数理解析研究所講究録
1041 巻 1998 年 119-126

RECURSIVE ENUMERABILITY IN SET THEORY

MASAMICHI WATE AND TAKEO YAKU

(Received Jun. 19, 1997)

ABSTRACT. The rank[1] of a set in the class V represents the complexity of the construction of the set with respect to power operation. The rank ω_1^{CK} of the set of all recursively enumerable ordinals was introduced in [2]. Accordingly, the rank ω_1^{CK} derives a certain constructible set $L_{\omega_1^{CK}}$ [1, 2, 3], where $L_{\omega_1^{CK}}$ is the set of all constructible sets with a rank less than ω_1^{CK} [2, 3].

The purpose of this paper is to investigate the ranks and the recursive enumerability in constructible sets.

We will extend recursive enumerability ω_1^{CK} from the ordinals Ord to the constructible sets L and define the set L^{re}. It is shown that L^{re} satisfies sevral axioms and does not satisfy remained axioms. It is also shown that $L^{re} \subset L_{\omega_1^{CK}}$. It is noted that the ranks of L^{re} and $L_{\omega_1^{CK}}$ are equal.

1. INTRODUCTION

Recursive enumerability seems to represent the limit of computers. Recursive enumerability in Ord, which is an extension of the natural numbers, has been introduced and studied by several authors. In this paper, we deal recursive enumerability in general sets, in order to investigate the limit of computers, beyond the ordinal numbers. A *set* denotes a collection which has an *upper bound*. A *class* denotes collection which may have no *upper bound*. The class V was defined in [1]. An ordinal is an extension of the set ω of natural numbers. The class L of *constructible sets* was defined in [1]. The set ω_1^{CK} of all *recursively enumerable ordinals* was defined in [2, 3], by the notion of Gödel numbering, which is also called the set of *constructive* ordinals. We note that $\omega_1^{CK} \subset Ord \subset L \subseteq V$, where Ord denotes the class of all ordinals. Gödel introduced the notion of *rank* in V [1], which is defined by power set and represents a complexity of construction of a set in V. Gödel also introduced in [1] $L_\alpha =^{def} \{x \in L \mid \text{rank}(x) < \alpha\}$, by the notion of "rank".

We deal with the relation of "rank" and "recursive enumerability" in the class L. We will extend *recursively enumerability* in the ordinals Ord and define a set L^{re} contained the class L, by Gödel numbering. The definition of L^{re} is regarded as reasonable, since it will be shown that $L^{re} \cap Ord = \omega_1^{CK}$.

In §2, we will review the Gödel numbering with respect to number theoretic functions, and the notion of recursively enumerable ordinals. In §3, we will define L^{re}

---

[1] 1991 *Mathematics Subject Classification*. Primary 04-04, 03E45.
Key words and phrases. ZFC, recursive enumerability, Gödel number, constructive ordinal.

and will show properties of L^{re} with respect to ZFC. In §4, we will show that $L^{re} \subset L_{\omega_1^{CK}}$, i.e. the recursive enumerability (Gödel numbering) does not correspond to the constructibility with a rank less than ω_1^{CK}. Hence, in L, the notion of recursive enumerability (Gödel numbering) is essentially different from the notion of constructibility restricted by the rank ω_1^{CK}, while, in Ord, the recursive enumerability and the constructibility restricted by the rank ω_1^{CK} are the same.

2. Preliminaries

In this section, we introduce preliminary definitions and methods [4, 5] and [2, 3].

A *Gödel numbering* of a certain set is an onto mapping from a certain subset of the natural numbers to the set, i.e. each member of the set corresponds to at least one natural number, and each natural number corresponds to at most one member of the set.

When a natural number n corresponds to a member a of the set, we call n a *Gödel number* of a, and we denote $n = \mathrm{gn}(a)$ or $a = \mathrm{ng}(n)$. The gn is not necessarily a one-valued function.

