数理解析研究所講究録
1036 巻 1998 年 139-159

MAXIMAL FUNCTIONS INEQUALITIES AND NAVIER-STOKES

MARCO CANNONE AND FABRICE PLANCHON

ABSTRACT. We present a survey of different algorithms for constructing global mild solutions in $\mathcal{C}([0,T);L^3(\mathbb{R}^3))$ to the Cauchy problem for the Navier-Stokes equations with an external force. A particular attention will be devoted to a method originally introduced by C. P. Calderón and involving some maximal function inequalities.

INTRODUCTION

There is a rich literature on the existence and uniqueness of mild strong local or global solutions of the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3$. The basic approach to tackling the problem is the following. One first transforms the Navier-Stokes equations, with the unknown velocity v and pressure p and initial velocity v_0 and external force ϕ,

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -(v\cdot\nabla)v - \nabla p + \phi \\ \nabla \cdot v = 0 \\ v(0) = v_0 \end{cases} \tag{0.1}$$

into the mild integral equation

$$v(t) = S(t)v_0 + B(v,v)(t) + \int_0^t S(t-s)\mathbb{P}\phi(s)ds, \tag{0.2}$$

where

$$B(v,u)(t) = -\int_0^t S(t-s)\mathbb{P}\nabla\cdot(v\otimes u)(s)ds, \tag{0.3}$$

$\mathbb{P}$ and S being respectively the projection onto the divergence free vector field and the heat semi-group. Then it is customary to obtain the existence and uniqueness of strongly continuous global ($T=\infty$) or local ($T<\infty$) solutions $v(t,x)\in\mathcal{C}([0,T);X)$ of (0.2), with X being an abstract Banach space, by means of the standard contraction algorithm. Of course, the main difficulty in applying such an algorithm is to establish, *a priori*, the bicontinuity of the bilinear operator $B(v,u)(t)$ in $\mathcal{C}([0,T];X) \times \mathcal{C}([0,T];X) \longrightarrow \mathcal{C}([0,T];X)$.

A part from some particular function spaces [24,25,7,26], it is very hard (and in same cases even not true [29]) to establish such a bicontinuity property for the the bilinear operator $B(v,u)$ in the case of a so-called [5] *limit space* $\mathcal{L}$ for the Navier-Stokes equations, whose norm is invariant under the transformation $f(\cdot) \longrightarrow \lambda f(\lambda\cdot)$, $\forall\lambda > 0$. Nevertheless, it was shown by several authors that even if the bilinear operator $B(v,u)(t)$ turns out *not* to be bicontinuous in a certain *limit space* $\mathcal{C}([0,T];\mathcal{L})$, this would not *necessarily* imply non-existence of mild solutions $v(t,x) \in \mathcal{C}([0,T];\mathcal{L})$ for the Navier-Stokes equations.

This fact was known since the papers by T. Kato and H. Fujita published in the sixties in which the *limit space* $\dot{H}^{\frac{1}{2}}(\Omega)$ in both a bounded domain Ω [22] or in the whole space $\mathbb{R}^3$ [16] is considered. The case of the Lebesgue *limit space* $L^3(\Omega)$, when Ω is the half space $\mathbb{R}^3_+$, was studied, on the other hand, by F. Weissler at the end of 1979 [37]. Two years later, Y. Giga and T. Miyakawa [19] considered the case $L^3(\Omega)$ for an arbitrary bounded domain Ω. Finally, in 1984, T. Kato published a celebrated paper [20] obtaining, as well, the existence of a strongly continuous solution to the Navier-Stokes equations with values in $L^3(\mathbb{R}^3)$. The idea pursued by all those authors is very clever, and easily leads in the case of $\mathbb{R}^3$ to the existence of a solution in $\mathcal{C}([0,T];L^3(\mathbb{R}^3))$ which is *local* for arbitrary initial data and *global* for small initial data in $L^3(\mathbb{R}^3)$. But, let us point it out from now, the uniqueness in $\mathcal{C}([0,T];L^3(\mathbb{R}^3))$ of such a solution was not known until the recent fundamental contribution of [14,15]. The reason will be clear in a while.

More explicitly, following [5], let us recall how one can circumvent the problem of the possible non-continuity of $B(v,u)(t)$ in $\mathcal{C}([0,T];L^3(\mathbb{R}^3))$ by showing that $B(v,u)(t)$ is nevertheless continuous in a suitable auxiliary space. More exactly, this space $\mathcal{K}_q$ is made up of the functions $v(t,x)$ such that

$$t^{\frac{\alpha}{2}} v(t,x) \in \mathcal{C}([0,T];L^q(\mathbb{R}^3)) \tag{0.4}$$

and

$$\lim_{t\longrightarrow 0} t^{\frac{\alpha}{2}} \|v(t)\|_q = 0 \tag{0.5}$$

q being fixed with $3 < q < \infty$ and $\alpha = \alpha(q) = 1 - \frac{3}{q}$.

It is easy to observe that not only is the bilinear operator $B(v,u)(t)$ bicontinuous in this auxiliary space, but also $S(t)v_0 \in \mathcal{K}_q$ as long as $v_0 \in L^3(\mathbb{R}^3)$ and that, if (0,4) and (0.5) are satisfied for *a certain* $3 < q < \infty$, then any solution of the integral mild equation (0.2) belongs to $\mathcal{K}_q$ for *all* q $(3 < q < \infty)$. Moreover, that is more important, it belongs to the natural space $\mathcal{N} = \mathcal{C}([0,T];L^3(\mathbb{R}^3))$ and tends strongly to v_0 in $L^3(\mathbb{R}^3)$. All this enables one to deduce, by means of a standard contraction procedure applied in a *certain* $\mathcal{K}_q$ $(3 < q < \infty)$, an existence theorem for local and global mild solutions for the Navier-Stokes equations with small initial data in $L^3(\mathbb{R}^3)$ [20] and appropriate smallness and regularity conditions on the external force [19].

With regard to this last point, we have to remark that, in the literature concerning the existence theory of strong solutions to the *non-stationary* Navier-Stokes equations, the external force ϕ was very often disregarded, apart from the initial works of T. Kato and H. Fujita [16,22], the papers by Y. Giga and T. Miyakawa [19,27] and a interesting result by J. Y. Chemin [9]. Our aim is, among others, to restate in the following pages this point of view.

