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LAYER HEAT POTENTIALS FOR A BOUNDED CYLINDER WITH FRACTAL LATERAL BOUNDARY

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1. Introduction

Let D be a bounded smooth domain in $\mathbf{R}^d$ and set

$$\Omega_D = D \times (0,T) \quad \text{and} \quad S_D = \partial D \times [0,T].$$

The double layer heat potential Φf of $f \in L^p(S_D)$ is defined by

$$\Phi f(X) = -\int_0^T \int_{\partial D} \langle \nabla_y W(X-Y), n_y \rangle f(Y) d\sigma(y) ds \tag{1.1}$$

for $X = (x,t) \in (R^d \setminus \partial D) \times \mathbf{R}$, where $\langle\ ,\ \rangle$ is the inner product in $\mathbf{R}^d$, n_y is the unit outer normal to ∂D, σ is the surface measure on ∂D and W is the fundamental solution for the heat operator, i.e.,

$$W(X) = W(x,t) = \begin{cases} \dfrac{\exp\left(-\frac{|x|^2}{4t}\right)}{(4\pi t)^{d/2}} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The double layer heat potential is important not only physically but also mathematically. For example, R. M. Brown proved that the solution to the initial-Dirichlet problem in a Lipschitz cylinder for the heat operator can be written by a double layer heat potential and the solution to the initial-Neumann problem in a Lipschitz cylinder for the heat operator is given by a single layer heat potential (cf. [B_1], [B_2]).

If D is a bounded domain with fractal boundary, then n_y and the surface measure can not be defined. But if D has a smooth boundary and f is a C^1-function on $\mathbf{R}^{d+1}$ with compact support, then we see by the Green formula that for $X = (x,t) \in D \times \mathbf{R}$

$$\begin{aligned} \Phi f(X) = &\int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y f(Y), \nabla_y W(X-Y) \rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} f(Y) \triangle_y W(X-Y) dy \end{aligned} \tag{1.2}$$

and for $X = (x,t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$

$$\Phi f(x) = -\int_0^T ds \int_D \langle \nabla_y f(Y), \nabla_y W(X-Y) \rangle dy - \int_0^T ds \int_D f(Y) \Delta_y W(X-Y) dy. \tag{1.3}$$

So we see that, if a function f defined on S_D can be extended to be a function $\mathcal{E}(f)$ on $\mathbf{R}^d \times [0,T]$ such that for each $t \in [0,T]$ the function $x \mapsto \mathcal{E}(f)(x,t)$ is a C^1-function on $\mathbf{R}^d \setminus \partial D$ and, for each $x \in \mathbf{R}^d \setminus \partial D$ and each j ($j = 1, 2, \cdots, d$) the function $t \mapsto \frac{\partial \mathcal{E}(f)}{\partial x_j}(x,t)$ is measurable, then the right-hand sides of (1.2) and (1.3) may be defined.

In this paper we assume that D is a bounded domain in $\mathbf{R}^d$ ($d \geq 2$) and ∂D is a β-set satisfying $d-1 \leq \beta < d$. Here, according to **[JW]** we say that a closed set F is a β-set if there exist a positive Radon measure μ on F and positive real numbers r_0, b_1, b_2 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap F) \leq b_2 r^\beta \tag{1.4}$$

for all $z \in F$ and all $r \leq r_0$, where $B(z,r)$ stands for the open ball in $\mathbf{R}^d$ with center z and radius r.

We note that, if D is a bounded Lipschitz domain, then ∂D is a $(d-1)$-set and the surface measure μ has the property (1.4) for $F = \partial D$ and $\beta = d-1$. Furthermore if ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set such that the β-dimensional Hausdorff measure $\mathcal{H}^\beta$ restricted to ∂D has the property (1.4) for $F = \partial D$ (cf. **[Hu]**).

