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THE QUASI-SAZONOV TOPOLOGY

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Abstract

In this paper, we define some topologies determined by μ-measurable seminorms. If the cylindrical measure μ is canonical p-stable, then the aforesaid topology becomes a Sazonov topology of an L^p-space $(1 < p < 2)$, which is different from the well-known one.

1 Introduction

We are going to construct a Sazonov topology which is different from the well-known type. The main tool to do it is the measurable norm due to Dudley-Feldman-Le Cam([1]). First we introduce a new notion " the quasi- Sazonov topology " and investigate the relation between the Sazonov topology and the quasi-Sazonov topology. Consequently, the quasi-Sazonov topology is the Sazonov topology if the space is of stable type 1. This result leads the interesting fact that there exists another Sazonov topology of an L^p-space if $1 < p < 2$. It is the analogy with that Hilbert spaces have two Sazonov topologies which have been shown by Sazonov , Gross and Kuo ([10], [2,3], [4]).

2 Preliminary and Notation

Let E be a real separable Banach space with topological dual E', $C(E)$ be the class of all cylindrical measures on E and $P(E)$ be the class of all Radon probability measures on E. Clearly, $P(E) \subset C(E)$.

By $\Phi(E')$ we denote the set of all complex valued functions ϕ satisfying the following three conditions : (1) $\phi(0) = 1$, (2) ϕ is positive definite and (3) for each finite dimensional subspace $G \subset E'$ the restriction of ϕ to G is continuous. Here G is endowed with the unique norm-topology.

If we denote by $\hat{\mu}$ the characteristic function of $\mu \in C(E)$, then we have

$$\Phi(E') = \{ \hat{\mu} \ ; \mu \in C(E)\}.$$

Now we explain the order of Radon probability measures, and the type and the cotype of cylindrical measures. For $\mu \in P(E)$, we define $\| \mu \|_p = (\int_E \| x \|^p d\mu(x))^{1/p}$, if $-\infty < p < \infty$, $p \neq 0$, $\| \mu \|_\infty =$ ess.sup $\| x \|$ (with respect to μ) and $\| \mu \|_0 = \exp \int_E \log \| x \| d\mu(x)$. For $p < 0$, $\| \mu \|_p$ is always finite.

Definition. *The Radon probability measure μ is of order p, if $\| \mu \|_p < \infty$.*

Clearly if $q \leq p$ and μ is of order p , then μ is of order q, since $\| \mu \|_p$ is an increasing function of p.

Let μ be a cylindrical measure on E and $\xi \in E'$. Then ξ is a continuous linear map from E into $\mathbf{R}$, then we can define the image measure $\xi(\mu) = \mu_\xi$ on $\mathbf{R}$.

Definition. *For $\mu \in C(E)$, we define $\| \mu \|_p^\star = \sup_{\|\xi\| \leq 1} \| \mu_\xi \|_p$ and $\| \mu \|_p^\circ = [\inf_{\|\xi\| \geq 1} \| \mu_\xi \|_p]^{-1}$. We say μ is of type p, if $\| \mu \|_p^\star < \infty$ and μ is of cotype p, if $\| \mu \|_p^\circ < \infty$.*

Here we introduce stable cylindrical measures. Let (Ω, P) be a probability measure space, and $1 < p \leq 2$ and p' be the conjugate index to p (i.e. $1/p' + 1/p = 1$). On $L^{p'}(\Omega, P)$ there exists the cylindrical measure γ_p such that $\hat{\gamma_p} = e^{-\|\xi\|^p}$ for $\xi \in L^p$. We say that γ_p is the canonical p-stable symmetric cylindrical measure. For $p = 2$, γ_2 is equal to γ, i.e. the canonical Gauss cylindrical measure.

The following result is well known ([12]). It is convenient to introduce the notation $\bar{p} = p$ if $p < 2$, or ∞ if $p = 2$.