We define $\langle n_0, \ldots, n_k \rangle = 2^{n_0} \cdots p_k^{n_k}$, where p_k is the $k+1$-th prime number. A function f from the n-tuples of the natural numbers to the natural nunbers is *genaral recursive* if it is obtained by finite applications of the following (1) $\sim$ (6).

(1) $C(x) = 0$,
(2) $S(x) = x + 1$,
(3) $U_i^n(x_1, \ldots, x_n) = x_i \ (1 \leq i \leq n)$
(4) $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$
when $g_1, \ldots, g_m$, and h are general recursive,
(5) $\begin{cases} f(0, x_2, \ldots, x_n) = g(x_2, \ldots, x_n) \\ f(x_1 + 1, x_2, \ldots, x_n) = h(f(x_1, x_2, \ldots, x_n), x_1, x_2, \ldots, x_n) \end{cases}$
when g and h are general recursive,
(6) $f(x_1, \ldots, x_n) = \mu y \ (g(y, x_1, \ldots, x_n) = 0)$
when g is general recursive, and for each n-tuple $x_1, \ldots, x_n$, there exists a y, such that $g(y, x_1, \ldots, x_n) = 0$,

where, $\mu y \ (g(y, x_1, \ldots, x_n) = 0)$ means minimum natural number y such that $g(y, x_1, \ldots, x_n) = 0$ for $x_1, \ldots, x_n$.

Now, we carry out the Gödel numbering of the general recursive functions according to the order of the construction of them. We define

(i) $\mathrm{gn}(C) = \langle 1 \rangle$,
(ii) $\mathrm{gn}(S) = \langle 2 \rangle$,
(iii) $\mathrm{gn}(U_i^n) = \langle 3, n, i \rangle$,
(iv) $\mathrm{gn}(f) = \langle 4, \mathrm{gn}(g_1), \ldots, \mathrm{gn}(g_m), \mathrm{gn}(h) \rangle$ when f is of the form (4),
(v) $\mathrm{gn}(f) = \langle 5, \mathrm{gn}(g), \mathrm{gn}(h) \rangle$ when f is of the form (5),
(vi) $\mathrm{gn}(f) = \langle 6, \mathrm{gn}(g) \rangle$ when f is of the form (6).

For example, in the case of $f(x) = 1$, $\mathrm{gn}(f) = \langle 4, \langle 1\rangle, \langle 2\rangle\rangle$, since $f(x) = S(C(x))$, and in the case of $f(x,y) = x+y$, $\mathrm{gn}(f) = \langle 5, \langle 3,1,1\rangle, \langle 4, \langle 3,3,1\rangle, \langle 2\rangle\rangle\rangle$, since f is gotten by the application of the form (5) for $g(y) = U_1^1(y)$ and $h(z,x,y) = S(U_1^3(z,x,y))$. Both of $\langle 1\rangle$ and $\langle 4, \langle 3,1,1\rangle, \langle 1\rangle\rangle$ are Gödel numbers of the constant function $f(x) = 0$, since $C(U_1^1(x)) = C(x) = 0$. Thus, a Gödel number of a general recursive function is determined but not uniquely. This technic of Gödel numbering owes [4, 5]. When $\mathrm{gn}(f) = e$, we write $f = \{e\}$, i.e. $\forall n \in \omega\ f(n) = \{e\}(n)$ [4].

S.C.Kleene introduced the set ω_1^{CK} of recursively enumerable ordinals and its Gödel numbering O, in [2, 3]. He named the members of ω_1^{CK} "constructive ordinals". The set ω_1^{CK} of the constructive ordinals and the set O of natural numbers which express them (the ordinal which a natural number n is expressed by $|n|$), are defined by simultaneos induction as follows:

1. $1 \in O$ and $|1| = 0$, i.e. the natural number 1 expresses the ordinal 0,
2. if $x \in O$ then $2^x \in O$ and $|2^x| = |x| + 1$, i.e. if a natural number x expresses an ordinal α, then the natural number 2^x expresses the ordinal $\alpha + 1$,
3. if x is a Gödel number of a certain partial recursive function, and there is a sequence of integers $y_0, y_1, \ldots, y_n, \ldots$ such that $y_n \in O$, $|y_n| = n$, $\{x\}(y_n)$ is defined, $\{x\}(y_n) \in O$ for each $n \in \omega$, and $|\{x\}(y_0)| < |\{x\}(y_1)| < \cdots < |\{x\}(y_n)| < \ldots$, then $3 \cdot 5^x \in O$ and $|3 \cdot 5^x| = \lim_{n\to\infty} |\{x\}(y_n)|$, i.e. if a natural number x is a Gödel number of such a function f, then the natural number $3 \cdot 5^x$ expresses the limit ordinal $\alpha = \lim_{n\to\infty} \alpha_n$, where $\alpha_n = |f(y_n)|$. Then $\omega_1^{CK} =^{def} |O| =^{def} \sup_{n\in O} |n|$.

ω is the set of all natural numbers. $l, m, n, i, j, k, \ldots$ are variables which range over ω. ω_1^{CK} is the set of all recursively enumerable ordinals. Ord is the class of all ordinal numbers. $\alpha, \beta, \gamma, \ldots$ are variables which range over Ord. V is the class of all sets, i.e.

1. $V_0 = \emptyset$,
2. $V_{\beta+1} = V_\beta \cup p(V_\beta)$,
3. $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ (α is a limit ordinal),
4. $V = \bigcup_{\alpha\in Ord} V_\alpha$.

$x, y, z, \ldots$ are variables which range over V. L is the class of all constructible sets. i.e.

1. $L_0 = \emptyset$,
2. $L_{\beta+1} = L_\beta \cup \{x | x \subseteq L_\beta \wedge (x \text{ is gotten by Gödel operation})\}$,
3. $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ (α is a limit ordinal),
4. $L = \bigcup_{\alpha\in Ord} L_\alpha$.

3. The Set L^{re} and Its Properties

In this section, we introduce the set L^{re}, and describe its properties in the axiomatic set theory ZFC.

Definition Let f be a general recursive function with one variable such that if $f(x) = 0$ then also $f(x+1) = 0$. If the range of f except 0 is a family of Gödel numbers of all members of a set a, we define that a Gödel number of f is a Gödel number of the set a. Namely, if $a = \{\text{ng}(\{e\}(x)) | \{e\}(x) > 0\}$ then $e = \text{gn}(a)$.

In this way, Gödel numbers of sets are defined inductively, but it is impossible to do Gödel numbering about all sets. So, we define the set L^{re} be all sets which have Gödel numbers.

It is familiar by the Löwenheim-Scolem theorem that there is a countable model of ZFC. However, we can see that L^{re} is not only a countable model, but also an effective model of ZFC, if we restrict the axiom of power set and replacement (separation).

Proposition 1. *L^{re} satisfies the following axioms in ZFC:*

(Axiom of Empty), *(Axiom of Pairing)*,
(Axiom of Extensionality), *(Axiom of Union)*,
(Axiom of Infinity), *(Axiom of Regularity)*,
(Axiom of Choice).

Let $A(x, y, x_1, \ldots, x_n)$ be a formula such that $\forall x \exists ! y A(x, y, c_1, \ldots, c_n)$ for constants $c_1, \ldots, c_n$. Putting $y = f(x)$ if and only if $A(x, y, c_1, \ldots, c_n)$, f is a function from the class of all sets V to V. But, there is not necessarily an algorithm to get y from a given x. For example, let $A(x, y)$ be the formula

$$(x \in \omega \wedge \exists z\ T_1(x, x, z) \wedge y = x) \vee (\neg(x \in \omega \wedge \exists z\ T_1(x, x, z)) \wedge y = 0)$$

(cf. [4] about T_1). Then y is determined uniquely from x for all x, but there is no algorithm to get y from x. Hence, $\{y \in \omega | \exists x\ A(x, y)\}$ is in $V_{\omega+1}$, possibly is in $L_{\omega+1}$, but is not in L^{re}. However, if we modify the axiom of replacement as follows, L^{re} satisfies the modified replacement.