An important property, that was proved very recently by G. Furioli, P.G. Lemarié and E. Terraneo [14,7] by means of Besov type estimates, is, as we announced, the uniqueness of strong solutions in the natural space $\mathcal{C}([0,T); L^3(\mathbb{R}^3))$ and, more generally, in the natural space $\mathcal{C}([0,T); \mathcal{L})$ constructed by means of a *limit* space $\mathcal{L}$ [15]. Before that result, in fact, the uniqueness of the aformentioned solutions was only avaliable in the "ad hoc" auxiliary spaces, in which the point fixed algorithm turns out be to well-defined. To be more explicit, in the case of $L^3(\mathbb{R}^3)$, the uniqueness was known under the supplementary condition (0.5) [see 21].

The aim of this paper is to provide the reader with two alternative algorithms for the study of the mild integral equation (0.2) with initial data in $L^3(\mathbb{R}^3)$ and an external force in an appropriate space, which are different from the one we described so far. In particular, new conditions —to be satisfied by the external force— will be derived here. In the following pages we will limit ourselves to stating our main results, their proofs and more precise results will be presented in [8].

The first algorithm is very simple and reads as follows. Let us introduce the auxiliary space $\mathcal{M}$ defined by the finiteness of the following norm

$$|||v|||_{\mathcal{M}} = \| \sup_{0<t<T} |v(t)| \|_3 \tag{0.6}$$

i.e. the norm which is obtained after interchanging the sup and the L^3 norms in the so-called natural norm

$$|||v|||_{\mathcal{N}} = \sup_{0<t<T} \|v(t)\|_3 \tag{0.7}$$

of $\mathcal{C}([0,T); L^3(\mathbb{R}^3))$.

This idea of interchanging time and space in the mixed norms was originally introduced in a series of papers by C. P. Calderón [2,3,4] and was independently used by one of the autors [5] with the same aim of carring out abstract contraction mapping theorems for the Navier-Stokes equations. In fact, it is quite easy to prove that the bilinear operator $B(v,u)(t)$ is bicontinuous in $\mathcal{M}$. By itself, this property is very surprising because $\mathcal{M}$ is a limit space, which means that its norm is invariant under the similarity transformation $\lambda v(\lambda^2 t, \lambda x)$. For instance, as we already remarked, it is not known whether or not the same holds true in $\mathcal{N}$. Moreover, a simple application of the maximal function inequality implies that $S(t)v_0 \in \mathcal{M}$ if and only if $v_0 \in L^3(\mathbb{R}^3)$. Finally, as in [2,3,4,5], the usual contraction mapping argument can be applied to deduce an existence and uniqueness theorem of local mild solutions for the Navier-Stokes equations with initial data in $L^3(\mathbb{R}^3)$ that are global if the data and the external force are small in appropriate spaces. Here the condition on the external force will be of integral type both in time and space variables. Further, we will be able to reach to corresponding limit exponent $q = \frac{3}{2}$ in the space variable for the external force, as in the case for the stationary Navier-Stokes equations [28].

The second method for solving the Navier-Stokes equations in $L^3(\mathbb{R}^3)$ that we will present here is based on an remark contained in Giga [18] and in Kato [20]. The reader will find in [23] a similar approach in a more general framework, involving, in fact, a larger class of functional spaces. Here the idea is to consider the auxiliary space $\mathcal{G}_q$ defined by the supplementary condition that $v(t) \in L^{\frac{2}{\alpha}}([0,T); L^q(\mathbb{R}^3))$, where $3 < q \leq 9$ and $\alpha = 1 - \frac{3}{q}$. Once again both the bilinear and the linear terms turn out to be well-defined in that space as long as the initial data is in $L^3(\mathbb{R}^3)$. But, as we will see, it is in fact sufficient, to ensure these properties, to take the initial data in a certain Besov space. Here the equivalence (for $T = \infty$) will be: $S(t)v_0 \in \mathcal{G}_q$ if and only if $v_0 \in \dot{B}_q^{-\alpha,\frac{2}{\alpha}}(\mathbb{R}^3)$ $(3 < q \leq 9, \alpha = 1 - \frac{3}{q})$.

The paper is organized in the following way. Section I contains the basic definitions. Section II is devoted to presenting the first algorithm for solving the Navier-Stokes equations in the $L^3(\mathbb{R}^3)$ frame as originally introduced in [2,3,4,5]. Finally, Section III contains another method for solving the Navier-Stokes equations in $L^3(\mathbb{R}^3)$ along the lines of Kato and Ponce's paper.

I. SOME DEFINITIONS

We study the Cauchy problem for the Navier-Stokes equations governing the time evolution of the velocity $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ and the pressure $p(t,x)$ of an incompressible fluid filling all of $\mathbb{R}^3$ in presence of an external force $\phi(t,x)$

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -(v\cdot\nabla)v - \nabla p + \phi \\ \nabla \cdot v = 0 \\ v(0) = v_0 \end{cases} \tag{1.1}$$

Here, the external force $\phi(t,x)$ will be considered as arising from a potential

$V(t,x)$ in such a way that [9,28]

$$\phi = \nabla \cdot V \tag{1.2}$$

which means, that

$$\phi_j = \sum_{k=1}^{3} \partial_k V_{kj} \quad j = 1,2 \text{ and } 3 \tag{1.3}$$

We will focus our attention on the existence of solutions $v(t,x)$ to (1.1) in the space $\mathcal{C}([0,T); L^3(\mathbb{R}^3))$ that are strongly continuous functions of $t \in [0,T)$ with values in the Banach space $L^3(\mathbb{R}^3)$ of vector distributions. According on whether T will be *bounded* ($T < \infty$) or *unbounded* ($T = \infty$) we will obtain respectively *local* or *global* solutions.

Here and in the following we say that a vector $a = (a_1, a_2, a_3)$ belongs to a function space X if $a_j \in X$ holds for every $j = 1,2,3$ and we put $\|a\| = \max_{1 \leq j \leq 3} \|a_j\|$.

Before introducing the appropriate functional setting, let us transform the system (1.1) into a single integral equation. In order to do this, let us first introduce the projection operator $\mathbb{P}$ onto the divergence free vector field. We let $\partial_j = -i\frac{\partial}{\partial x_j}$, ($i^2 = -1$) and we denote by $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ for $j = 1,2$ and 3 the Riesz transformation.

