数理解析研究所講究録
1036 巻 1998 年 101-109

Hypoellipticity for Operators of Infinitely Degenerate Egorov Type

NILS DENCKER (University of Lund)
YOSHINORI MORIMOTO (Kyoto University)
TATSUSHI MORIOKA (Osaka University)

§1. Introduction and Result

We study the hypoellipticity for the operator

(1) $$P = D_t + i\alpha(t)\, b(t,\, x,\, D_x) \quad \text{in } \mathbf{R}_t \times \mathbf{R}^n_x,$$

where $i = \sqrt{-1}$ and $\alpha(t)$ is a C^∞ function satisfying

(2) $$\alpha_I := \int_I \alpha(t)dt > 0 \quad \text{for any interval } I \subset \mathbf{R}.$$

Here $b(t,x,\xi) \in C^\infty(\mathbf{R}_t\,,\ S^1_{1,0}(\mathbf{R}^n_x))$ is a classical symbol for any fixed t. We assume the principal symbol b_1 of b is real valued. We denote the coordinates of $T^*(\mathbf{R}_t \times \mathbf{R}^n_x)$ by $(t,x;\tau,\xi)$, $t\,,\tau \in \mathbf{R}$ and $x\,,\xi \in \mathbf{R}^n$. We assume the following conditions (H.1) and (H.2).

(H.1) $(\tau,\ b_1(t,x,\xi))$ satisfies the so-called Hörmander's bracket condition (C.H), that is , for any $\rho \in \operatorname{Char} P$ there exist a positive integer m and $(k(1),k(2),...,k(m)) \in \{0,1\}^m$ such that

$$(H_{r_{k(1)}} \cdots H_{r_{k(m-1)}} r_{k(m)})(\rho) \neq 0 \ ,$$

where $r_0 = \tau$, $r_1 = b_1$ and H_q is the Hamilton vector field of q.

(H.2) $(\partial_t b_1)(t,x,\xi) \geq 0$ for $(t,x,\xi) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$.

Theorem 1. *If P of the form (1) satisfies (H.1) and (H.2) then P is hypoelliptic in $\mathbf{R}_t \times \mathbf{R}^n_x$.*

We can relax the assumption (H.1) by assuming the logarithmic regularity estimate as follows:

(H.3) For any $\varepsilon > 0$ and any compact $K \subset \mathbf{R}_t \times \mathbf{R}^n_x$ there exists a constant $C = C(\varepsilon, K)$ such that

(3) $\quad ||(\log\langle D_x\rangle)u||^2 \leq \varepsilon(||D_t u||^2 + ||b_1(t,x,D_x)u||^2) + C||u||^2 \quad$ for any $u \in C_0^\infty(K)$,

where $\langle\xi\rangle = (2+|\xi|^2)^{1/2}$ and $||\cdot||$ denotes the usual norm of $L^2(\mathbf{R}_t \times \mathbf{R}_x^n)$. We remark that (H.3) follows from (H.1).

Theorem 2. *The operator P of the form (1) is hypoelliptic in $\mathbf{R}_t \times \mathbf{R}_x^n$ if (H.2) and (H.3) are fulfilled. Furthermore, for any $\bar{\rho}_0 = (t_0, x_0; \tau_0, \xi_0) \in T^*(\mathbf{R}_t \times \mathbf{R}_x^n) \setminus 0$ and any real s*

(4) $$v \in \mathcal{E}'(\mathbf{R}_t \times \mathbf{R}_x^n),\ Pv \in H_{s+1}^{\ell oc}(\bar{\rho}_0) \ \Rightarrow\ v \in H_s^{\ell oc}(\bar{\rho}_0),$$

where $v \in H_s^{\ell oc}(\bar{\rho}_0)$ means that there exists a classical symbol $a(t,x,\tau,\xi) \in S_{1,0}^0(\mathbf{R}_{t,x}^{n+1})$ such that $a \neq 0$ in a conic neighborhood of $\bar{\rho}_0$ and $a(t,x,D_t,D_x)v \in H_s(\mathbf{R}_t \times \mathbf{R}_x^n)$.