Axiom 1. Modified Replacement Let $f(x, z_1, \ldots, z_n)$ be general recursive function such that when constants $c_1, \ldots, c_n$ are given, for each set a, $f(\text{gn}(a), c_1, \ldots, c_n)$ is a Gödel number of a certain set. Then

$$\forall u \exists v \forall r \forall s (s \in v \equiv r \in u \wedge f(\text{gn}(r), c_1, \ldots, c_n) = \text{gn}(s))$$

This axiom is a special case of usual axiom, since $y = f(x, z_1, \ldots, z_n)$ is expressed by a certain formula of set theory when f is general recursive.

Proposition 2. *L^{re} satisfies the modified replacement.*

The Axiom of Separation does not necessary hold as well as Axiom of Replacement. For example, nevertheless $\omega \in L^{re}$, the set $\{x \in \omega | \exists y \in \omega T_1(x, x, y)\}$ is not in L^{re}, since there is no algorithm to decide whether $\exists y \in \omega T_1(x, x, y)$ for given $x \in \omega$. So, we will mofify Axiom of Separation as well as Axiom of Replacement as follows. Then we can show that Modified Axiom of Separation is derived from Modified Axiom of Replacement in L^{re}, as well as in ZFC.

Axiom 2. Modified Separation Let $A(x, z_1, \ldots, z_n)$ be a general recursive predicate. Then, $\forall x \exists y \forall w (w \in y \equiv w \in x \wedge A(\mathrm{gn}(w), c_1, \ldots, c_n))$ for each fixed $c_1, \ldots, c_n$.

Proposition 3. *Modified Separation is derived from Modified Replacement.*

It seems natural to define power set of a set a in L^{re} as $p_{re}(a) = \{x | x \subseteq a \wedge x \in L^{re}\}$. However, $p_{re}(a)$ may not have a Gödel number itself, even though each element of it has a Gödel number. In the case of a being an infinite set, putting

$$f(n) = \begin{cases} \mu y \ (\mathrm{ng}(y) \subseteq a) & \text{if } n = 0, \\ \mu y \ (\mathrm{ng}(y) \subseteq a \wedge y > f(n-1)) & \text{otherwise,} \end{cases}$$

f is certainly a mapping from natural numbers to natural numbers and the range of f is a set of Gödel numbers of the elements of $p_{re}(a)$. But, the formula $\mathrm{ng}(y) \subseteq a$ may not be general recursive, and hence f may not so, for there is not necessarily an algorithm to deside whether $a \subseteq b$ or not, for two sets a and b.

On the other hand, since the subsets of $\{x | x \subseteq a \wedge x \in L^{re}\}$ in L^{re} are countable, we can express them $a_1, a_2, \ldots, a_n, \ldots$. Putting $p_{re}(a) = \bigcup_{n=1}^{\infty} a_n$, $p_{re}(a)$ determines uniquely. But, $f(n) = \mathrm{gn}(a_n)$ is not necessarily general recursive, hence $p_{re}(a) \notin L^{re}$ in general.

However, if we weaken the axiom of power set as following, L^{re} satisfies the modified axiom of power set, too:

Axiom 3. Modified axiom of power set For a given $a \in L^{re}$, there is a set $b \in L^{re}$ such that each element of b is a subset of a, and b contains all finite subset of a.

Proposition 4. *L^{re} satisfies the modified axiom of power set.*

Now, we classify L^{re} to hierarchy. Define

$$\begin{cases} L_0^{re} = \emptyset, \\ L_{\beta+1}^{re} = \{a \in L^{re} | a \subseteq L_\beta^{re}\}, \\ L_\alpha^{re} = \bigcup_{\beta < \alpha} L_\beta^{re} \quad (\alpha \text{ is a limit number}). \end{cases}$$

L_α^{re} may not be in L^{re}, in general.