For an arbitrary vector field $v(x) = (v_1(x), v_2(x), v_3(x))$ on $\mathbb{R}^3$, we set

$$z(x) = \sum_{j=1}^{3} (R_j v_j)(x) \tag{1.4}$$

and define the operator $\mathbb{P}$ by

$$(\mathbb{P}v)_k(x) = v_k(x) - (R_k z)(x), \qquad k = 1,2 \text{ and } 3. \tag{1.5}$$

As such, $\mathbb{P}$ is a pseudo-differential operator of degree zero and is an orthogonal projection onto the kernel of the divergence operator. Making use of this projection operator $\mathbb{P}$ and the heat semi-group

$$S(t) = \exp(t\Delta), \tag{1.6}$$

it is now a straightforward procedure to reduce the classical partial differential system (1.1) into the mild integral equation

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (v \otimes v)(s)ds + \int_0^t S(t-s)\mathbb{P}\nabla \cdot V(s)ds \tag{1.7}$$

In the following Sections a particular attention will be devoted to the study of the bilinear operator

$$B(v,u)(t) = -\int_0^t S(t-s)\mathbb{P}\nabla\cdot(v\otimes u)(s)ds \tag{1.8}$$

More precisely, arguing componentwise as in [5], we will limit ourselves to the study of its scalar version given by

$$B(f,g)(t) = -\int_0^t (t-s)^{-2}\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * (fg)(s)ds \tag{1.9}$$

where $f = f(t,x)$ and $g = g(t,x)$ are two *scalar* fields and $\Theta = \Theta(x)$ is an analytic function of x which is $O(|x|^{-4})$ at infinity and whose integral is zero. To simplify, we can consider Θ as the function whose Fourier transform is given by

$$\hat{\Theta}(\xi) = |\xi|e^{-|\xi|^2}. \tag{1.10}$$

In an analogous way, the linear operator L involving the external force defined by

$$L(V)(t) = \int_0^t S(t-s)\mathbb{P}\nabla\cdot V(s)ds \tag{1.11}$$

will be treated in the following simplified scalar form

$$L(F) = \int_0^t (t-s)^{-2}\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * F(s)ds \tag{1.12}$$

where $F = F(t,x)$ is a scalar field. In particular, we notice that

$$B(f,g) = -L(fg) \tag{1.13}$$

which, in the following, will allow us to treat both the bilinear term B and the linear one L in exactly the same way.

Before solving the corresponding simplified scalar integral Navier-Stokes equation, we need to introduce some functional spaces. In order to do that we have to recall here some notations and definitions.

Let's start with the Littlewood-Paley decomposition. To this end, we take an arbitrary function φ in the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ and whose Fourier transform $\hat{\varphi}$ is such that

$$0 \le \hat{\varphi}(\xi) \le 1, \quad \hat{\varphi}(\xi) = 1 \text{ if } |\xi| \le \frac{3}{4}, \quad \hat{\varphi}(\xi) = 0 \text{ if } |\xi| \ge \frac{3}{2} \tag{1.14}$$

and let

$$\psi(x) = 8\varphi(2x) - \varphi(x) \tag{1.15}$$

$$\varphi_j = 2^{3j}\varphi(2^j x), \quad j \in \mathbb{Z} \tag{1.16}$$

$$\psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z} \tag{1.17}$$

We denote by S_j and Δ_j respectively the convolution operators with φ and ψ. Finally the set $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ is a Littlewood- Paley decomposition, so that

$$I = S_0 + \sum_{j\geq 0} \Delta_j = \sum_{j\in\mathbb{Z}} \Delta_j. \tag{1.18}$$

This decomposition is very useful for we can define (independently of the choice of the initial function φ) the following Besov [1,30,13] and Triebel-Lizorkin [34,35,13] spaces.

Definition I.1. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (homogeneous) Besov space $\dot{B}_q^{s,p}(\mathbb{R}^3)$ if and only if*

$$\left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\Delta_j f\|_q)^p\right)^{\frac{1}{p}} < \infty \tag{1.19}$$

Definition I.2. *Let $0 < p \leq \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (homogeneous) Triebel-Lizorkin space $\dot{F}_q^{s,p}(\mathbb{R}^3)$ if and only if*

$$\left\|\left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j f|)^p\right)^{\frac{1}{p}}\right\|_q < \infty \tag{1.20}$$

The next four propositions are of paramount importance because they give definitions for the Besov and Triebel-Lizorkin norms in terms of the heat semi-group $S(t)$ [that appears in (1.6)] and in terms of the function Θ [that appears in (1.9) and (1.12)] or, more precisely, in terms of the scaled function Θ_t defined by

$$\Theta_t = \frac{1}{t^3}\Theta\left(\frac{\cdot}{t}\right) \tag{1.21}$$

For the detailed proofs the reader will consult [30] and for a more general characterization [34,35,13].

Proposition I.1. *Let $1 \leq p, q \leq \infty$ and $s < 1$, then the quantities*

$$\left(\sum_{j \in \mathbb{Z}} (2^{sj} \|\Delta_j f\|_q)^p \right)^{\frac{1}{p}} \tag{1.22}$$

and

$$\left(\int_0^\infty (t^{-s} \|\Theta_t f\|_q)^p \frac{dt}{t} \right)^{\frac{1}{p}} \tag{1.23}$$

are equivalent and will be referred to in the sequel by $\|f\|_{\dot{B}_q^{s,p}(\mathbb{R}^3)}$.

Proposition I.2. *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $s < 1$, then the quantities*

$$\left\| \left(\sum_{j \in \mathbb{Z}} (2^{sj} |\Delta_j f|)^p \right)^{\frac{1}{p}} \right\|_q \tag{1.24}$$

and

$$\left\| \left(\int_0^\infty (t^{-s} |\Theta_t f|)^p \frac{dt}{t} \right)^{\frac{1}{p}} \right\|_q \tag{1.25}$$

are equivalent and will be referred in the sequel by $\|f\|_{\dot{F}_q^{s,p}(\mathbb{R}^3)}$.

The next two equivalences, Propositions I.3-4, are even more useful because they allow to pass from Δ_j to S_j (and from the discrete set S_j to the continuous $S(t)$ one).

Proposition I.3. *Let $1 \leq p, q \leq \infty$ and $s < 0$, then the quantities*

$$\left(\sum_{j \in \mathbb{Z}} (2^{sj} \|\Delta_j f\|_q)^p \right)^{\frac{1}{p}}, \tag{1.26}$$

$$\left(\sum_{j \in \mathbb{Z}} (2^{sj} \|S_j f\|_q)^p \right)^{\frac{1}{p}}, \tag{1.27}$$

$$\left(\int_0^\infty (t^{-\frac{s}{2}} \|S(t) f\|_q)^p \frac{dt}{t} \right)^{\frac{1}{p}}, \tag{1.28}$$

and

$$\left(\int_0^\infty (t^{-s}\|\Theta_t f\|_q)^p \frac{dt}{t}\right)^{\frac{1}{p}} \tag{1.29}$$

are equivalent and will be referred to in the sequel by $\|f\|_{\dot{B}_q^{s,p}(\mathbb{R}^3)}$.