We give some historical remarks concerning our result. First we recall the definition of subelliptic operators. Namely, a classical pseudodifferential operator P of order m is called subelliptic with loss of δ derivatives if $0 < \delta < 1$ and if

$$v \in \mathcal{E}'(\mathbf{R}^{n+1}),\ \ Pv \in H_s^{\ell oc}(\mathbf{R}^{n+1}) \ \Rightarrow\ v \in H_{s+m-\delta}^{\ell oc}(\mathbf{R}^{n+1}).$$

The characterization of subelliptic operators was laboriously studied by Egorov[3] and it was completely proved by Hörmander[4] (see also [5] Chapter27) that P is subelliptic if and only if the principal symbol p of P satisfies the Nirenberg-Treves condition $(\overline{\Psi})$ and (C.H) condition with $r_0 = \mathrm{Re}\, p$ and $r_1 = \mathrm{Im}\ p$. After multiplication with elliptic operator and a canonical transformation, the principal symbol p has the form microlocally

$$p = \tau + iq(t,x,\xi),\ \ q(t,x,\xi) \text{ real valued}$$

and for this form the condition $(\overline{\Psi})$ is stated as

(5) $$q(t,x,\xi) > 0 \text{ and } s > t \ \Rightarrow\ q(s,x,\xi) \geq 0.$$

It follows from (H.2) that P of the form (1) satisfies the condition (5) (and hence $(\overline{\Psi})$). We remark that $(\overline{\Psi})$ is necessary for P of the form (1) to be hypelliptic because the adjoint operator P^* is then locally solvable (see [5] Theorem 27.4.7). In the theory of subelliptic operators, the operator

(6) $$D_t + it^{2k}(D_{x_1} + t^{2j+1}x_1^{2m}|D_x|) \quad \text{in} \quad \mathbf{R}^3, \quad (k,j,m \text{ non-negative integers})$$

is an important model because, roughly speaking, any subelliptic operators can be reduced to this opeartor and the Mizohata one after several microlocalization arguments. So we shall call the operator of (6), Egorov type, even in the case where $t^{2k}, t^{2j+1}x_1^{2m}$ are replaced by other (infinitely) degenerate functions. It should be noted that almost all contents of subelliptic theory are required in order to prove the subelliptic estimate for the simple model (6).

Our Theorem 1 shows that the operator

(7) $$P_1 = D_t + i\alpha(t)(D_{x_1} + t^{2j+1}x_1^{2m}|D_x|) \quad \text{in} \quad \mathbf{R}^3$$

is hypoelliptic if $\alpha(t) > 0$ for $t \neq 0$. In [9],[10], the hypoellipticity for infinitely degenerate Egorov type operators was studied by the second author, but it was not shown there that the operator P_1 is hypoelliptic when $\alpha(t)$ has a zero of infinite order at $t = 0$. The difficulty comes from the fact that L^2 *a priori* estimate seems to be not satisfied for this P_1, in general. Indeed, Lerner [6] showed that L^2 *a priori* estimate does not hold for some version of infinitely degenerate Egorov type operators though it satisfies $(\overline{\Psi})$, (whose adjoint operator is a counter example to L^2 local solvability of operators satisfying (Ψ) condition). Recently, the first author [1] showed that Lerner's counter example is locally solvable with loss of at most two derivatives and developped the method in [2]. We shall prove Theorem 2 by using the fundamental estimate given in [2], instead of L^2 *a priori* estimate. The proof of Theorem 2 in the next section is based on a method similar to that of [11] Theorem 8.