Proposition 1. *The cylindrical measure γ_p is of type q and of cotype q for all $q < \bar{p}$.*

Next we consider the case of general Banach spaces. For $1 < p \leq 2$, let T be an operator from E' into some L^p. Then the function $\phi(\xi) = \exp(- \| T\xi \|^p)$ for $\xi \in E'$ is the characteristic function of a symmetric cylindrical measure μ on E. We say that μ is a p-stable symmetric cylindrical measure on E.

Remark. Note that for instance γ_p is a p-stable symmetric cylindrical measure on $L^{p'}$.

Let μ be a p-stable symmetric cylindrical measure with an operator T. We denote by $\Lambda_p(E', L^p)$ the set of operators T for which μ is a Radon probability measure. In this case T is called to be a Λ_p-operator.

In this section we will close with the relation between operators and cylindrical measures. Althought the results about this matter are not very deep they are crucial for our subsequent investigations. They allow us to translate properties of cylinderical measures into the language of operators and vice versa.

Let A be a linear mapping from E' into $L^0(\Omega, P)$. Then it is easy to verify that the family $\{\text{dist}(A(\xi_1), \cdots, A(\xi_n))\}$ is consistent. Hence, it defines a cylindrical measure μ^A on E with

$$\mu^A_{\xi_1 \cdots \xi_n} = \text{dist}(A(\xi_1), \cdots, A(\xi_n))$$

for all $\{\xi_1, \cdots, \xi_n\} \subset E'$. Note that

$$\text{dist}(A(\xi_1), \cdots, A(\xi_n))(B) = P(\{(A(\xi_1), \cdots, A(\xi_n)) \in B\})$$

for $B \in B(\mathbf{R}^n)$. Thus every linear mapping A from E' into $L^0(\Omega, P)$ generates a cylindrical measure on E. The converse is true as well ([5]):

Proposition 2. *Let μ be an arbitrary cylindrical measure on E. Then there exists a linear mapping A from E' into an appropriate space $L^0(\Omega, P)$ such that $\mu = \mu^A$.*

An operator A from E' into $L^p(\Omega, P)$, $0 \le p < \infty$, is decomposed if there exists an E-valued random variable ϕ with

$$A\xi(\omega) = \langle \phi(\omega), \xi \rangle, \quad P - a.e.,$$

for all $\xi \in E'$.

Proposition 3. *A cylindrical measure μ^A admits a Radon extension iff A is decomposed by an E-valued random variable ϕ. Moreover, the Radon extension coincides with dist(ϕ) .*

Remark. The operator A is said to be a linear random function associated with μ.

3 The Quasi-Sazonov Topology

First we introduce the notion " Sazonov topology ".

Definition. *Let τ be a vector topology defined on E'.*

(1) *If the following statement [$\hat{\mu} \in \Phi(E')$ is τ-continuous $\Longrightarrow \mu \in P(E)$] is satisfied, then τ is called a sufficient Sazonov topology (in shorter, SS-topology) .*

(2) *If the statement [$\mu \in P(E) \Longrightarrow \hat{\mu}$ is τ-continuous] is satisfied, then τ is called a necessary Sazonov topology (in shorter, NS-topology).*

(3) *Let τ be an SS-topology and an NS-topology, then τ is called to be a Sazonov topology (in shorter, S-topology). If there exists an S-topology on E', then E is said to be an S-space.*

Here we present a few examples of S-spaces.

Examples

1. The finite dimensional vector space $\mathbf{R}^n$ is an S-space. Usual Euclidean topology is an S-topology (Bochner's theorem).

2. Every real separable Hilbert space is an S-space. We have two S-topologies denoted by τ_{HS} and τ_m. τ_{HS} is the weakest topology satisfying that every Hilbert-Schmidt operator defined on the aforesaid Hilbert space is continuous. τ_m is the weakest topology satisfying that every γ- measurable seminorm is continuous, where γ is the canonical

Gauss cylindrical measure. Here the notion " γ-measurable " was introduced by Gross (not Dudley-Feldman-Le Cam, however these coincide with each other).