Let $0 < \alpha \leq 1$ and F be a closed set in $\mathbf{R}^d$. We denote by $\Lambda_\alpha(F \times [0,T])$ the Banach space of all continuous functions f on $F \times [0,T]$ such that $f(\cdot,t)$ is α-Hölder continuous for every $t \in [0,T]$ with norm

$$\|f\|_{\infty,\alpha} = \sup_{X \in F \times [0,T]} |f(X)| + \sup_{x,y \in F, x \neq y, t \in [0,T]} \frac{|f(x,t) - f(y,t)|}{|x-y|^\alpha}.$$

Further let $0 < \alpha$, $\lambda \leq 1$. We also denote by $\Lambda_{\alpha,\lambda}(F \times [0,T])$ the Banach space of all $f \in \Lambda_\alpha(F \times [0,T])$ such that f is λ-Hölder continuous with respect to the time variable with norm

$$\|f\|_{\infty,\alpha,\lambda} = \|f\|_{\infty,\alpha} + \sup_{x \in F, t,s \in [0,T], t \neq s} \frac{|f(x,t) - f(x,s)|}{|t-s|^\lambda}.$$

We will prove the following lemma in §3.

Lemma 1.1. *Let $d-1 \le \beta < d$ and F be a compact β-set in $\mathbf{R}^d$ satisfying (1.4) and $F \subset B(0, R/2)$. Then there exists a bounded operator $\mathcal{E}$ from $\Lambda_\alpha(F \times [0,T])$ to $\Lambda_\alpha(\mathbf{R}^d)$ with the following properties:*

(i) $\mathcal{E}(f)(\cdot, t)$ is a C^1-function on $\mathbf{R}^d \setminus F$ for each $t \in [0,T]$, and both of $\mathcal{E}(f)(x,\cdot)$ and $\left(\frac{\partial \mathcal{E}(f)}{\partial x_j}\right)(x,\cdot)$ $(j = 1, \cdots, d)$ are measurable for each $x \in \mathbf{R}^d$ and for each $x \in \mathbf{R}^d \setminus F$, respectively,

(ii) $\mathcal{E}(f) = f$ on F and supp $\mathcal{E}(f)(\cdot, t) \subset B(0, 2R)$ for each $t \in [0,T]$.

(iii)

$$\left|\frac{\partial \mathcal{E}(f)}{\partial y_i}(y,s)\right| \le c\|f\|_{\infty,\alpha}\mathrm{dist}(y, \partial D)^{\alpha-1}, \quad \left|\frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_k}(y,s)\right| \le c\|f\|_{\infty,\alpha}\mathrm{dist}(y, \partial D)^{\alpha-2}$$

for every $(y,s) \in (\mathbf{R}^d \setminus F) \times [0,T]$.

(iv) If $f \in \Lambda_{\alpha,\lambda}(F \times [0,T])$, then $\mathcal{E}(f) \in \Lambda_{\alpha,\lambda}(\mathbf{R}^d \times [0,T])$.

Using Lemma 1.1 we define, for $f \in \Lambda_\alpha(S_D)$,

$$\begin{aligned}\Phi f(X) = &\int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y,s), \nabla_y W(X-Y)\rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \mathcal{E}(f)(y,s) \Delta_y W(X-Y) dy\end{aligned} \tag{1.5}$$

for $X = (x,t) \in D \times \mathbf{R}$ and

$$\begin{aligned}\Phi f(X) = -&\int_0^T ds \int_D \langle \nabla_y \mathcal{E}(f)(y,s), \nabla_y W(X-Y)\rangle dy \\ -&\int_0^T ds \int_D \mathcal{E}(f)(y,s) \Delta_y W(X-Y) dy\end{aligned} \tag{1.6}$$

for $X = (x,t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$.