§2. Proof of Theorem 2

We note that P is hypoelliptic in $\Omega = \{(t,x) \in \mathbf{R}_t \times \mathbf{R}_x^n; \alpha(t) > 0\}$, more precisely, P is microhypoelliptic at any $\bar{\rho} = (t_0, x_0; \tau_0, \xi_0) \in T^*(\Omega) \setminus 0$. In fact, it follows from (H.2) and Fefferman-Phong inequality that for any compact $K \subset \Omega$ there exists a $C_K > 0$ such that

$$\begin{aligned} ||Pu||^2 &= ||D_t u||^2 + ||\alpha b u||^2 + 2\text{Re } (\alpha(\partial_t b)u, u) + 2\text{Re } ((\partial_t \alpha/\alpha)(\alpha b)u, u) \\ &\geq ||D_t u||^2 + \frac{1}{2}||\alpha b u||^2 - C_K ||u||^2, \quad u \in C^\infty(K), \end{aligned}$$

where we used Schwartz's inequality to estimate the fourth term in the middle, in view of $\alpha \geq \exists c_K > 0$ on K. Together with (H.3), the above estimate shows that for any $\varepsilon > 0$ and any compact $K \subset \Omega$ there exists another $C(\varepsilon, K) > 0$ such that

$$|| \left(\log \sqrt{D_t^2 + |D_x|^2 + 2}\right) u||^2 \leq \varepsilon ||Pu||^2 + C(\varepsilon, K)||u||^2, \quad u \in C^\infty(K).$$

By means of Theorem 1 of [7] and its proof, we see the micro-hypoellipticity of P at any $\bar{\rho} = (t_0, x_0; \tau_0, \xi_0) \in T^*(\Omega) \setminus 0$, namely,

(2.1) $$v \in \mathcal{E}'(\mathbf{R}_{t,x}^{n+1}),\ Pv \in H_s^{\ell oc}(\bar{\rho}_0) \;\Rightarrow\; v \in H_s^{\ell oc}(\bar{\rho}_0)$$

It suffices to show (4) of Theorem 2 in the case where $\bar{\rho} = (t_0, x_0; 0, \xi_0)$ with $\alpha(t_0) = 0$, because

(2.2) $$P \text{ is microlocally elliptic in } \{\, (t, x; \tau, \xi)\, ;\, \tau \neq 0\}.$$

For the brevity we assume $(t_0, x_0) = (0, 0)$ and $|\xi_0| = 1$. Take $\Phi(\tau, \xi) \in S^0_{1,0}(\mathbf{R}_t \times \mathbf{R}^n_x)$ such that $\Phi = 1$ in $\{|\tau| \le \delta|\xi|\}$ and $\Phi = 0$ in $\{|\tau| \ge 2\delta|\xi|\}$ for a small $\delta > 0$, which will be chosen later on. In order to cut the space $\mathbf{R}^n_x$ we choose an $h(x) \in C_0^\infty(\mathbf{R}^n_x)$ function such that $0 \le h \le 1$, $h(x) = 1$ for $|x| \le 1/5$ and $h(x) = 0$ for $|x| \ge 7/24$, and set $h_\delta(x) = h(x/\delta)$. For the conical cutting in $\mathbf{R}^n_\xi$, we define the following:

Definition. For $\delta > 0$ and $\xi_0 \in \mathbf{R}^n$ $(|\xi_0| = 1)$ we say that a function $\psi(\xi) \in C^\infty(\mathbf{R}^n)$ belongs to Ψ_{δ,ξ_0} if $0 \le \psi \le 1$ satisfies

$$\begin{cases} \psi(\xi) = 1 & \text{for } |\xi/|\xi| - \xi_0| \le \delta/12 \text{ and } |\xi| \ge 2/3, \\ \psi(\xi) = 0 & \text{for } |\xi/|\xi| - \xi_0| \ge \delta/10 \text{ or } |\xi| \le 1/2, \\ \psi(\xi) = \psi(\xi/\lambda) & \text{for } 0 < \lambda \le 1 \text{ and } |\xi| \ge 1. \end{cases}$$