Proposition I.4. *Let* $1 \le p \le \infty$, $1 \le q < \infty$ *and* $s < 0$, *then the quantities*

$$\left\|\left(\sum_{j\in\mathbb{Z}}(2^{sj}|\Delta_j f|)^p\right)^{\frac{1}{p}}\right\|_q, \tag{1.30}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}(2^{sj}|S_j f|)^p\right)^{\frac{1}{p}}\right\|_q, \tag{1.31}$$

$$\left\|\left(\int_0^\infty (t^{-\frac{s}{2}}|S(t)f|)^p \frac{dt}{t}\right)^{\frac{1}{p}}\right\|_q, \tag{1.32}$$

and

$$\left\|\left(\int_0^\infty (t^{-s}|\Theta_t f|)^p \frac{dt}{t}\right)^{\frac{1}{p}}\right\|_q \tag{1.33}$$

are equivalent and will be referred in the sequel by $\|f\|_{\dot{F}_q^{s,p}(\mathbb{R}^3)}$.

Before ending this Section we will recall here a classical result concerning the existence of fixed point solution to abstract functional equations (for a proof see [5])

Lemma I.1. *Let* X *be an abstract Banach space with norm* $\|\ \|$, $L : X \longrightarrow X$ *a linear operator such that for any* $x \in X$

$$\|L(x)\| \le \lambda\|x\| \tag{1.34}$$

and $B : X \times X \longrightarrow X$ *a bilinear operator such that for any* $x_1, x_2 \in X$,

$$\|B(x_1,x_2)\| \le \eta\|x_1\|\|x_2\|, \tag{1.35}$$

then for any λ, $0 \le \lambda < 1$ *and for any* $y \in X$ *such that*

$$4\eta\|y\| < (1-\lambda)^2 \tag{1.36}$$

the equation

$$x = y + L(x) + B(x,x) \tag{1.37}$$

has a solution x in X. Moreover, this solution is the only one such that

$$\|x\| \leq \frac{1-\lambda-\sqrt{(1-\lambda)^2-4\eta\|y\|}}{2\eta}. \tag{1.38}$$

In particular the solution is such that

$$\|x\| \leq \frac{2\|y\|}{1-\lambda}. \tag{1.39}$$

II. THE FIRST ALGORITHM

In this and the following Section we will study the integral mild Navier-Stokes equation

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\mathbb{P}\nabla\cdot(v\otimes v)(s)ds + \int_0^t S(t-s)\mathbb{P}\nabla\cdot V(s)ds \tag{2.1}$$

in the (natural) function space

$$\mathcal{N} = \mathcal{C}([0,T);L^3(\mathbb{R}^3)) \tag{2.2}$$

whose elements are continuous functions $v(t,x)$ from $[0,T)$ in $L^3(\mathbb{R}^3)$ such that

$$|||v|||_{\mathcal{N}} = \sup_{0<t<T} \|v(t,x)\|_3 \tag{2.3}$$

is finite.

For the convenience of the reader we will state here the existence [20] and uniqueness [14] local and global theorem with data in $L^3(\mathbb{R}^3)$ that we discussed in the introduction, and we will briefly sketch a variant of its proof as it was given in [5,31,6]. The idea is to take advantage of the simplified structure of the bilinear operator in its scalar form, as in Eq. (1.9). In particular, the divergence $\nabla\cdot$ and heat $S(t)$ operators will be treated as a single convolution operator [5]. This is why no explicit conditions on the gradient of the unknown function v and no restriction on q (namely $3 < q < 6$) will be required here, as they were indeed in the original paper of T. Kato [20].

In order to proceed we have to recall the definition of the auxiliary space $\mathcal{K}_q$. More exactly, this space $\mathcal{K}_q$ is made up by the functions $v(t,x)$ such that [5]

$$t^{\frac{\alpha}{2}} v(t,x) \in \mathcal{C}([0,T); L^q(\mathbb{R}^3)) \tag{2.4}$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \|v(t)\|_q = 0, \tag{2.5}$$

with q being fixed in $3 < q \leq \infty$ and $\alpha = \alpha(q) = 1 - \frac{3}{q}$. In the case $q = 3$, it is also convenient to define the space $\mathcal{K}_3$ as the natural space $\mathcal{N}$ with the additional condition that its elements $v(t,x)$ satisfy

$$\lim_{t\to 0} \|v(t)\|_3 = 0. \tag{2.6}$$

The theorem in question is the following.

Theorem II.1. *Let $3 < p \leq 6$, $3 < q < \frac{3p}{6-p}$, $\beta = 1 - \frac{3}{p}$ and $\alpha = 1 - \frac{3}{q}$ be fixed. There exist two constants $\delta_q > 0$ and $\delta_p > 0$ such that for any initial data $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ in the sense of distributions, and any external force V such that*

$$\sup_{0<t<T} t^{\frac{\alpha}{2}} \|S(t)v_0\|_q < \delta_q, \tag{2.7}$$

$$\sup_{0<t<T} t^{\beta} \|V(t)\|_{\frac{p}{2}} < \delta_p, \tag{2.8}$$

and

$$\lim_{t\to 0} t^{\beta} \|V(t)\|_{\frac{p}{2}} = 0, \tag{2.9}$$

then there exists a unique solution $v(t,x)$ of the equation (2.1) belonging to $\mathcal{N}$, which tends strongly to v_0 as time goes to zero. Moreover, this solution belongs to all spaces $\mathcal{K}_q$ for all $3 < q < \frac{3p}{6-p}$. In particular, (2.7) holds for arbitrary $v_0 \in L^3(\mathbb{R}^3)$ provided we consider T small enough, and as well if $T = \infty$, provided the norm of v_0 in the Besov space $\dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ is smaller than δ_q.

The existence part of proof of this theorem is a consequence of the following lemmata that we recall here.