Furthermore we also define the operator K by

$$Kf(Z) = \frac{1}{2}\left(I_1(Z) + I_2(Z)\right), \tag{1.7}$$

where

$$\begin{aligned}I_1(Z) = &\int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y,s), \nabla_y W(Z-Y)\rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(Z-Y) dy \\ &+ f(Z) \int_{(\mathbf{R}^d \setminus \overline{D}) \times \{0\}} W(Z-Y) dy\end{aligned}$$

and

$$I_2(Z) = -\int_0^T ds \int_D \langle \nabla \mathcal{E}(f)(y,s), \nabla_y W(Z-Y)\rangle dy$$
$$-\int_0^T ds \int_D (\mathcal{E}(f)(Y) - f(Z))\Delta_y W(Z-Y)dy - f(Z)\int_{D\times\{0\}} W(Z-Y)dy$$

Under these notations we will prove the following theorem in §3.

Theorem. *Assume that D is a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set. If $0 \le \beta - (d-1) < \alpha < 1$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$, then, for each $Z \in \partial D \times [0,T]$,*

$$\lim_{X\to Z, X\in D\times(0,T)} \Phi f(X) = Kf(Z) + \frac{1}{2}f(Z) \tag{1.8}$$

and

$$\lim_{X\to Z, X\in(\mathbf{R}^d\setminus\overline{D})\times(0,T)} \Phi f(X) = Kf(Z) - \frac{1}{2}f(Z). \tag{1.9}$$

Thus we see that our double layer heat potentials have the same boundary behavior as the usual ones for a bounded cylinder with smooth lateral boundary.

Remark. In this paper we shall treat the double layer heat potentials of Hölder continuous functions on S_D. But under a similar consideration we can also the double layer heat potentials of functions in a Besov space on S_D and prove that they have the parabolically non-tangential limit at a.e. $Z \in S_D$.

2. Properties of W

In this section we recall and study properties of the function W. To do so, we use the parabolic metric δ defined by

$$\delta(X,Y) = \left(|x-y|^2 + |t-s|\right)^{1/2} \quad \text{for } X = (x,t) \text{ and } Y = (y,s).$$

Lemma 2.1. (i) $W(X) \le c\delta(X,0)^{-d}$,
(ii) $|\nabla_x W(X)| \le c\delta(X,0)^{-d-1}$ *if* $X \neq 0$,
(iii) $|\frac{\partial^2}{\partial x_i \partial x_j} W(X)| \le c\delta(X,0)^{-d-2}$, $|\frac{\partial}{\partial t} W(X)| \le c\delta(X,0)^{-d-2}$ *if* $X \neq 0$,
(iv) $|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} W(X)| \le c\delta(X,0)^{-d-3}$, $|\frac{\partial^2}{\partial x_j \partial t} W(X)| \le c\delta(X,0)^{-d-3}$ *if* $X \neq 0$,
(v) $|W(X-Y) - W(Z-Y)| \le c\delta(X,Z)^\epsilon\{\delta(X,Y)^{-d-\epsilon} + \delta(Z,Y)^{-d-\epsilon}\}$ *if* $0 \le \epsilon \le 1$ *and* $X \neq Y$, $Z \neq Y$,
(vi) $|\nabla_y W(X-Y) - \nabla_y W(Z-Y)| \le c\delta(X,Z)^\epsilon\{\delta(X,Y)^{-d-1-\epsilon} + \delta(Z,Y)^{-d-1-\epsilon}\}$ *if* $0 \le \epsilon \le 1$ *and* $X \neq Y$, $Z \neq Y$.

Proof. The assertions (i), (ii), (iii) and (iv) are well known (cf. [**B2**, p.5]). The assertions (v) and (vi) will be shown by the same method as in the proof of Lemma 2.3 in [**W2**]. □

Let D_0 be a bounded piecewise smooth domain in $\mathbf{R}^d$ and u, v be smooth functions on $\overline{D_0} \times [0, \rho]$. Using the divergence theorem, we obtain

(2.1)
$$\int_0^\rho \int_{D_0} (uL^*v - vLu)\, dxdt$$
$$= \int_0^\rho dt \int_{\partial D_0} \langle u\nabla_x v - v\nabla_x u, n_x\rangle d\sigma(x) - \int_{D\times\{t=0\}} uvdx + \int_{D\times\{t=\rho\}} uvdx,$$

where
$$L = \Delta - \frac{\partial}{\partial t} \quad \text{and} \quad L^* = \Delta + \frac{\partial}{\partial t}.$$