Let $v \in \mathcal{E}'(\mathbf{R}^{n+1}_{t,x})$ and $Pv \in H^{\ell oc}_{s+1}(\bar{\rho}_0)$. If $\psi(\xi) \in \Psi_{70\delta,\xi_0}$ and $\delta > 0$ is sufficiently small, then we can find $\chi(t) \in C_0^\infty(\mathbf{R})$ such that $\chi = 1$ in a neighborhood of $t = 0$, supp $\chi' \subset \{t;\, \alpha(t) > 0\}$ and

$$\psi(D_x)h_{10\delta}(x)\chi(t)\Phi(D_t, D_x)Pv \in H_{s+1}. \tag{2.3}$$

Note that

$$\psi h_{10\delta}(x)P\chi\Phi v = \psi h_{10\delta}(x)\chi\Phi P v - \psi h_{10\delta}(x)[P,\chi]\Phi v - \psi h_{10\delta}(x)\chi[P,\Phi]v,$$

and that the second and third terms in the right hand side belong to H_{s+1} and H_{s+2}, respectively, by means of (2.1) and (2.2). If $w = \chi\Phi v$ then it follows from (2.3) that

$$\langle D_x\rangle^{s+1}\psi(D_x)h_{10\delta}(x)Pw \in L^2(\mathbf{R}_t \times \mathbf{R}^n_x). \tag{2.4}$$

Since $v \in H_{-N}$ for a large $N > 0$,

$$\langle D_x\rangle^{-N}\Phi v \in L^2(\mathbf{R}_t \times \mathbf{R}^n_x) \text{ and hence } \langle D_x\rangle^{-N} w \in L^2(\mathbf{R}_t \times \mathbf{R}^n_x). \tag{2.5}$$

To complete the proof of (4), we shall show for a suitable $\tilde{\psi}(\xi) \in \Psi_{\delta,\xi_0}$

$$\langle D_x\rangle^{s}\tilde{\psi}(D_x)h_\delta(x)w \in L^2(\mathbf{R}_t \times \mathbf{R}^n_x). \tag{2.6}$$

To this end, we use the Weyl calculus of pseudodifferential operator and by changing the lower order terms of b, if necessary, we can write

$$P = D_t + i\alpha(t)b^w(t, x, D_x),$$

where $b^w(t, x, D_x)$ is a pseudo-differential operator with a Weyl symbol, that is,

$$b^w(t, x, D_x)u = (2\pi)^{-n}\int e^{i(x-y)\xi}b(t, \frac{x+y}{2}, \xi)u(y)dyd\xi, \quad u \in \mathcal{S}(\mathbf{R}^n).$$

Furthermore, we consider the microlocalized operator at $\rho_0 = (0, \xi_0)$ with a parameter $0 < \lambda \leq 1$ as follows:

$$P_\lambda^w = D_t + i\alpha(t) b_\lambda^w(t, x, D_x),$$

where $b_\lambda^w(t, x, D_x)$ is a pseudo-differential operator with a Weyl symbol

$$b_\lambda(t, x, \xi) = b(t, x, \xi) h_{100\delta}(\lambda\xi - \xi_0).$$

We apply Theorem A.2 of Dencker [2] by setting $A(t) = \alpha(t)$ and $B(t) = b_\lambda^w(t, x, D_x)$. Since (A.3) of [2] follows from (H.2), we have

Lemma 1. *There exists constants C_0 and $T_0 > 0$ independent of $0 < \lambda \leq 1$ such that*

$$||u||^2 \leq C_0 \left\{ \text{Im } (P_\lambda^w u, b_\lambda^w u) + ||P_\lambda^w u||^2 \right\} \tag{2.7}$$

for any $u(t, x) \in \mathcal{S}(\mathbf{R}_t \times \mathbf{R}_x^n)$ having support where $|t| \leq T_0$.

