easy to see that $\Psi[y_j] \to \Psi[y]$ uniformly on $[T_*, \tau(T)]$. Consequently, we conclude that Ψ is continuous on Y. This completes the proof.

For each $y \in Y$, we assign the function $\varphi[y]$ as follows:

$$\varphi[y](t) = \begin{cases} \dfrac{y(T)}{1+h(T)} & \text{if } h(T) < 1, \\ \Psi[y](t) & \text{if } h(T) \geq 1, \end{cases} \qquad t \in [T_*, T].$$

Lemma 2.

(i) *For each* $y \in Y$, $\varphi[y]$ *satisfies*

$$\varphi[y](T) + h(T)\varphi[y](\tau(T)) = y(T);$$

(ii) *Suppose that* $\{y_j\}_{j=1}^{\infty}$ *is a sequence in* Y *converging to* $y \in Y$ *uniformly on every compact subinterval of* $[T_*, \infty)$. *Then* $\varphi[y_i]$ *converges to* $\varphi[y]$ *uniformly on* $[T_*, T]$.

Proof. It is obvious that (i) and (ii) hold for the case $h(T) < 1$. For the case $h(T) \geq 1$, (i) and (ii) follow from (ii) and (iii) of Lemma 1.

For each $y \in Y$, we define the function $\Phi[y]$ as follows:

$$\Phi[y](t) = \begin{cases} \displaystyle\sum_{i=0}^{m}(-1)^i[h(t)]^i y(\tau^i(t)) + (-1)^{m+1}[h(t)]^{m+1}\varphi[y](\tau^{m+1}(t)), \\ \qquad\qquad t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)], \quad m = 0, 1, \dots, \\ \varphi[y](t), \qquad\qquad t \in [T_*, T]. \end{cases}$$

Lemma 3. *Let* $y \in Y$.

(i) $\Phi[y]$ *is continuous on* $[T_*, \infty)$;

(ii) $\Phi[y]$ *satisfies*

$$\Phi[y](t) + h(t)\Phi[y](\tau(t)) = y(t), \quad t \geq T;$$

(iii) *For* $t \in [\tau(T), \infty)$ *with* $h(t) \geq 1$,

$$\Phi[y](t) = \Psi[y](t);$$

(iv) Φ *is continuous on* Y *in the* $C[T_*, \infty)$*-topology.*

Proof. (i) It is easy to see that $\Phi[y]$ is continuous on

$$[T_*, \infty)\backslash\{\tau^{-m}(T) : m = 0, 1, 2, \dots\}.$$

From (i) of Lemma 2, it follows that

$$\lim_{t \to T-0} \Phi[y](t) = \varphi[y](T) = y(T) - h(T)\varphi[y](\tau(T)) = \lim_{t \to T+0} \Phi[y](t)$$

and that if $m \geq 1$, then

$$\begin{aligned}
&\lim_{t\to\tau^{-m}(T)-0} \Phi[y](t) \\
&= \sum_{i=0}^{m-1}(-1)^i[h(\tau^{-m}(T))]^i y(\tau^{i-m}(T)) + (-1)^m[h(\tau^{-m}(T))]^m\varphi[y](T) \\
&= \sum_{i=0}^{m-1}(-1)^i[h(\tau^{-m}(T))]^i y(\tau^{i-m}(T)) \\
&\quad + (-1)^m[h(\tau^{-m}(T))]^m[y(T) - h(T)\varphi[y](\tau(T))] \\
&= \sum_{i=0}^{m}(-1)^i[h(\tau^{-m}(T))]^i y(\tau^{i-m}(T)) \\
&\quad + (-1)^{m+1}[h(\tau^{-m}(T))]^{m+1}\varphi[y](\tau^{(m+1)}(\tau^{-m}(T))) \\
&= \lim_{t\to\tau^{-m}(T)+0} \Phi[y](t).
\end{aligned}$$

Consequently, $\Phi[y]$ is continuous on $[T_*, \infty)$.