If $Lu = L^*v = 0$ in $D_0 \times (0, \rho)$, then (2.1) implies

$$\text{(2.2)} \quad \int_0^\rho dt \int_{\partial D_0} \langle u\nabla_x v - v\nabla_x u, n_x\rangle d\sigma(x) - \int_{D_0\times\{t=0\}} uvdx + \int_{D_0\times\{t=\rho\}} uvdx = 0.$$

Let $X = (x, t)$ $(0 \le t \le T)$ be an exterior point of $D_0 \times (0, T)$. Then, setting $u = 1$ and $v(Y) = W(X - Y)$ and noting that $W(X - Y) = 0$ for $Y = (y, T)$, we deduce from (2.2)

$$\text{(2.3)} \qquad \int_0^T ds \int_{\partial D_0} \langle \nabla_y W(X - Y), n_y\rangle d\sigma(y) - \int_{D_0\times\{s=0\}} W(X - Y)dy = 0.$$

Hereafter we assume that D is a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set satisfying $\overline{D} \subset B(O, R/2)$.

Let us use the Whitney decomposition to approximate D and $\mathbf{R}^d \setminus \overline{D}$ (cf. [**S**, p.167]). Let $\mathcal{V}(D)$ be the Whitney decomposition of D and define

$$A_n = \cup_{k=k_0}^n \cup_{Q\in\mathcal{V}_k(D)} Q,$$

where $\mathcal{V}_k(D) = \{Q \in \mathcal{V}(D); Q \text{ is a } k\text{-cube}\}$ and k_0 is the smallest integer k such that $\mathcal{V}_k(D) \neq \emptyset$.

Similarly we also define

$$B_n = \left(\cup_{k=-\infty}^n \cup_{Q\in\mathcal{V}_k(\mathbf{R}^d\setminus\overline{D})} Q\right).$$

Then we have the following lemma.

Lemma 2.2. *Set*

$$g_n(X) = \int_0^T \int_{A_n} \Delta_y W(X-Y)dy \quad \text{and} \quad h_n(X) = \int_0^T \int_{B_n} \Delta_y W(X-Y)dy.$$

Then $\lim_{n\to\infty} g_n(X)$ *and* $\lim_{n\to\infty} h_n(X)$ *exist on* $\mathbf{R}^d \times [0,T]$ *and for* $X \in \mathbf{R}^d \times (0,T]$

$$\lim_{n\to\infty} g_n(X) = \int_{D\times\{0\}} W(X-Y)dy - \chi_D(X)$$

and

$$\lim_{n\to\infty} h_n(X) = \int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}} W(X-Y)dy - \chi_{\mathbf{R}^d\setminus\overline{D}}(X)$$

Proof. Let $X = (x,t) \in \mathbf{R}^d \times (0,T]$ and $t > \rho > 0$. Applying (2.2) to $A_n \times (0,\rho)$, we have

$$\int_0^\rho ds \int_{\partial A_n} \langle \nabla_y W(X-Y), n_y \rangle d\sigma(y) - \int_{A_n\times\{0\}} W(X-Y)dy$$
$$+ \int_{A_n\times\{\rho\}} W(X-Y)dy = 0.$$

Using the divergence theorem for A_n in $\mathbf{R}^d$, we have

$$\int_0^\rho ds \int_{A_n} \Delta_y W(X-Y)dy - \int_{A_n\times\{0\}} W(X-Y)dy$$
$$+ \int_{A_n\times\{\rho\}} W(X-Y)dy = 0.$$

As $\rho \to t$ and $n \to \infty$, we obtain,

$$\lim_{n\to\infty} g_n(X) = \int_{D\times\{0\}} W(X-Y)dy - \chi_D(X).$$

On the other hand $g_n(X) = 0$ for $t = 0$. Hence $\lim_{n\to\infty} g_n(X)$ exists for each $X \in \mathbf{R}^d \times [0,T]$.