We may assume supp $\chi \subset \{|t| < T_0\}$ by taking a small $\delta > 0$. Let $H_\delta(x, D_x; \lambda)$ denote the usual pseudodifferential operators with symbol $H_\delta = h_\delta(x) h_\delta(\lambda\xi - \xi_0)$. By (H.3) we have

Lemma 2. *Let $\delta > 0$ be a number chosen in the above and let T_0 be the same as in Lemma 1. For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ independent of $0 < \lambda \leq 1$ such that*

$$\begin{aligned} (\log \lambda)^2 ||\sqrt{\alpha} u||^2 &\leq \varepsilon \left\{ \text{Im } (P_\lambda^w u, b_\lambda^w u) + ||P_\lambda^w u||^2 \right\} \\ &\quad + C_\varepsilon \left\{ ||\sqrt{\alpha} u||^2 + \lambda^{-1} ||(1 - H_{20\delta}(x, D_x; \lambda)) u||^2 \right\}, \end{aligned} \tag{2.8}$$

for any $u(t, x) \in \mathcal{S}(\mathbf{R}_t \times \mathbf{R}_x^n)$ having support where $|t| \leq T_0$.

Proof. Substitute $\sqrt{\alpha(t)} H_{40\delta}(x, D; \lambda) u$ into (3). Then we have

$$\begin{aligned} (\log \lambda)^2 ||h_{40\delta}(\lambda D - \xi_0) h_{40\delta}(x) \sqrt{\alpha} u||^2 &\leq \varepsilon (||D_t u||^2 + ||\sqrt{\alpha} b_\lambda^w(t, x, D_x) u||^2 + C ||u||^2) \\ &+ C_\varepsilon ||\sqrt{\alpha} u||^2, \end{aligned}$$

because λ^{-1} is equivalent to $|\xi|$ on supp $h_{40\delta}(\lambda\xi - \xi_0)$ and $(\sqrt{\alpha})'$ is bounded. Note that supp $H_{20\delta} \cap$ supp $(1 - H_{40\delta}) = \emptyset$ and

$$||D_t u||^2 \leq 2 \left(||P_\lambda^w u||^2 + ||\alpha b_\lambda^w u||^2 \right).$$

Since it follows from (H.2) that

$$(\alpha b_\lambda^w u, b_\lambda^w u) = \mathrm{Im}\ (P_\lambda^w u, b_\lambda^w u) - \frac{1}{2}\mathrm{Re}\ ((\partial_t b_\lambda)^w u, u) \le \mathrm{Im}\ (P_\lambda^w u, b_\lambda^w u) + C||u||^2$$

we have the desired estimate (2.8) by using (2.7). Q.E.D.

Let $\varphi(x,\xi;\lambda) = 1 - H_{2\delta}(x,\xi,\lambda)$. It is clear that

$$\left\{ \begin{array}{l} \varphi = 0 \text{ on supp } H_\delta(x,\xi;\lambda), \\ \varphi = 1 \text{ outside of supp } H_{2\delta}(x,\xi;\lambda) \end{array} \right. \tag{2.9}$$

For an integer $\ell > 2s + 2N + 4$ we set

$$K(x,\xi;\lambda) = \lambda^{\ell\varphi(x,\xi;\lambda)} = e^{\ell\varphi(x,\xi;\lambda)\log\lambda}.$$

If $K_\beta^\alpha(x,\xi;\lambda)$ denotes $\partial_\xi^\alpha D_x^\beta K(x,\xi;\lambda)$ and if $0 < \lambda \le 1$ then

$$|\log\lambda|^{-|\alpha|-|\beta|}\lambda^{-|\alpha|}K_\beta^\alpha(x,\xi;\lambda)h_{10\delta}(\lambda\xi - \xi_0)$$

belongs to a bounded set of $S(1,g_0)$, where $g_0 = (\log\langle\xi\rangle)^2|dx|^2 + (\log\langle\xi\rangle)^2\langle\xi\rangle^{-2}|d\xi|^2$ and $\langle\xi\rangle^2 = 2 + |\xi|^2$. It follows from (2.9) that

$$\lambda^{-\ell}\sigma^w([K^w(x,D,\lambda), h_{10\delta}(x)])h_{10\delta}(\lambda\xi - \xi_0), \quad \lambda^{-\ell}\sigma^w([K^w(x,D,\lambda), h_{10\delta}(\lambda D - \xi_0)])$$

belong to a bounded set of $S(1,g_0)$ uniformly for $0 < \lambda \le 1$. By the same reason we have the following formulae modulo L^2 bounded operator uniformly for $0 < \lambda \le 1$