3. Banach spaces do not have always S-topologies. It follows from results of Mouchtari ([8]) that a real separable Banach space E is an S-space if E is embeded in L^0 and has a metric approximation property (m.a.p.). In this case the S-topology is τ_0. Conversely, if E is an S-space, then E is embeded in L^0 and is of cotype 2. In particular, an $L^p(1 \leq p < 2)$ space is the S-space and the S-topology is $\tau_q(0 \leq q < p)$ ([5,9]). Topologies $\tau_q(0 \leq q < 2)$ will be explained in succeeding sections.

Remark. Every real separable Banach space has an SS-topology ([6]) and an NS-topology.

Our main result is to generalize the above example 2 to the case of $L^p(1 < p < 2)$spaces.

Now we start to explain a new notion " Quasi-Sazonov topology".

Let μ be a cylindrical measure on E and T be an associated linear operator with μ, i.e. $\mu = \mu^T$. Note that μ is of type 1 iff T is a continuous linear operator from E' into $L^1(\Omega, P)$. We denote by $C_1(E)$ the set of all type 1 cylindrical measures and by $T(E')$ the set of all continuous linear operators of E' into an appropriate space $L^1(\Omega, P)$.

Definition. *Let τ be a vector topology defined on E'.* (1) *If the following statement* [$T \in T(E')$ *is τ-continuous from E' into L^1* $\Longrightarrow \mu \in P(E)$] *is satisfied, then τ is called a quasi-sufficient Sazonov topology (in shorter, QSS-topology).*

(2) *If the statement* [$\mu \in C_1(E) \cap P(E) \Longrightarrow T$ *is τ-continuous from E' into L^1*] *is satisfied, then τ is called a quasi-necessary Sazonov topology (in shorter,*

QNS-topology).

(3) *Let* τ *be a QSS-topology and a QNS-topology, then* τ *is called to be the quasi-Sazonov topology (in shorter , QS-topology*).

Before we show the relation between the Sazonov topology and the quasi- Sazonov topology, we state a theorem due to Nikishin([7]).

Theorem 1. *Let* $0 \leq q \leq p \leq 2$ *and* X *be a quasi-Banach space of stable type* p. *Every continuous linear operator* T *from* X *into* L^q *can be factorized in the following way;*

$$T : X \longrightarrow L^q \quad \leadsto \quad T = T_g \circ S, \quad S : X \to L^p, \quad T_g : L^p \to L^q$$

S *is continuous and* T_g *a multiplication by* g *in* L^r, $1/q = 1/p + 1/r$.

Remark. If $0 < p \leq 2$, then we say that the Banach space E has stable type p provided that

$$(\sum_{i=1}^{\infty} \| x_i \|^p)^{1/p} < \infty \quad (\{x_i\} \subset E)$$

implies the almost everywhere existence of $\sum_{i=1}^{\infty} \theta_i^{(p)} x_i$, where $\{\theta_i^{(p)}\}$ is a sequence if independent real-valued random variables with

$$\widehat{\theta_i^{(p)}}(t) = e^{-|t|^p}, \quad t \in \mathbf{R}, \ 0 < p < 2$$

and

$$\widehat{\theta_i^{(2)}}(t) = e^{-|t|^2/2}, \quad t \in \mathbf{R}.$$

Now we start to explain one of main theorems.

Theorem 2. *Let* τ *be a vector topology defined on* E'.

(1) *If τ is an SS-topology, then τ is a QSS-topology.*

(2) *Suppose that E' is of stable type 1 , then a QSS-topology coincides with an SS-topology.*

(3) *If τ is a QNS-topology, then τ is an NS-topology.*

Proof. (1) Suppose that τ is an SS-topology. Let μ be a type 1 cylindrical measure on E, and T be an associated linear operator with μ. If T is τ-continuous from E' into L^1, then T is τ-continuous from E' into L^0. This means that $\hat{\mu}$ is τ-continuous from E' into the complex plane $\mathbf{C}$. Therefore $\mu \in P(E)$.