Similarly we can also prove the conclusion for h_n.

□

3. Double layer heat potentials

In this section we first prove Lemma 1.1 in §1.

Proof of Lemma 1.1 We use the extension operator $\mathcal{E}_0$ in [S, p.172] and choose a C^∞-function ϕ_0 such that

$$\phi_0 = 1 \quad \text{on } B(0,R), \quad \operatorname{supp}\phi_0 \subset B(0,2R) \quad \text{and } 0 \le \phi_0 \le 1.$$

We define

$$\mathcal{E}(f)(x,t) = \mathcal{E}_0(f(\cdot,t))(x)\phi_0(x)$$

for $f \in \Lambda_\alpha(F)$ and $(x,t) \in (\mathbf{R}^d \setminus F) \times [0,T]$ and

$$\mathcal{E}(f)(x,t) = f(x,t) \quad \text{on } (x,t) \in F \times [0,T].$$

Then properties (i), (ii), (iii) follow from the definition and (13) on p.174 in [S]. Since the operator $\mathcal{E}_0$ is linear, positive and maps the constant function 1 to 1, (iv) is also valid. □

In [W1] we gave the following lemma.

Lemma A. *Let δ, k be non-negative numbers satisfying $d-\beta > \delta$ and $d-\delta-k > 0$. Then*

$$\int_{B(z,r)} \operatorname{dist}(y,\partial D)^{-\delta}|y-z|^{-k}dy \le cr^{d-\delta-k}$$

for every $z \in \partial D$ and $r > 0$.

We next show that the double layer heat potential defined by (1.5) and (1.6) converges.

Lemma 3.1. *Let $0 \le \beta-(d-1) < \alpha < 1$ and $f \in \Lambda_\alpha(S_D)$. Then Φf is caloric in $(\mathbf{R}^d \setminus \partial D) \times \mathbf{R}$.*

Proof. Set, for $X = (x,t) \in D \times \mathbf{R}$,

$$J_1(X) = \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \langle \nabla_y \mathcal{E}(f)(y,s), \nabla_y W(X-Y)\rangle dy \tag{3.1}$$

and let $X_0 = (x_0,t_0) \in D$. Choose $\rho > 0$ satisfying $\overline{B(x_0,2\rho)} \subset D$. If $X = (x,t) \in B(x_0,\rho) \times \mathbf{R}$, then we deduce from Lemmas 2.1 and 1.1 and Lemma A

$$|J_1(X)| \le \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \operatorname{dist}(y,\partial D)^{\alpha-1}\delta(X,Y)^{-1-d}dy \le c_1\rho^{-1-d}\|f\|_{\infty,\alpha},$$

whence J_1 converges locally uniformly in D. We denote by g_1 the integrand of the right-hand side on (3.1). Since

$$|\nabla_y \frac{\partial^2}{\partial x_i \partial x_j} W(X-Y)| \le c_2\delta(X,Y)^{-d-3} \text{ and } |\nabla_y \frac{\partial}{\partial t} W(X-Y)| \le c_3\delta(X,Y)^{-d-3},$$

we see that the integral of Lg_1 over $(\mathbf{R}^d\setminus\overline{D})\times[0,T]$ also converges locally uniformly on D. Therefore J_1 satisfies the heat equation in $D\times\mathbf{R}$.

Next, set

$$J_2(X)=\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}\mathcal{E}(f)(y,s)\Delta W(X-Y)dy.$$

Using Lemma 1.1, (iii), we can show by the above method that J_2 also converges locally uniformly in D and satisfies the heat equation. Thus we conclude that $\Phi f=J_1+J_2$ has the same properties in $D\times\mathbf{R}$. We can show that Φf also has the same properties in $(\mathbf{R}^d\setminus\overline{D})\times R$. □

Using Lemma 2.1, (iv), (v) and Lemma A, we can prove the following lemma by a similar method to that in the proof of [**W1**, Lemma 3.3].