Proposition 5. *For each α, $L_\alpha^{re} \subseteq L_\alpha$.*

Remark Of course, $V_\omega = L_\omega = L_\omega^{re}$, since any finite set have a Gödel number e such that $\{e\}(n) = 0$, for all sufficiently large n and this is general recursive. And $L_\alpha^{re} \subseteq L_\alpha \subseteq V_\alpha$, in general.

4. Main Theorem

Now, we describe a few Lemmas to prove Main Theorem.

Lemma 1. *If $\alpha < \omega_1^{CK}$, then $\alpha \in L^{re}$.*

Proof Suppose that α is an ordinal less than ω_1^{CK}. By induction on the construction of α.
Case 1. $\alpha = 0$. Clearly, $0 = \emptyset$ is in L^{re}.
Case 2. $\alpha = \beta+1$. By the hypothesis of induction, β is in L^{re}. Therefore, $\alpha = \beta \cup \{\beta\}$ is in L^{re}.
Case 3. α is a limit number. Then, there is a strictly increasing sequence converging to α ($\beta_0 < \beta_1 < \cdots < \beta_n < \cdots < \alpha$) and a partial recuesive function $\{y\}$ such that $|\{y\}(y_n)| = \beta_n$ when $|y_n| = n$. By the hypothesis of induction, $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ are in L^{re}. Therefore, there is a general recursive function g, such that for each fixed n, the range of $g(n, m)$ is a set of Gödel numbers of all ordinals less than β_n.

```
begin i := 0;
  for j := 0 to ∞ do
  for k := 0 to j do
    if g(k, j − k) > 0 then begin f(i) := g(k, j − k); i := i + 1 end
end.
```

Then the function f is general recursive and $\mathrm{gn}(f)$ is a Gödel number of α. □

Lemma 2. *If $\alpha \in L^{re}$, then $\alpha < \omega_1^{CK}$.*

Proof Suppose that α is an ordinal in L^{re}. By induction on the construction of α.
Case 1. $\alpha = 0$. Then $|1| = 0$ and $1 \in O$.
Case 2. $\alpha = \beta + 1$. By the hypothesis of induction, there is a natural number y in O such that $|y| = \beta$. Then, $|2^y| = \alpha$ and $2^y \in O$.
Case 3. α is a limit number, and there is a general recursive function f such that the range of f is the set of Gödel numbers of ordinals less than α. Since α is a limit number, $f(n) > 0$ for each n. By the hypothesis of induction, there is a natural number $y_n \in O$ such that $|y_n| = \mathrm{ng}(f(n))$, for each n. Put

$$\begin{cases} g(1) = y_0, \\ g(2^n) = y_{\mu m(|y_m| > |g(n)|)}. \end{cases}$$

Then, the function g is partial recursive. So, g has a Gödel number z. Hence $3 \cdot 5^z$ is in O and $|3 \cdot 5^z| = \alpha$. □

Lemma 3. *If $x \in L^{re}$, then $rank(x) \in L^{re}$*

Proof Let $\alpha = \text{rank}(x)$. By transfinite induction. We can assume that α is a limit ordinal, since it is clear in the case of being an isolated ordinal. Let e be a Gödel number of x. Then, $\forall n\ \text{rank}(\text{ng}(\{e\}(n))) < \alpha$, hence $\text{rank}(\text{ng}(\{e\}(n))) \in L^{re}$. So, we put $r_{\{e\}(n)} = \text{gn}(\text{rank}(\text{ng}(\{e\}(n))))$. Since $r_{\{e\}(n)}$ is a Gödel number of an ordinal number, the range of $\{r_{\{e\}(n)}\}(m)$ $(m = 0, 1, \dots)$ is the set of the Gödel numbers of all ordinal numbers less than $\text{ng}(r_{\{e\}(n)})$. So, we put

begin $i := 0$;
  for $j := 0$ to ∞ do
  for $k := 0$ to j do
    if $\{r_{\{e\}(k)}\}(j-k) > 0$ then begin $r_e(i) := \{r_{\{e\}(k)}\}(j-k)$; $i := i+1$ end
end.