(ii) An easy computation shows that (ii) follows.

(iii) If $h(T) < 1$, then there is no number $t \in [\tau(T), \infty)$ such that $h(t) \geq 1$ (recall the choice of T). Assume that $h(T) \geq 1$. Then

$$\Phi[y](t) = \varphi[y](t) = \Psi[y](t) \quad \text{for } t \in [\tau(T), T].$$

We suppose that there is an integer $m \geq 0$ such that $\Phi[y](t) = \Psi[y](t)$ for all $t \in [\tau^{-(m-1)}(T), \tau^{-m}(T)]$ with $h(t) \geq 1$. In view of (ii) of Lemma 3 and (3.2), we find that if $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$ and if $h(t) \geq 1$, then

$$\Phi[y](t) = y(t) - h(t)\Phi[y](\tau(t)) = y(t) - H(t)\Psi[y](\tau(t)) = \Psi[y](t).$$

By induction, we conclude that $\Phi[y](t) = \Psi[y](t)$ for $t \in [\tau(T), \infty)$ with $h(t) \geq 1$.

(iv) Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Lemma 2 implies that $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on $[T_*, T]$. It suffices to prove that $\Phi[y_j] \to \Phi[y]$ uniformly on $I_m \equiv [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m = 0, 1, 2, \ldots$. Since $|h(t)| \leq \lambda$ on $[t_0, \infty)$ for some $\lambda \geq 1$, we observe that

$$\begin{aligned}
&\sup_{t\in I_m} |\Phi[y_j](t) - \Phi[y](t)| \\
&\leq \sum_{i=0}^{m} \lambda^i \sup_{t\in I_m} |y_j(\tau^i(t)) - y(\tau^i(t))| \\
&\quad + \lambda^{m+1} \sup_{t\in I_m} |\varphi[y_j](\tau^{m+1}(t)) - \varphi[y](\tau^{m+1}(t))| \\
&\leq \lambda^m \sum_{i=0}^{m} \sup_{t\in I_{m-i}} |y_j(t) - y(t)| + \lambda^{m+1} \sup_{t\in[T_*,T]} |\varphi[y_j](t) - \varphi[y](t)|.
\end{aligned}$$

Then, $\sup_{t\in I_m}|\Phi[y_j](t)-\Phi[y](t)|\to 0$ as $j\to\infty$, so that $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on I_m for $m=0,1,2,\ldots$.

Lemma 4. *Let $\{t_j\}_{j=0}^{\infty}$ be a sequence satisfying $\lim_{j\to\infty}t_j=\infty$ and $|h(t_j)|\le\nu<1$, $j=1,2,\ldots$ for some $\nu>0$. Then $\lim_{t\to\infty}\Phi[y](t_j)=0$ for each $y\in Y$.*

Proof. Let $\varepsilon>0$. Since $\lim_{t\to\infty}y(t)=0$, there is an integer $p\ge 1$ such that

$$\frac{y(\tau^{-p}(T))}{1-\nu}<\frac{\varepsilon}{3}.$$

There exists an integer $q\ge 1$ such that

$$\frac{y(T)\nu^{r-p+1}}{1-\nu}<\frac{\varepsilon}{3}\quad\text{and}\quad \nu^{r+1}\sup_{t\in[T_*,T]}|\varphi[y](t)|<\frac{\varepsilon}{3}\quad\text{for all } r\ge p+q.$$