Lemma 3.2. *Let $0\le\beta-(d-1)<\alpha<1$ and $f\in\Lambda_{\alpha,\alpha/2}(S_D)$. Then both of the function J_1 defined by (3.1) and the function J_3 defined by*

$$J_3(X)=\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}(\mathcal{E}(f)(Y)-\mathcal{E}(f)(X))\Delta W(X-Y)dy$$

are are continuous on $\mathbf{R}^d\times[0,T]$. Furthermore the function J_1' (resp. J_3') obtained by replacing $\mathbf{R}^d\setminus\overline{D}$ with D in the definition of J_1 (resp. J_3) is also continuous on $\mathbf{R}^d\times[0,T]$.

Lemma 3.3. *Let $0\le\beta-(d-1)<\alpha<1$ and $g\in\Lambda_{\alpha,\alpha/2}(\mathbf{R}^d\times[0,T])$ such that $g(\cdot,t)\in C^1(\mathbf{R}^d)$, supp $g(\cdot,s)\subset B(0,r_0)$ for every $t\in[0,T]$ and $\frac{\partial g}{\partial x_j}(x,\cdot)$ is bounded for every $x\in\mathbf{R}^d$. Let $X=(x,t)\in\mathbf{R}^d\times(0,T]$ and set, for $0<\rho\le T$,*

$$\begin{aligned}A_\rho g(X)=&\int_0^\rho ds\int_{\mathbf{R}^d\setminus\overline{D}}\langle\nabla g(Y),\nabla_y W(X-Y)\rangle dy\\&+\int_0^\rho ds\int_{\mathbf{R}^d\setminus\overline{D}}(g(Y)-g(X))\Delta_y W(X-Y)dy\\&+g(X)\int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}}W(X-Y)dy\end{aligned}$$

and

$$\begin{aligned}B_\rho g(X)=&-\int_0^\rho ds\int_D\langle\nabla g(Y),\nabla_y W(X-Y)\rangle dy\\&-\int_0^\rho ds\int_D(g(Y)-g(X))\Delta_y W(X-Y)dy\\&-g(X)\int_{D\times\{0\}}W(X-Y)dy.\end{aligned}$$

Then

$$A_T g(X) = B_T g(X) + g(X) \text{ for } X \in \mathbf{R}^d \times (0,T] \tag{3.2}$$

Proof. To simplify the notation, we use $A_\rho(x)$ and $B_\rho(X)$ instead of $A_\rho g(X)$ and $B_\rho g(X)$, respectively. We first show (3.2) in case $D = D_0$ is a bounded piecewise smooth domain. Let $X = (x,t)$ and set, for $0 < \rho < t$,

$$I_\rho(X) = -\int_0^\rho ds \int_{\partial D_0} g(Y)\langle \nabla_y W(X-Y), n_y\rangle d\sigma(y).$$

The Green formula for D_0 yields

$$\begin{aligned} I_\rho(X) = &-\int_0^\rho ds \int_{D_0} \langle \nabla g(Y), \nabla_y W(X-Y)\rangle dy \\ &-\int_0^\rho ds \int_{D_0} (g(Y)-g(X))\triangle_y W(X-Y)dy \\ &-g(X)\int_0^\rho ds \int_{D_0} \triangle_y W(X-Y)dy \end{aligned} \tag{3.3}$$

From (2.2) we deduce

$$\begin{aligned} &\int_0^\rho ds \int_{D_0} \triangle_y W(X-Y)dy \\ &= \int_{D_0\times\{0\}} W(X-Y)dy - \int_{D_0\times\{\rho\}} W(X-Y)dy, \end{aligned}$$

whence

$$\int_0^t ds \int_{D_0} \triangle_y W(X-Y)dy = \int_{D_0\times\{0\}} W(X-Y)dy - \chi_{D_0}(x).$$