(2) Let T be a linear random function associated with $\mu \in C(E)$ and τ be a QSS-topology. Suppose that T is τ-continuous from E' into L^0 .

Using Nikishin's Theorem, T is factorized in the following way :

$$T = T_g \circ S \quad ; \quad S : E' \longrightarrow L^1, \quad T_g : L^1 \longrightarrow L^0$$

S is τ-continuous from E' into L^1 and T_g a multiplication by g in L^0. Let ν be the cylindrical measure associated with S. ν is of type 1. Since τ is a QSS-topology, we have $\nu \in P(E)$. This means S is a decomposed operator. Then there exists an E-valued random variable ϕ such that $S\xi = \langle \phi(\cdot), \xi \rangle$ for every $\xi \in E'$. $g\phi$ is also the E-valued random variable, then T is a decomposed operator and $\mu \in P(E)$.

(3) Assume that τ is a QNS-topology. Let μ be a cylindrical measure and T be an associated linear random function with μ. If μ is extensible to a Radon probability measure, then T is a decomposed operator from E' into L^0. We have $T\xi = \langle \phi(\cdot), \xi \rangle$ for some E-valued random variable $\phi(\cdot)$. Then we define a random variable

$$\psi(\omega) \equiv \phi(\omega) / \parallel \phi(\omega) \parallel \quad (set\ \psi(\omega) = 0\ if\ \phi(\omega) = 0)$$

and an operator $S \in L(E', L^1)$ by $S\xi \equiv \langle \psi(\cdot), \xi \rangle, \xi \in E'$. It follows that $T = XS$ where $X \in L(L^1, L^0)$ is defined by $Xf \equiv \| \phi(\cdot) \| f$. We denote by ν the associated cylindrical measure with S, then ν is of type 1. Since S is decomposed operator, ν is a Radon probability and so S is τ-continuous from E' into L^1. This implies that T is τ- continuous from E' into L^0. Then τ is an NS-topology.

This completes the proof. □

For $1 < p < \infty$, the space L^p is of stable type 1. Therefore we get the following corollary.

Corollary. *Every QS-topology on L^p for $1 < p < \infty$ is an S-topology.*

4 Topology M_p

We will define the new topology which is constructed by the canonical p-stable symmetric cylindrical measure γ_p .

Let $1 < p \le 2$ and p' be the conjugate index to p.

Definition. *Let N be the family of all continuous seminorms defined on $L^{p'}$ that are γ_p-measurable* ([1]). *We denote by M_p the weakest topology such that every seminorm belonging to N becomes to be continuous.*

Remark. M_2 is equal to τ_m (which is defined by Gross and Kuo).

Here we have the main theorem as follows.

Theorem 3. *Let $1 < p \leq 2$ and p' be the conjugate index to p. The topology M_p is an S-topology of L^p.*

Proof. First we show that M_p is a QSS-topology of L^p. Let λ be a type 1 cylindrical measure on L^p and T be the associated linear random function from $L^{p'}$ into L^1. Suppose that T is M_p - continuous. There exists a seminorm $s(\cdot)$ belonging to N. For any $\epsilon > 0$, there exists a $\delta > 0$ such that $s(x') < \delta$ implies

$$\int |T(x')(\omega) - T(0)(\omega)| dP < \epsilon.$$

Therefore T is decomposed as follows :

$$T = \Phi \circ i \qquad i : L^{p'} \longrightarrow L^{p'}_{s(\cdot)} \qquad \Phi : L^{p'}_{s(\cdot)} \longrightarrow L^1,$$

where $L^{p'}_{s(\cdot)}$ means the associated Banach space with $s(\cdot)$ and i is the canonical injection of $L^{p'}$ into $L^{p'}_{s(\cdot)}$. Let i' be the dual operator of i. i' is 1-summing operator. Because γ_p - measurability of $s(\cdot)$ implies that the image measure $i(\gamma_p)$ is able to extensible to a Radon measure and also γ_p is of cotype 1. By Theorem of Schwartz ([11]), i' is 1-summing. Also L^p is reflexive and so i' is 1-Radonifying operator.