Let $m\ge p+q$. Then $\tau^{m-p}(t)\ge\tau^{-p}(T)$ for $t\in[\tau^{-m}(T),\tau^{-(m+1)}(T)]$. In view of the monotonicity of y, we see that if $t\in[\tau^{-m}(T),\tau^{-(m+1)}(T)]$ and $|h(t)|\le\nu$, then

$$\begin{aligned}|\Phi[y](t)|&\le\sum_{i=0}^{m}\nu^i y(\tau^i(t))+\nu^{m+1}|\varphi[y](\tau^{m+1}(t))|\\&\le\sum_{i=0}^{m-p}\nu^i y(\tau^i(t))+\sum_{i=m-p+1}^{m}\nu^i y(\tau^i(t))+\frac{\varepsilon}{3}\\&\le y(\tau^{m-p}(t))\sum_{i=0}^{m-p}\nu^i+y(T)\nu^{m-p+1}\sum_{i=0}^{p-1}\nu^i+\frac{\varepsilon}{3}\\&\le\frac{y(\tau^{-p}(T))}{1-\nu}+\frac{y(T)\nu^{m-p+1}}{1-\nu}+\frac{\varepsilon}{3}<\varepsilon.\end{aligned}$$

This implies that $|\Phi[y](t)|<\varepsilon$ for $t\in[\tau^{-(p+q)}(T),\infty)$ with $|h(t)|\le\nu$ and hence the conclusion follows.

Lemma 5. *Let $m=0,1,2,\ldots$. If t satisfies $t\ge\tau^{-m}(T)$ and $0\le h(t)\le 1$, then*

$$\left|\sum_{i=0}^{m}(-1)^i[h(t)]^i y(\tau^i(t))\right|\le 2y(\tau^m(t)),\quad y\in Y. \tag{3.7}$$

Proof. Let $t\ge\tau^{-m}(T)$ and $0\le h(t)\le 1$. Put

$$A(t)\equiv\sum_{i=0}^{m}(-1)^i[h(t)]^i y(\tau^i(t)).$$

It is easy to see that (3.7) holds for $m=0$ and 1. If $m\ge 3$ is odd, we can rewrite $A(t)$ as

$$\begin{aligned}A(t)=y(t)-&\sum_{j=1}^{(m-1)/2}[h(t)]^{2j-1}[y(\tau^{2j-1}(t))-h(t)y(\tau^{2j}(t))]\\&-[h(t)]^m y(\tau^m(t))\end{aligned}$$

and

$$A(t) = \sum_{j=0}^{(m-1)/2} [h(t)]^{2j}[y(\tau^{2j}(t)) - h(t)y(\tau^{2j+1}(t))].$$

If $m \geq 2$ is even, we can rewrite $A(t)$ as

$$A(t) = y(t) - \sum_{j=1}^{m/2}[h(t)]^{2j-1}[y(\tau^{2j-1}(t)) - h(t)y(\tau^{2j}(t))]$$

and

$$A(t) = \sum_{j=0}^{(m/2)-1} [h(t)]^{2j}[y(\tau^{2j}(t)) - h(t)y(\tau^{2j+1}(t))] + [h(t)]^m y(\tau^m(t)).$$

Since y is nondecreasing on $[T,\infty)$, we see that

$$y(t) - h(t)y(\tau(t)) \leq [1 - h(t)]y(t), \quad t \geq \tau^{-1}(T).$$

Hence, for the case where $m \geq 3$ is odd, we have

$$\begin{aligned}
A(t) &\geq - \sum_{j=1}^{(m-1)/2} [h(t)]^{2j-1}[1 - h(t)]y(\tau^{2j-1}(t)) - [h(t)]^m y(\tau^m(t)) \\
&\geq - \sum_{j=1}^{(m-1)/2} [h(t)]^{2j-1}[1 - h(t)]y(\tau^m(t)) - [h(t)]^m y(\tau^m(t)) \\
&= y(\tau^m(t)) \sum_{i=1}^{m}(-1)^i[h(t)]^i \\
&= - y(\tau^m(t))h(t)\frac{1 - [-h(t)]^m}{1 + h(t)} \geq -2y(\tau^m(t)).
\end{aligned}$$

In the same way, we can show that $A(t) \leq 2y(\tau^m(t))$ for the case where $m \geq 3$ is odd, and that $-2y(\tau^m(t)) \leq A(t) \leq 2y(\tau^m(t))$ for the case where $m \geq 2$ is even.