Lemma II.1. *If $v_0 \in L^3(\mathbb{R}^3)$, then $S(t)v_0 \in \mathcal{K}_q$ for any $3 < q \leq \infty$. In particular this implies (when $T = \infty$) the continuous embedding*

$$L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3) \quad 3 < q \leq \infty. \tag{2.10}$$

Lemma II.2. *The bilinear operator $B(f,g)(t)$ is bicontinuous from $\mathcal{K}_q \times \mathcal{K}_q \longrightarrow \mathcal{K}_q$ for any $3 < q < \infty$.*

Once these two lemmata are applied for a *certain* q, $3 < q < \frac{3p}{6-p}$, one can easily deduce, provided (2.7-9) are satisfied and via the fixed point algorithm Lemma I.1 ($\lambda = 0$), the existence of *a solution* $v(t,x) \in \mathcal{N}$ that tends strongly to v_0 at zero and belongs to $\mathcal{K}_q$ for *all* $3 < q < \frac{3p}{6-p}$.

The latter properties being a consequence of the following generalization of Lemma II.2, applied to both the bilinear B and the linear L terms.

Lemma II.3. *The bilinear operator $B(f,g)(t)$ is bicontinuous from $\mathcal{K}_q \times \mathcal{K}_q \longrightarrow \mathcal{K}_p$ for any $3 \leq p < \frac{3q}{6-q}$ if $3 < q < 6$; any $3 \leq p < \infty$ if $q = 6$; and $\frac{q}{2} \leq p \leq \infty$ if $6 < q < \infty$.*

The proof of the uniqueness of the solution in $\mathcal{N}$ requires a more careful study of the bilinear term. We will not give a proof of this fundamental result here and refer the reader to [14,7,26].

Instead, we will concentrate on a very elegant variant of the existence part of the proof of Theorem II.1. In order to proceed we need to introduce the function space $\mathcal{M}$ whose elements belong to $\mathcal{N}$ and are such that

$$|||v|||_{\mathcal{M}} = \|\sup_{0<t<T} |v(t,x)|\|_3 \tag{2.11}$$

is finite.

Recall that $\mathcal{M}$ is continuously embedded in $\mathcal{N}$, because of the following elementary inequality

$$\sup_{0<t<T} \|v(t,x)\|_3 \leq \|\sup_{0<t<T} |v(t,x)|\|_3 \tag{2.12}$$

The method we will pursue here, is to solve the Eq. (2.1) in $\mathcal{M}$. This will be possible because, at variance with $\mathcal{N}$ [29], the bilinear operator is bicontinuous in $\mathcal{M}$. More precisely, the following two lemmata hold true [2,3,4,5].

Lemma II.4. *$S(t)v_0 \in \mathcal{M}$ if and only if $v_0 \in L^3(\mathbb{R}^3)$.*

Lemma II.5. *The bilinear operator $B(f,g)(t)$ is bicontinuous from $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$.*

Finally, a straightforward application of the fixed point algorithm, Lemma I.1 ($\lambda = 0$), implies the existence of a global solution to (2.1) in $\mathcal{M}$ with initial data v_0 sufficiently small in $L^3(\mathbb{R}^3)$, say

$$\|v_0\|_3 < \delta \tag{2.13}$$

and external force $V(t,x)$ such that

$$\left\| \sup_{0<t<\infty} \left| \int_0^t S(t-s)\mathbb{P}\cdot\nabla V(s)ds \right| \right\|_3 < \delta \tag{2.14}$$

However, because the bicontinuity constant arising in Lemma II.5 does not depend on T and in general the condition

$$\lim_{T\to 0} \| \sup_{0<t<T} |S(t)v_0| \|_3 \neq 0 \tag{2.15}$$

holds if $v_0 \in L^3(\mathbb{R}^3)$, $v_0 \not\equiv 0$, there is no evidence to guarantee that such a global solution is strongly continuous at the origin (thus unique), and —which is intimately related— that such a solution exists locally in time for an arbitrary initial data v_0 in $L^3(\mathbb{R}^3)$ and appropriate external force.

This is why we have to use here the same trick as introduced in [5]. More precisely, we will, instead of looking for a solution $v(t,x) \in \mathcal{M}$ to (2.1) via the point fixed Lemma I.1 ($\lambda = 0$), look for a solution

$$w(t,x) = v(t,x) - S(t)v_0 \in \mathcal{M} \tag{2.16}$$

via the point fixed Lemma I.1 ($\lambda \neq 0$). More precisely, we will solve the equation

$$\begin{aligned} w(t,x) = \tilde{B}(S(t)v_0, S(t)v_0) + &\int_0^t S(t-s)\mathbb{P}\cdot\nabla V(s)ds \\ &+2\tilde{B}(w, S(t)v_0) + \tilde{B}(w,w) \end{aligned} \tag{2.17}$$

where the symmetric bilinear operator $\tilde{B}$ is defined, in terms of B, by

$$\tilde{B}(v,u)(t) = \frac{B(v,u)(t) + B(u,v)(t)}{2}. \tag{2.18}$$

We can now take advantage of the particular structure of the heat semi-group appearing in (2.17). More exactly, we can generalize the previous lemmata and obtain the following ones

Lemma II.6. *Let $\alpha = 1 - \frac{3}{q}$ and $3 < q < \infty$ be fixed. Then*

$$\| \sup_{0<t<T} t^{\frac{\alpha}{2}} |S(t)v_0| \|_q \leq C_q \|v_0\|_3, \tag{2.19}$$

and in particular, if $v_0 \in L^3(\mathbb{R}^3)$, the l.h.s. of (2.19) tends to zero as T tends to zero.

Lemma II.7. *Let $\alpha = 1 - \frac{3}{q}$, $3 < q < 6$, and $f(t,x) = S(t)f_0$, with $f_0 = f_0(x)$, then the following estimate holds for the bilinear operator*

$$\|B(S(t)f_0, S(t)f_0)\|_{\mathcal{M}} \leq C_q \| \sup_{0<t<T} t^{\frac{\alpha}{2}} |S(t)f_0| \|_q^2. \tag{2.20}$$

Lemma II.8. *Let $\alpha = 1 - \frac{3}{q}$, $3 < q < \infty$, and $f(t,x) = S(t)f_0$, with $f_0 = f_0(x)$, and $g = g(t,x)$ then the following estimate holds for the bilinear operator*

$$\|B(S(t)f_0, g)\|_{\mathcal{M}} \leq C'_q \|g\|_{\mathcal{M}} \| \sup_{0<t<T} t^{\frac{\alpha}{2}} |S(t)f_0| \|_q \tag{2.21}$$

We can now state the following existence and uniqueness theorem of [2,3,4,5] as:

Theorem II.2. *Let $3 < q < 6$, $3 \leq p < 6$, $\alpha = 1 - \frac{3}{q}$ and $\beta = 1 - \frac{3}{p}$ be fixed. There exist two constants $\delta_q > 0$ and $\delta_p > 0$ such that for any initial data $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ in the sense of distributions, and any external force V such that*

$$\| \sup_{0<t<T} t^{\frac{\alpha}{2}} |S(t)v_0| \|_q < \delta_q \tag{2.22}$$

and that, if $3 < p < 6$,

$$\left(\int_0^T \|V(t)\|_{\frac{p}{2}}^{\frac{1}{\beta}} \right)^{\beta} < \delta_p \tag{2.23}$$

or that, if $p = 3$,

$$\| \sup_{0<t<T} |V(t)| \|_{\frac{3}{2}} < \delta_3 \tag{2.24}$$

and

$$\lim_{T \to 0} \| \sup_{0<t<T} |V(t)| \|_{\frac{3}{2}} = 0 \tag{2.25}$$

then there exists a unique solution $v(t,x)$ of the equation (2.1) belonging to $\mathcal{N}$, which tends strongly to v_0 as time goes to zero. Moreover, this solution belongs to the space $\mathcal{M}$. In particular, (2.22) holds for arbitrary $v_0 \in L^3(\mathbb{R}^3)$ provided we consider T small enough, and as well if $T = \infty$, provided the norm of v_0 in the Triebel-Lizorkin space space $\dot{F}_q^{-\alpha,\infty}(\mathbb{R}^3)$ is smaller than δ_q.

The existence part of the proof is now a consequence of Lemma I.1, while its uniqueness counter-part follows once again from [14].

In order to appreciate the result we have just stated, let us now concentrate in comparing the hypotheses that arise in the statements of Theorem III.1 and Theorem III.2, this will clarify conditions (2.23-25) on the external force.

First of all, it is not difficult to see that, for any $T > 0$ and $3 \leq q \leq \infty$, $\alpha = 1 - \frac{3}{q}$,

$$\sup_{0<t<T} t^{\frac{\alpha}{2}} \|S(t)v_0\|_q \leq \| \sup_{0<t<T} t^{\frac{\alpha}{2}} |S(t)v_0| \|_q \tag{2.26}$$

which corresponds, for $T = \infty$, to the well-known embedding

$$\dot{F}_q^{-\alpha,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{-\alpha,\infty}(\mathbb{R}^3). \tag{2.27}$$

This circumstance indicates that, as far as the initial data v_0 is concerned, condition (2.22) is stronger than (2.7). However, with regard to the external force, we are able to reach here the limit value $\frac{3}{2}$ in space variable, which was not the case for Theorem III.1. The main difference being represented by Lemma II.5, that does not apply when $\mathcal{M}$ is replaced by $\mathcal{N}$ [29].

To be more explicit, let us remark that in Theorem II.1 the conditions to be verified by the external force in order to allow to use Lemma I.1 are in fact much weaker, and can be written as

$$\sup_{0<t<T} t^{\frac{\alpha}{2}} \left\| \int_0^t S(t-s)\mathbb{P}\nabla \cdot V(s)ds \right\|_q < \delta_q \tag{2.28}$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \left\| \int_0^t S(t-s)\mathbb{P}\nabla \cdot V(s)ds \right\|_q = 0, \tag{2.29}$$

to be verified for a fixed $3 < q < \frac{3p}{6-p}$ and $\alpha = 1 - \frac{3}{q}$. Of course, a simple application of Lemma II.3 (via Eq. (1.13)) guarantees that these conditions (2.28-29) hold as long as (2.8-9) are verified.

Let us now pass to Theorem II.2. In this case, the weaker natural conditions to be satisfied are

$$\left\| \sup_{0<t<T} \left| \int_0^t S(t-s)\mathbb{P}\nabla \cdot V(s)ds \right| \right\|_3 < \delta_q \tag{2.30}$$

and

$$\lim_{T\to 0} \left\| \sup_{0<t<T} \left| \int_0^t S(t-s)\mathbb{P}\nabla \cdot V(s)ds \right| \right\|_3 = 0. \tag{2.31}$$

A possible stronger condition is simple given by

$$\| \sup_{0<t<T} |V(t)| \|_{\frac{3}{2}} < \delta_q, \tag{2.32}$$

and

$$\lim_{T\to 0} \| \sup_{0<t<T} |V(t)| \|_{\frac{3}{2}} = 0, \tag{2.33}$$

which is a consequence of Lemma II.5. Here, at variance with Theorem II.1, the index $\frac{3}{2}$ arises naturally, as in the stationary Navier-Stokes equations [28].

But we can obtain more and generalize this result ($p = 3$). In fact, if $0 \leq \beta \leq 1$,

$$\sup_{0<t<T} \left| \int_0^t (t-s)^{-2} \Theta \left(\frac{x}{\sqrt{t-s}} \right) * V(s) ds \right| \leq C_\beta \left(\int_0^T |V(t)|^{\frac{1}{\beta}} \right)^\beta * \frac{1}{|x|^{2+2\beta}} \tag{2.34}$$

so that finally

$$\left\| \sup_{0<t<T} \left| \int_0^t (t-s)^{-2} \Theta \left(\frac{x}{\sqrt{t-s}} \right) * V(s) ds \right| \right\|_3 \leq C_\beta \left(\int_0^T \|V(t)\|_{\frac{p}{2}}^{\frac{1}{\beta}} \right)^\beta \tag{2.35}$$

whenever $3 < p < 6$ and $\beta = 1 - \frac{3}{p}$.

Of course, asking V to be integrable in time will automatically ensure Eq. (2.31) except when $p = 3$.

In other words, when $3 < p < 6$ and $\beta = 1 - \frac{3}{p}$, we were able to replace the conditions of Theorem II.1, namely

$$\sup_{0<t<T} t^\beta \|V(t)\|_{\frac{p}{2}} < \delta_p \tag{2.36}$$

and

$$\lim_{t \to 0} t^\beta \|V(t)\|_{\frac{p}{2}} = 0 \tag{2.37}$$

by the single assumption

$$\left(\int_0^T \|V(t)\|_{\frac{p}{2}}^{\frac{1}{\beta}} \right)^\beta < \delta_p \tag{2.38}$$

which is neither weaker nor stronger, but just easier and more natural to deal with because it makes no distinction among the points (the origin in particular) of the interval $[0, T]$.