$$\begin{aligned} \lambda^{-\ell+1}P_\lambda^w K^w H_{10\delta} &\equiv \lambda^{-\ell+1}H_{10\delta}PK^w \\ &\equiv \lambda^{-\ell+1}\{H_{10\delta}K^w P + i\alpha(t)H_{10\delta}[b^w, K^w]\} \\ &\equiv \lambda^{-\ell+1}\{K^w h_{10\delta}(\lambda D - \xi_0)h_{10\delta}(x)P + i\alpha(t)[b^w, K^w]H_{10\delta}\}. \end{aligned} \tag{2.10}$$

It follows from the expansion formula of the Weyl calculus (see Theorem 18.5.4 of [5]) that

$$\left\{ \begin{array}{l} \lambda^{-1}(\log\lambda)^{-2}K^{-1}h_{100\delta}(\lambda\xi - \xi_0)\times \\ \{\ \sigma^w([b^w, K^w]) - i\ell\ \log\lambda\ \sigma^w((H_\varphi b)^w K^w)\ \} \\ \in\ S(1,g_0) \text{ uniformly for } 0 < \lambda \le 1. \end{array} \right. \tag{2.11}$$

Here $\sigma^w(A)$ denotes the Weyl symbol of pseudodifferential operators of A. It follows from (2.10) and (2.11) that for any $u \in \mathcal{S}(\mathbf{R}_t \times \mathbf{R}^n_x)$

$$\begin{aligned} \operatorname{Im}\,(P^w_\lambda K^w H_{10\delta}u, b^w_\lambda K^w H_{10\delta}u) \;\le\;& \operatorname{Im}(\bar{b}^w_\lambda K^w h_{10\delta}(\lambda D-\xi_0)h_{10\delta}(x)Pu, K^w H_{10\delta}u) \\ -\;& \ell(\log\lambda)\operatorname{Im}\,((H_\varphi b)^w\alpha(t)K^w H_{10\delta}u, b^w_\lambda K^w H_{10\delta}u) \\ +\;& C_\ell\{(\log\lambda)^2\|\sqrt{\alpha}K^w H_{10\delta}u\|^2 + \lambda^{2s+1}\|\Lambda^{-N}u\|^2\}, \end{aligned}$$

where $\Lambda = \langle D_x\rangle$. Use the Schwartz inequality in the first term of the right hand side. Then for any $\mu > 0$ there exists a $C_\mu > 0$ such that

$$\begin{aligned} &\operatorname{Im}(\bar{b}^w_\lambda K^w h_{10\delta}(\lambda D-\xi_0)h_{10\delta}(x)Pu, K^w H_{10\delta}u) \le \mu\|K^w H_{10\delta}u\|^2 \\ &\quad + C_\mu\left\{\|\lambda^{-1}h_{10\delta}(\lambda D-\xi_0)h_{10\delta}(x)Pu\|^2 + \lambda^{2s+1}\|\Lambda^{-N}u\|^2\right\}. \end{aligned}$$

Since the principal symbol of $\bar{b}^w_\lambda(H_\varphi b)^w$ is real valued, we also obtain

$$\begin{aligned} &|\ell(\log\lambda)\operatorname{Im}\,((H_\varphi b)^w\alpha K^w H_{10\delta}u, b^w_\lambda K^w H_{10\delta}u)| \le \\ &\mu\|K^w H_{10\delta}u\|^2 + C_\mu\left\{(\log\lambda)^2\|\alpha K^w H_{10\delta}u\|^2 + \lambda^{2s+1}\|\Lambda^{-N}u\|^2\right\}. \end{aligned}$$