Lemma 6. *Let $y \in Y$. Then $\lim_{t\to\infty} \Phi[y](t) = 0$.*

Proof. Assume that $\lim_{t\to\infty} \Phi[y](t) = 0$ does not hold. Then we first claim that there is a sequence $\{t_j\}_{j=1}^{\infty}$ such that

$$\text{(3.8)} \qquad \begin{cases} \lim_{j\to\infty} t_j = \infty, \quad \lim_{j\to\infty} \Phi[y](t_j) \text{ exists in } \mathbb{R} \cup \{\infty, -\infty\}\backslash\{0\}, \\ 0 < h(t_j) < 1 \quad \text{for } j \geq 1 \quad \text{and} \quad \lim_{j\to\infty} h(t_j) = 1. \end{cases}$$

By assumption there is a sequence $\{s_j\}_{j=1}^{\infty}$ for which $s_j \to \infty$ and $\Phi[y](s_j) \to c \in \mathbb{R}\cup\{\infty, -\infty\}\backslash\{0\}$ as $j \to \infty$. Since $-1 < \mu \leq h(t) \leq \lambda$ for $t \geq t_0$, there is a subsequence $\{t_j\}_{j=1}^{\infty}$ of $\{s_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} h(t_j) = d \in [\mu, \lambda]$. Lemma 4 implies that $d \geq 1$. It can be shown that $h(t_j) < 1$, $j \geq j_0$ for some j_0. Otherwise, there exists a subsequence $\{\tilde{t}_j\}_{j=1}^{\infty}$ of $\{t_j\}_{j=1}^{\infty}$ such that $h(\tilde{t}_j) \geq 1$ for all j. From (iii) of Lemma 3 and (ii) of Lemma 1, it follows that

$$|c| = \left|\lim_{j\to\infty} \Phi[y](\tilde{t}_j)\right| = \left|\lim_{j\to\infty} \Psi[y](\tilde{t}_j)\right| \leq \lim_{j\to\infty} \eta(\tau^{-1}(\tilde{t}_j)) = 0,$$

which is a contradiction. Since $d \geq 1$, we see that $d = 1$, so that $0 < h(t_j) < 1$, $j \geq j_1$ for some $j_1 \geq j_0$. This proves the existence of $\{t_j\}_{j=1}^{\infty}$ satisfying (3.8).

Suppose that $\{t_j\}_{j=1}^{\infty}$ is a sequence satisfying (3.8). Let $\varepsilon > 0$ be arbitrary. There is an integer $p \geq 1$ such that

$$\eta(t) < \varepsilon, \quad t \geq \tau^{-p-1}(T).$$

There is a number $\delta > 0$ such that if $s_1, s_2 \in [\tau^{-p}(T), \tau^{-(p+1)}(T)]$ with $|s_1 - s_2| < \delta$, then

$$|\Phi[y](s_1) - \Phi[y](s_2)| < \varepsilon. \tag{3.9}$$

Consider the mapping $N : [\tau^{-p}(T), \infty) \longrightarrow \mathbb{N}\cup\{0\}$ such that

$$\tau^{N(t)}(t) \in [\tau^{-p}(T), \tau^{-(p+1)}(T)) \quad \text{for } t \geq \tau^{-p}(T).$$

We note that $\lim_{t\to\infty} N(t) = \infty$. It is easily verified that $\{t_j\}_{j=1}^{\infty}$ has a subsequence $\{u_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \tau^{N(u_j)}(u_j) \quad \text{exists in } [\tau^{-p}(T), \tau^{-(p+1)}(T)].$$

Put $\overline{u} = \lim_{j\to\infty} \tau^{N(u_j)}(u_j)$. Then we find that

$$h(\overline{u}) = \lim_{j\to\infty} h(\tau^{N(u_j)}(u_j)) = \lim_{j\to\infty} h(u_j) = 1.$$