Our point of view is motivated here by a previous result of J.Y. Chemin [9] who was able to replace the conditions

$$\sup_{0<t<T} t^{\frac{3}{4}} \|V(t)\|_{\dot{H}^1} < \delta \tag{2.39}$$

and

$$\lim_{t \to 0} t^{\frac{3}{4}} \|V(t)\|_{\dot{H}^1} = 0 \tag{2.40}$$

originally given in the papers of T. Kato and H. Fujita [22,16] by the single assumption

$$\left(\int_0^T \|V(t)\|^2_{\dot{H}^{\frac{1}{2}}} dt\right)^{\frac{1}{2}} < \delta. \tag{2.41}$$

In particular, if the value $p = 6$ were allowed in Theorem II.2, then condition (2.38) for $p = 6$ would give

$$\left(\int_0^T \|V(t)\|^2_3 dt\right)^{\frac{1}{2}} < \delta \tag{2.42}$$

that is a weaker assumption than (2.41), because of the Sobolev embedding

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3). \tag{2.43}$$

We will see in the following pages how to obtain such a limit case $p = 6$.

III. THE SECOND ALGORITHM

As we recalled in the previous section, the method for finding a strongly continuous solution with values in $L^3(\mathbb{R}^3)$ makes use of an *ad hoc* auxiliary subspace of functions that are continuous in the t-variable and take values in a Lebesgue space in the x-variable. Moreover, Y. Giga proved in [18] that not only does the solution under consideration belong to $L^\infty_t(L^3_x)$ and $\mathcal{K}_q$ but also it belongs to $\mathcal{G}_q = L_t^{\frac{2}{\alpha}}(L^q_x)$ for all q in the interval $3 < q \leq 9$ and $\alpha = 1 - \frac{3}{q}$.

At this point, one can naturally ask oneself whether these spaces $\mathcal{G}_q$ can be used –independently– as auxiliary *ad hoc* subspaces. This question arises also in view of the fact that $L^p_t(L^q_x)$ estimates (and, more generally, the so-called Strichartz estimates [33]) are frequently used for the study of other well-known non-linear PDE's, like the Schrödinger one or the wave equation.

Even if this doesn't lead here to a breakthrough as in the case of the Schrödinger equation, making use *directly* of $L^p_t(L^q_x)$ estimates for Navier-Stokes is indeed possible. This was proved by T. Kato and G. Ponce in [23], where, in fact, the authors consider the case of a much larger functional class, including the $\mathcal{G}_q$ one.

In what follows, we will focus our attention only on the latter case and prove an existence theorem of local (resp. global) strong solutions in $\mathcal{C}([0,T); L^3(\mathbb{R}^3))$ with initial data and external force (resp. small enough) in a certain Besov space. In particular, the condition on the external force will be of integral type, as in the previous section, but it will arise more naturally here and with the limit case $p = 6$ allowed.

The "Besov language" will provide a very convenient and powerful tool, needed to overcome difficulties which were absent in the previous section. Let's start by recalling the spaces $\mathcal{G}_q$. Precisely, $\mathcal{G}_q$ is made up by the functions $v(t,x) \in L_t^{\frac{2}{\alpha}}(L_x^q)$ such that

$$\|f\|_{\mathcal{G}_q} =: \left(\int_0^T \|f(t,x)\|_q^{\frac{2}{\alpha}} dt\right)^{\frac{\alpha}{2}} < \infty \tag{3.1}$$

T being, as usual, either finite or infinite, and $\alpha = \alpha(q) = 1 - \frac{3}{q}$.

We are now in the position of stating the following

Lemma III.1. *Let $3 < q \leq 9$ and $\alpha = 1 - \frac{3}{q}$ be fixed. Then*

$$\left(\int_0^T \|S(t)v_0\|_q^{\frac{2}{\alpha}} dt\right)^{\frac{\alpha}{2}} \leq C_q \|v_0\|_3, \tag{3.2}$$

where the integral in the l.h.s. of (3.2) tends to zero as T tends to zero provided $v_0 \in L^3(\mathbb{R}^3)$.

In short, our lemma says that if $v_0 \in L^3(\mathbb{R}^3)$, then $S(t)v_0$ is in $\mathcal{G}_q$, and therefore we are allowed to work within that functional framework.

Let us now concentrate our attention on the bilinear operator [31].

Lemma III.2. *The bilinear operator $B(f,g)(t)$ is bicontinuous from $\mathcal{G}_q \times \mathcal{G}_q \longrightarrow \mathcal{G}_p$ for any $3 < p < \frac{3q}{6-q}$ if $3 < q < 6$; any $3 < p < \infty$ if $q = 6$; and $\frac{q}{2} \leq p \leq \infty$ if $6 < q < \infty$.*

Proposition III.1. *The bilinear operator $B(f,g)(t)$ is bicontinuous from $\mathcal{G}_6 \times \mathcal{G}_6 \longrightarrow \mathcal{N}$. In fact, $B(f,g)$ takes its values in $\mathcal{C}(0,T;\dot{B}_3^{0,2}(\mathbb{R}^3))$, which is a proper subset of $\mathcal{C}(0,T;L^3(\mathbb{R}^3))$.*

We would like to remark here that a variant of this result was recently applied in [14] in the proof of the uniqueness theorem for strong $L^3(\mathbb{R}^3)$ solutions (see also [7] for more comments).

We are now in the position of proving the following theorem ([31])

Theorem III.1. *Let $4 < p \leq 6$, $3 < q < \min(9, \frac{3p}{6-p})$, $\beta = 1 - \frac{3}{p}$ and $\alpha = 1 - \frac{3}{q}$ be fixed. There exist two constants δ_q and δ_p such that for any initial data $v_0 \in L^3(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$ in the sense of distributions, and external force V such that*

$$\left(\int_0^T \|S(t)v_0\|_q^{\frac{2}{\alpha}} dt\right)^{\frac{\alpha}{2}} < \delta_q \tag{3.3}$$

and

$$\left(\int_0^T \|V(s)\|_{\frac{p}{2}}^{\frac{1}{\beta}}\, dt\right)^{\beta} < \delta_p \tag{3.4}$$

then there exists a unique solution $v(t,x)$ of the equation (2.1) belonging to $\mathcal{N}$, which tends strongly to v_0 as time goes to zero. Moreover, this solution belongs to all the spaces $\mathcal{G}_q$ ($3 < q < \min(9, \frac{3p}{6-p})$) and is such that the fluctuation $w(t,x)$ defined in (2.8) satisfies

$$w \in \mathcal{C}([0,T); \dot{B}_3^{0,2}(\mathbb{R}^3)) \tag{3.5}$$

and

$$w \in L^2((0,T); L^\infty(\mathbb{R}^3)) \tag{3.6}$$

Finally, (3.3) holds for arbitrary $v_0 \in L^3(\mathbb{R}^3)$ provided we consider T small enough, and as well if $T = \infty$, provided the norm of v_0 in the Besov space $\dot{B}_q^{-\alpha,\frac{2}{\alpha}}(\mathbb{R}^3)$ is smaller than δ_q.