Hence we see that

$$\begin{aligned} \operatorname{Im}\,(P^w_\lambda K^w H_{10\delta}u, b^w_\lambda K^w H_{10\delta}u) \;\le\;& 2\mu\|K^w H_{10\delta}u\|^2 \\ +\;& C_\mu\left\{\|\lambda^{-1}h_{10\delta}(\lambda D-\xi_0)h_{10\delta}(x)Pu\|^2\right. \\ +\;& (\log\lambda)^2\,\|\sqrt{\alpha}K^w H_{10\delta}u\|^2 \\ +\;& \left.\lambda^{2s+1}\|\Lambda^{-N}u\|^2\right\}. \end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned} \|P^w_\lambda K^w H_{10\delta}u\|^2 \;\le\;& 2\|h_{10\delta}(\lambda D-\xi_0)h_{10\delta}(x)Pu\|^2 \\ +\;& C\left\{(\log\lambda)^2\,\|\alpha K^w H_{10\delta}u\|^2\right. \\ +\;& \left.\lambda\|K^w H_{10\delta}u\|^2 + \lambda^{2s+1}\|\Lambda^{-N}u\|^2\right\}. \end{aligned} \tag{2.13}$$

Let $u \in \mathcal{S}(\mathbf{R}_t \times \mathbf{R}^n_x)$ satisfy

$$\operatorname{supp} u \subset \{|t| \le T_0\}. \tag{2.14}$$

Substitute $K^w H_{10\delta}u$ into (2.7) and (2.8). Choose $\mu = 1/(4C_0)$ in (2.12). In view of (2.12) and (2.13), there exists a small $\lambda_0 > 0$ such that

$$\begin{aligned} ||K^w H_{10\delta}u||^2 &\le C(||\lambda^{-1}h_{10\delta}(\lambda D_x - \xi_0)h_{10\delta}(x)Pu||^2 \\ &+ \lambda^{2s+1}||\Lambda^{-N}u||^2) \quad \text{if } 0 < \lambda < \lambda_0 \end{aligned}$$

Since it follows from (2.9) that the symbol of $K^w H_{10\delta} = 1$ on supp H_δ, we have for $0 < \lambda < \lambda_0$

$$\begin{aligned} &||h_\delta(\lambda D_x - \xi_0)h_\delta(x)u||^2 \le \\ &\quad C\left(||\lambda^{-1}h_{10\delta}(\lambda D_x - \xi_0)h_{10\delta}(x)Pu||^2 + \lambda^{2s+1}||\Lambda^{-N}u||^2\right). \end{aligned}$$

Multiplying $\lambda^{-2s}(1+\kappa\lambda^{-1})^{-2(N+s+2)}$ with a parameter $\kappa > 0$ by both sides, we have

$$\begin{aligned} &||h_\delta(\lambda D_x - \xi_0)(1+\kappa\Lambda)^{-(N+s+2)}\Lambda^s h_\delta(x)u||^2 \le \\ &\quad C(||h_{10\delta}(\lambda D_x - \xi_0)(1+\kappa\Lambda)^{-(N+s+2)}\Lambda^{s+1}h_{10\delta}(x)Pu||^2 + \lambda||\Lambda^{-N}u||^2) \end{aligned}$$

because λ^{-1} is equivalent to $|\xi|$ on supp $h_\delta(\lambda\xi - \xi_0)$. Integrate λ from 0 to λ_0 after dividing both sides by λ. Then by means of Proposition 1.7 of [8] we have for suitable $\psi_\delta(\xi) \in \Psi_{\delta,\xi_0}$ and $\tilde{\psi}_\delta(\xi) \in \Psi_{70\delta,\xi_0}$,

$$\begin{aligned} &||(1+\kappa\Lambda)^{-(N+s+2)}\Lambda^s\psi_\delta(D_x)h_\delta(x)u||^2 \le \\ &\quad C(||(1+\kappa\Lambda)^{-(N+s+2)}\Lambda^{s+1}\tilde{\psi}_\delta(D_x)h_{10\delta}(x)Pu||^2 + ||\Lambda^{-N}u||^2). \end{aligned}$$