Let I be the identity map of L^1 and ν be the associated cylindrical measure with I. Since $\phi'(\nu)$ is the associated cylindrical measure with ϕ, which is of type 1. Hence $i'(\phi'(\nu))$ is extensible to a Radon measure. Since $i'(\phi'(\nu)) = \lambda$, the desired result is gotten.

Therefore M_p is an SS-topology.

On the other hand L^p has an NS-topology τ_0 and M_p is stronger than τ_0. This means M_p is an NS-topology. The proof is completed. □

5 M_p and τ_p

Throught this section, $1 < p \leq 2$ and $1/p + 1/p' = 1$.

First we explain some notions. Recall that a measure $\mu \in P(E)$ is p-stable symmetric iff there exist a probability space (Ω, P) and an operator $T \in L(E', L^p(\Omega, P))$ such that $\hat{\mu}(\xi) = \exp(- \| T\xi \|^p), \xi \in E'$, and T is a Λ_p-operator and denote by $\Lambda_p(E', L^p)$ the set of all Λ_p-operators.

Let $\Pi_p(E, G)$ be the set of all p-summing operators from E into G , where G is a Banach space.

Moreover, $\Lambda_p^{dual}(L^{p'}, E)$ denotes the set of operators from $L^{p'}$ into E for which the dual operator belongs to $\Lambda_p(E', L^p)$. Equivalently, $S \in \Lambda_p^{dual}(L^{p'}, E)$ iff $S(\gamma_p)$ extends to a Radon measure on E.

Next we introduce some topologies generated by stable measures. For any $q \in (0, 2]$ we define a vector topology τ_q on E' by the following neighbourhood basis of zero :

$$\{\{\xi \in E'; \| S\xi \| \leq 1\}; S \in \Lambda_q(E', L^q)\}.$$

For $q = 0$, τ_0 is generated by all decomposed operators S from E' into some $L^0(\Omega, P)$.

For later reference we will state some propositions ([5], [13]).

Proposition 4. *For any* $0 < q < p$ *we have*

$$\Pi_q(L^{p'}, L^p) = \Lambda_p(L^{p'}, L^p)$$

Proposition 5. *If $0 < q < p$ and $1 < p \leq 2$, then*

$$\Pi_q(L^{p'}, E) \subset \Lambda_p^{dual}(L^{p'}, E).$$

Proposition 6. *If $2 < q < \infty$ and $1 < p \leq 2$, then $\Lambda_p^{dual}(L^{p'}, L^q)$ is not included in $\Pi_p(L^{p'}, L^q)$.*

Proposition 7. *If $0 \leq r < q \leq 2$, then τ_r is stronger than τ_q.*

Remark. We denote by $\tau_r > \tau_q$ the above case.

Proposition 8. *If E has stable type q, $0 < q \leq 2$, then it follows that $\tau_q = \tau_0$ on E' .*

Here we recall the following theorem.

Theorem 4. *Every real separable Hilbert space has two S-topologies that are τ_{HS} and τ_m satisfying that $\tau_{HS} < \tau_m$ and $\tau_{HS} \neq \tau_m$.*

Remark. Note that $\tau_{HS} = \tau_0$ and $\tau_m = M_2$.

It follows from Propositions 5, 8 that $M_p > \tau_0 = \tau_q$ if $q < p$. Proposition 6 implies that $M_p \neq \tau_0$.

Therefore we have the next theorem.

Theorem 5. *For $1 < p \leq 2$, the L^p- space has two S-topologies that are τ_0 and M_p satisfying that $\tau_0 < M_p$ and $\tau_0 \neq M_p$.*

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