There exists an integer j_0 such that $u_j \geq \tau^{-p}(T)$ and $|\tau^{N(u_j)}(u_j) - \overline{u}| < \delta$ for $j \geq j_0$. From (ii) of Lemma 3, we observe that

$$\begin{aligned}\Phi[y](t) &= y(t) - h(t)\Phi[y](\tau(t)) \\ &= y(t) - h(t)y(\tau(t)) + [h(t)]^2\Phi[y](\tau^2(t)) \\ &= \sum_{i=0}^{m-1}(-1)^i[h(t)]^i y(\tau^i(t)) + (-1)^m[h(t)]^m\Phi[y](\tau^m(t))\end{aligned} \tag{3.10}$$

for $t \geq \tau^{-m+1}(T)$. Since $h(\overline{u}) = 1$, we have

$$\begin{aligned}&|\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| \\ &\leq \left|\sum_{i=0}^{N(u_j)-1}(-1)^i[h(u_j)]^i y(\tau^i(u_j))\right| + \left|\sum_{i=0}^{N(u_j)-1}(-1)^i y(\tau^i(\tau^{-N(u_j)}(\overline{u})))\right| \\ &\quad + \left|[h(u_j)]^{N(u_j)}\Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\tau^{N(u_j)}(\tau^{-N(u_j)}(\overline{u})))\right|.\end{aligned} \tag{3.11}$$

Lemma 5 implies that if $j \geq j_0$, then

$$\left|\sum_{i=0}^{N(u_j)-1} (-1)^i [h(u_j)]^i y(\tau^i(u_j))\right| \leq 2y(\tau^{N(u_j)-1}(u_j)) \tag{3.12}$$
$$\leq 2\eta(\tau^{N(u_j)-1}(u_j)) < 2\varepsilon$$

and

$$\left|\sum_{i=0}^{N(u_j)-1} (-1)^i y(\tau^i(\tau^{-N(u_j)}(\overline{u})))\right| \leq 2y(\tau^{N(u_j)-1}(\tau^{-N(u_j)}(\overline{u}))) \tag{3.13}$$
$$\leq 2\eta(\tau^{-1}(\overline{u})) < 2\varepsilon.$$

From (iii) of Lemma 3, (ii) of Lemma 1 and the fact that $h(\overline{u}) = 1$, it follows that

$$|\Phi[y](\overline{u})| = |\Psi[y](\overline{u})| \leq \eta(\tau^{-1}(\overline{u})) < \varepsilon.$$

Then we observe that

$$\begin{aligned} &|[h(u_j)]^{N(u_j)}\Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\tau^{N(u_j)}(\tau^{-N(u_j)}(\overline{u})))| \\ &\leq |[h(u_j)]^{N(u_j)}||\Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\overline{u})| \\ &\quad + |[h(u_j)]^{N(u_j)} - 1||\Phi[y](\overline{u})| \\ &\leq |\Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\overline{u})| + 2|\Phi[y](\overline{u})| < 3\varepsilon, \quad j \geq j_0, \end{aligned} \tag{3.14}$$

because of (3.9). Combining (3.11)–(3.14), we obtain

$$|\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| < 7\varepsilon, \quad j \geq j_0.$$

This means that

$$\lim_{j\to\infty} |\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| = 0.$$

On the other hand, in view of (iii) of Lemma 3 and (ii) of Lemma 1, we see that

$$\lim_{j\to\infty} |\Phi[y](\tau^{-N(u_j)}(\overline{u}))| \leq \lim_{j\to\infty} \eta(\tau^{-N(u_j)-1}(\overline{u})) = 0.$$

From (3.8) it follows that

$$\lim_{j\to\infty} |\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| \quad \text{exists and is not equal to 0.}$$

This is a contradiction. The proof is complete.

Proposition mentioned in Section 2 follows from Lemmas 3 and 6.

Acknowledgment. The author would like to thank Professor M. Naito for many helpful suggestions.

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