Since $w = \chi\Phi v$ satisfies (2.5), one can find a sequence $\{\tilde{u}_j\}$ in $\mathcal{S}(\mathbf{R}_t \times \mathbf{R}_x^n)$ satisfying $\Lambda^{-N}\tilde{u}_j \to \Lambda^{-N}\Phi v$ in $L^2(\mathbf{R}_t \times \mathbf{R}_x^n)$, $(j \to \infty)$ and supp $\hat{\tilde{u}}_j \subset \{|\tau| < |\xi|\}$. If $u_j = \chi(t)\tilde{u}_j$ then u_j satisfies (2.14) and

$$\Lambda^{-N}u_j \to \Lambda^{-N}w \text{ and } \Lambda^{-(N+1)}Pu_j \to \Lambda^{-(N+1)}Pw \text{ in } L^2(\mathbf{R}_t \times \mathbf{R}_x^n),\ (j \to \infty)$$

because $\Lambda^{-(N+1)}D_t u_j = (D_t\chi)\Lambda^{-(N+1)}\tilde{u}_j + \chi(\Lambda^{-(N+1)}D_t\tilde{u}_j)$. Letting $j \to \infty$ in the above estimate with $u = u_j$, we get for any fixed $\kappa > 0$

$$||(1+\kappa\Lambda)^{-(N+s+2)}\Lambda^s\psi_\delta(D_x)h_\delta(x)w||^2 \le C(||\Lambda^{s+1}\tilde{\psi}_\delta(D_x)h_{10\delta}(x)Pw||^2 + ||\Lambda^{-N}w||^2)$$

because of (2.4) and (2.5). Letting $\kappa \to 0$ we get (2.6), and so (4) of Theorem 2. For an open conic ω in $T^*(\mathbf{R}^{n+1})$ we say $u \in H_s(\omega)$ if $u \in H_s^{\ell oc}(\bar{\rho})$ for any $\bar{\rho} \in \omega$. It follows from (4) and the usual covering arguments that for any open conic sets ω_0, ω with $\overline{\omega_0} \subset \omega$

$$Pu \in H_{s+1}(\omega) \quad \Rightarrow \quad u \in H_s(\omega_0)\ .$$

This shows the microhypoellipticity of P. Thus the proof of Theorem 2 is completed.

References

[1] N. Dencker, The solvability of non L^2 solvable operators. Journées "Équations aux dérivées partielles" St. Jean-de-Monts. (1996).

[2] N. Dencker, A class of solvable operators, "Geometrical Optics and Related Topics"(F. Colombini and N. Lerner, eds.), Birkhäuser, Boston, Mass., USA, (1997), 141-178.

[3] Yu. V. Egorov, Subelliptic operators, Russian Math. Surveys **30:2** (1975), 59-118, **30:3** (1975),55-105.

[4] L. Hörmander, Subelliptic operators, Seminar on Singularities of Solution of Linear Partial Differential Equations, Princeton University Press 1979, 3-49.

[5] L. Hörmander The analysis of linear partial differential operators I-IV., Springer Verlag 1983-1985

[6] N. Lerner, Non-solvability in L^2 for a first order operator satisfying condition (Ψ). Ann. of Math., **139**, (1994), 363-393.

[7] Y. Morimoto, Criteria for hypoellipticity of differential operators, Publ. RIMS Kyoto Univ., **22** (1986), 1129-1154.

[8] Y. Morimoto, A criterion for hypoellipticity of second order differential operators, Osaka J. Math., **24** (1987), 651-675.

[9] Y. Morimoto, Hypoelliptic operators of principal type with infinite degeneracy. Tsukuba J. Math.,**19**, (1995), 187-200.

[10] Y. Morimoto, Local solvability and hypoellipticity for pseudo-differential operators of Egorov type with infinite degeneracy. Nagoya Math. J., **139**, (1995), 151-171.

[11] Y. Morimoto and T. Morioka, The positivity of Schrödinger operators and the hypoellipticity of second order degenerate elliptic operators, Bull. Sc. Math. 2^e série, **121**, (1997), 507-547.

[12] L. Nirenberg and F. Treves, On local solvability of linear partial differential equations. Part I. Necessary conditions. Part II. Sufficient conditions, Correction. Comm. Pure Appl. Math., **23**, (1970), **23**, (1970), 1-38 and 459-509, **24**, (1971), 279-288.