Keeping in mind the previous propositions and remarks, the existence proof of that theorem is easily carried out. As far as the uniqueness part of the proof, this follows again from [14]. Finally, as we already observed in Section II, if we take $p = 6$ in Theorem III.1 we find a weaker condition on the external force than the one given in [9].

ACKNOWLEDGMENTS

M.C. would like to thank T. Miyakawa for pointing him out the papers by C. P. Calderón.

References

1. J. Bergh and J. Löfstrom, *Interpolation spaces, An Introduction*, Springer-Verlag, 1976.
2. C. P. Calderón, *Existence of weak solutions for the Navier-Stokes equations with initial data in L^p*, Trans. Amer. Math. Soc. **318 (1)** (1990), 179-200.
3. C. P. Calderón, *Addendum to the paper: "Existence of weak solutions for the Navier-Stokes equations with initial data in L^p"*, Trans. Amer. Math. Soc. **318 (1)** (1990), 201-207.
4. C. P. Calderón, *Initial values of solutions of the Navier-Stokes equations*, Proc. Amer. Math. Soc. **117 (3)** (1993), 761-766.
5. M. Cannone, *Ondelettes, Paraproduits et Navier-Stokes*, Diderot Editeur, 1995.
6. M. Cannone, *A generalisation of a theorem by Kato on Navier-Stokes equations*, Revista Matematica Iberoamericana (to appear).
7. M. Cannone and F. Planchon, *On the regularity of the bilinear term for solutions of the incompressible Navier-Stokes equations in $\mathbb{R}^3$*, Revista Matematica Iberoamericana (submitted).
8. M. Cannone and F. Planchon, *On the non stationary Navier-Stokes equations with an external force*, Differential and Integral Equations (submitted).

9. J. Y. Chemin, *Remarques sur l'existence globale pour le système de Navier-Stokes incompressible*, SIAM J. Math. Anal. **23** (1992), 20-28.
10. R. Coifman, P. L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.
11. E. B. Fabes, B. F. Jones and N. M. Rivière, *The initial value problem for the Navier-Stokes equations with data in L^p*, Arch. Rat. Mech. Anal. **45** (1972), 222-240.
12. E. B. Fabes, Lewis and N. M. Rivière, *Boundary value problems for the Navier-Stokes equations*, Am. Jour. of Math. **99** (1977), 601-625.
13. M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, vol. 79, Monograph in the CBM-AMS, 1991.
14. G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo, *Sur l'unicité dans $L^3(\mathbb{R}^3)$ des solutions mild des équations de Navier-Stokes*, C. R. Acad. Sci. Paris (1997) (to appear).
15. G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo, *Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces fonctionnels limites pour Navier-Stokes*, in preparation.
16. H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I*, Arch. Rat. Mech. Anal. **16** (1964), 269-315.
17. Y. Giga, *Solutions for semi linear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Diff. Equat. **61** (1986), 186-212.
18. Y. Giga, *Regularity criteria for weak solutions of the Navier-Stokes system*, Proceedings of Symposia in Pure Mathematics **45 (1)** (1986), 449-453.
19. Y. Giga and T. Miyakawa, *Solutions in L^r of the Navier-Stokes initial value problem*, Arch. Mech. Rat. Anal. **89** (1985), 267–281.
20. T. Kato, *Strong L^p solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Math. Zeit. **187** (1984), 471-480.
21. 加藤 敏夫, *Well-posedness* について, 数学 **48** (1996), 298-300.
22. T. Kato and H. Fujita, *On the non-stationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova **32** (1962), 243-260.
23. T. Kato and G. Ponce, *The Navier-Stokes equations with weak initial data*, Int. Math. Res. Not. **10** (1994), 435-444.
24. Y. Le Jan and A.S. Sznitman, *Cascades aléatoires et équations de Navier-Stokes*, C. R. Acad. Sci. Paris **324 (7)** (1997), 823-826.
25. Y. Le Jan and A.S. Sznitman, *Stochastic cascades and 3-dimensional Navier-Stokes equations* (to appear).
26. Y. Meyer, *Wavelets, paraproducts and Navier-Stokes equations* (to appear).
27. T. Miyakawa, *On the initial value problem for the Navier-Stokes equations in L^p spaces*, Hiroshima Math. J. **11** (1981), 9-20.
28. T. Miyakawa, *On L^1-Stability of Stationary Navier-Stokes Flows in $\mathbb{R}^n$*, J. Math. Sci. Univ. Tokyo **4** (1997), 67-119.
29. F. Oru, *Phd Thesis*, in preparation.
30. J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, 1976.
31. F. Planchon, *Solutions Globales et Comportement Asymptotique pour les Equations de Navier-Stokes*, PhD Thesis, Ecole Polytechnique, France, 1996.
32. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
33. R. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of solutions of the wave equations*, Duke Math. Journal **44** (1977), 705-714.
34. H. Triebel, *Theory of Function Spaces*, Monograph in mathematics, Birkhauser, vol. 78, 1983.

35. H. Triebel, *Theory of Function Spaces II*, Monograph in mathematics, Birkhauser, vol. 84, 1992.
36. F. B. Weissler, *Local existence and non existence for semilinear parabolic equation in L^p*, Indiana Univ. Math. Journ. **29** (1980), 79-102.
37. F. B. Weissler, *The Navier-Stokes initial value problem in L^p*, Arch. Rat. Mech. Anal. **74** (1981), 219-230.

Marco Cannone
URA 1321 du CNRS,
UFR Mathématiques,
Université Denis Diderot Paris VII,
2 Place Jussieu,
75251 Paris Cedex 05, France.
e-mail: cannone@math.jussieu.fr

Fabrice Planchon
Program for Applied and Computational Mathematics,
Princeton University,
Princeton NJ 08544-1000, USA.
e-mail: fabrice@math.princeton.edu