Then, r_e is a general recursive function, and the Gödel number of r_e is a Gödel number of $\text{rank}(x)$, since $\text{ng}(r_e)$ is a limit ordinal. □

Remark (1) There is no algorithm to get $\text{rank}(x)$ for given x. To insist $\alpha = \text{rank}(x)$, we must show $x \in V_{\alpha+1} - V_\alpha$, by the definition of "rank". That is $x \subseteq V_\alpha$ and $x \notin V_\alpha$. But we have no algorithm to decide whether $x \subseteq V_\alpha$ or not.

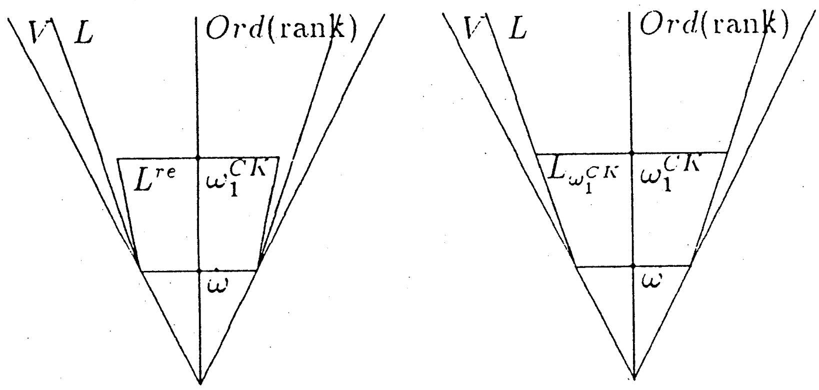
(2) $\text{rank}(L^{re}) = \text{rank}(L_{\omega_1^{CK}}) = \omega_1^{CK}$.

Main theorem *$L^{re} \subset L_{\omega_1^{CK}}$ and if $\alpha < \omega_1^{CK}$, $L^{re} \not\subseteq L_\alpha$.*

Proof Let x be in L^{re}. Then $\text{rank}(x)$ is in L^{re} by Lemma 3. So, $\text{rank}(x) < \omega_1^{CK}$ by Lemma 2. Therefore, x is in $L_{\omega_1^{CK}}$. Namely, $L^{re} \subseteq L_{\omega_1^{CK}}$. It is easy from a comment in §3 that L^{re} is a proper subset of $L_{\omega_1^{CK}}$

Let $\alpha < \omega_1^{CK}$. Then α is in L^{re} by Lemma 1, but α is not in L_α. Hence, $L^{re} \not\subseteq L_\alpha$.
□

The result of Main Theorem is shown in the following figure.



The rank ω_1^{CK} corresponds to the set of recursively enumerable ordinals. The rank ω corresponds to the set ω of the natural numbers.

ACKNOWLEDGEMENT

The authors would like to express their graduate to Prof. T. Miyadera, Prof. K. Sugita, and Prof. K. Tsuchida for helpful suggestions on the improvement of this paper.

REFERENCES

1. GÖDEL, K., *Consistency-proof for the generalized hypothesis*, Proc. Nat. Acad. **25**(1939)
2. KLEENE, S. C., *On notation for ordinal numbers*, J. Symbolic Logic **3**, pp. 150-155(1938)
3. KLEENE, S. C., *On the forms of the predicates in the theory of constructive ordinals (second paper)*, Amer. J. Math. **77**, pp. 405-428(1955)
4. KLEENE, S. C., *Recursive functionals and quantifiers of finite types I*, Trans. Amer. Math. Soc. **91**, pp. 1-52(1959)
5. KLEENE, S. C., *Recursive functionals and quantifiers of finite types II*, ibid. **108**, pp. 106-142(1963)

DEPARTMENT OF MATHEMATICS, COLLEGE OF HUMANITIES AND SCIENCES, NIHON UNIVERSITY, 3-25-40, SAKURAJOUSUI, SETAGAYA, TOKYO 156, JAPAN

E-mail: wate@math.chs.nihon-u.ac.jp

E-mail: yaku@math.chs.nihon-u.ac.jp