This and (3.3) imply

$$I_t(X) = B_t(X) + g(X)\chi_{D_0}(x) \text{ for } X \in (\mathbf{R}^d \setminus \partial D_0) \times (0,T]. \tag{3.4}$$

Similarly, using the Green formula for $B(0,r) \setminus \overline{D}_0$ and $r \to \infty$, we obtain

$$I_t(X) = A_t(X) - g(X)\chi_{\mathbf{R}^d\setminus\overline{D}_0}(x)$$

for $X \in (\mathbf{R}^d \setminus \partial D_0) \times (0,T]$. This and (3.4) lead to

$$A_t(X) = B_t(X) + g(X) \text{ for } X \in (\mathbf{R}^d \setminus \partial D_0) \times (0,T].$$

Noting that $A_t(X) = A_T(X)$ and $B_t(X) = B_T(X)$, we obtain (3.2) for $X \in (\mathbf{R}^d \setminus \partial D) \times (0,T]$. Since A_T and B_T are continuous on $\mathbf{R}^d \times (0,T]$ by Lemma 3.2, (3.2) holds for a bounded piecewise smooth domain $D = D_0$.

We next show (3.2) for a bounded domain such that ∂D is a β-set. We use (3.2) for $D_0 = A_n$. Since

$$\int_0^T \int_{\mathbf{R}^d} |\nabla g(Y)||\nabla_y W(X-Y)|dyds < \infty,$$
$$\int_0^T ds \int_{\mathbf{R}^d} |g(Y) - g(X)||\Delta_y W(X-Y)|dy < \infty$$

and

$$\int_{\mathbf{R}^d \times \{0\}} W(X-Y)dy < \infty,$$

we see that (3.2) holds for the domain D as $n \to \infty$. □

Lemma 3.4. *Let $0 \le \beta - (d-1) < \alpha$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$. Then (3.2) holds for $g = \mathcal{E}(f)$.*

Sketch of Proof. Let $f \in \Lambda_{\alpha,\alpha/2}(S_D)$ and $\{v_m\}$ be a mollifier on $\mathbf{R}^d$ such that supp $v_m \subset B(0,1/m)$. We define, for $Y = (y,s) \in \mathbf{R}^d \times [0,T]$,

$$g_m(Y) = \big(\mathcal{E}(f)(\cdot,s) * v_m\big)(y).$$

Lemma 3.3 yields

$$A_T g_m(X) = B_T g_m(X) + g_m(X) \text{ for } X \in \mathbf{R}^d \times (0,T].$$

Using $g_m(X) \to \mathcal{E}(f)(X)$ uniformly as $m \to \infty$ and Lemmas A, 1.1 and 2.1, we can show that

$$A_T g_m(X) \to A_T\mathcal{E}(f)(X)$$

and

$$B_T g_m(X) \to B_T\mathcal{E}(f)(X)$$

for $X \in \mathbf{R}^d \times [0,T]$ as $m \to \infty$.

□

We can also show the following lemma.

Lemma 3.5. *Let $0 \le \beta - (d-1) < \alpha < 1$. Then the operator K defined by (1.7) is a bounded operator from $\Lambda_{\alpha,\alpha/2}(S_D)$ to $\Lambda_{\alpha,\alpha/2}(S_D)$.*

Let us prove our thorem.

Proof of Theorem. Let $X \in D \times (0,T]$. Using Lemma 2.2, we have $\Phi f(X) = A_T f(X)$. Since $A_T f$ is continuous on $\mathbf{R}^d \times (0,T]$ by Lemma 3.2, we have

$$\lim_{X \to Z, X \in D \times (0,T)} \Phi f(X) = A_T f(Z).$$

On the other hand Lemma 3.4 yields

$$Kf(Z) = \frac{1}{2}\big(A_T f(Z) + B_T f(Z)\big) = A_T f(Z) - \frac{1}{2} f(Z).$$

Therefore we have (1.8). Similarly we can show (1.9). □

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