数理解析研究所講究録
1042 巻 1998 年 123-132

Kneading sequences for symmetric PL bimodal maps

大阪大学大学院理学研究科　中村里美　(Satomi Nakamura)
龍谷大学理工学部　岡宏枝　(Hiroe Oka)

1 PL bimodal map

PL bimodal maps are the piecewise linear maps on the interval with three monotone segments. We consider the family of symmetric bimodal maps on the interval $[-1,1]$, which map -1 and 1 to -1 and 1 respectively. A bimodal map has two turning points t_1 and t_2 on $(-1,1)$. The map is strictly increasing on $[-1,t_1]$ and on $[t_2,1]$, and strictly decreasing on $[t_1,t_2]$. Such maps are called $\{+-+\}$ type bimodal maps. Let λ be the slope on the first and third segments, and $-\mu$ be that on the second segment $(\lambda,\mu>0)$. The map is given by the formula

$$F_{\lambda,\mu}(x)=\begin{cases}\lambda x+\lambda-1 & (x\leq t_1)\\ -\mu x & (t_1\leq x\leq t_2)\\ \lambda x-(\lambda-1) & (x\geq t_2)\end{cases}$$

where $t_1=-t_2=\dfrac{1-\lambda}{\lambda+\mu}$. The parameters λ and μ are chosen from the set

$$D=\{(\lambda,\mu):\lambda>1,\mu>1,\frac{2}{\lambda}+\frac{1}{\mu}\geq 1\}.$$

Since the maps are bimodal and symmetric at the origin, $t_1<0$. Therefore $\lambda>1$. If $\mu<1$ then there exists an attracting fixed point. If $\mu=1$ then $x\in[t_1,t_2]\setminus\{0\}$ is 2-periodic. Therefore the condition $\mu>1$ is necessary for topological entropy to be positive. The last condition $\frac{2}{\lambda}+\frac{1}{\mu}\geq 1$ is required for the existence of an F-invariant interval.

Lemma 1.1 *Assume that $\lambda,\mu>1$. There exists an interval whose interior is invariant under F, if and only if $\dfrac{2}{\lambda}+\dfrac{1}{\mu}\geq 1$.*

If $\dfrac{2}{\lambda}+\dfrac{1}{\mu}<1$, then the invariant interval is a Cantor set. In this case, topological entropy is $\log 3$.

Now we consider another set of the parameters λ and μ

$$D_0=\{(\lambda,\mu)\in D:\frac{1}{\lambda}+\frac{1}{\mu}<1\}.$$

For $(\lambda,\mu)\in D\setminus D_0$ there exists an F-invariant interval, such that F maps the negative points and the positive points to positive points and negative points respectively.

Lemma 1.2 *For $(\lambda,\mu)\in D$, we have $F^2_{\lambda,\mu}(t_1)>0$, if and only if $\dfrac{1}{\lambda}+\dfrac{1}{\mu}<1$.*

2 Symbolic dynamics for the bimodal maps

2.1 Kneading sequences

We consider kneading sequences adapted to a special case of bimodal maps that are symmetric at the origin. Let f be a $\{+-+\}$ type bimodal map with two turning points $t_1<0$ and $t_2>0$. Let f^n is the nth iterate of f. The **orbit** of $x_0\in[f(t_2),f(t_1)]$ is

$$\begin{aligned}O(x_0)&=(x_0,f(x_0),f^2(x_0),\cdots)\\&=(x_0,x_1,x_2,\cdots).\end{aligned}$$

Its **itinerary** is

$$I(x_0)=(A_0A_1A_2\cdots),$$

where

$$A_i=\begin{cases}L & (f(t_2)\le x_i<t_1)\\ C_L & (x_i=t_1)\\ M & (t_1<x_it_2)\\ C_R & (x_i=t_2)\\ R & t_2<(x_i\le f(t_1)).\end{cases}$$

The itineraries $K_L(f), K_R(f)$ for the critical values $f(t_1), f(t_2)$ are called the **kneading sequences** of the map:

$$K_L(f) = I(f(t_1)), K_R(f) = I(f(t_2)).$$

For the itinerary $I(x_0) = (A_0A_1A_2\cdots)$, we denote the number of i's such that $A_i = M$ (for $i < n$) by θ_n, and let $\varepsilon_n = (-1)^{\theta_n}$. We denote a symbol by a capital letter without underline and a sequence by a capital letter with underline.

We define an order on the symbols and the sequences as follows.

(i) $L < C_L < M < C_R < R$.

(ii) let $\underline{S} = (A_0A_1A_2\cdots)$ and $\underline{T} = (B_0B_1B_2\cdots)$ be two different sequences. Let k be the smallest non-negative integer with $A_k \neq B_k$. We say $\underline{S} < \underline{T}$ if $A_k < B_k$ and $\varepsilon_k = 1$ or if $A_k > B_k$ and $\varepsilon_k = -1$ for the above k.

For $x, y \in [f(t_2), f(t_1)]$, it follows that

(i) if $I(x) < I(y)$ then $x < y$.

(ii) if $x < y$ then $I(x) \leq I(y)$.

$|\underline{A}|$ denote the cardinality of $\underline{A}$. When $|\underline{A}| = 0$, we write $\underline{A} = \phi$. If $|\underline{A}| > 0$, then $\underline{A} > \phi$. For $\underline{A} = (A_0A_1A_2A_3\cdots)$, define the **shift operator** σ by

$$\sigma(\underline{A}) = \begin{cases} \phi & \text{if } \ \underline{A} = C_L, C_R \ \text{ or } \ \phi \\ (A_1A_2A_3\cdots) & \text{otherwise} \end{cases}$$

We call a sequence $\underline{A}$ **maximal** if $\sigma^k(\underline{A}) \leq \underline{A}$ for $k = 1, 2, \cdots$. The kneading sequence $K_L(f)$ is maximal.

2.2 The products of the sequences

We say a sequence $\underline{A}$ is **even** or **odd** according to the parity of the number of M's it contains. We shall write $\underline{AB}$ for the concatenation of $\underline{A}$ and $\underline{B}$, and $\underline{A}^n = \underline{A}\cdots\underline{A}$ (n times) and $\underline{A}^\infty = \underline{AA}\cdots$. Let $\underline{A} \neq \phi$ and $\underline{B} \neq C_L$ or C_R. We define $\bar{\underline{A}}$ as follows:

$$\bar{A}_i = \begin{cases} R & (A_i = L) \\ C_R & (A_i = C_L) \\ M & (A_i = M) \\ C_L & (A_i = C_R) \\ L & (A_i = R) \end{cases}$$

We define $*$-**product** and $**$-**product** as follows :

(i) if $\underline{A}$ is even

$$\underline{A} * B_0B_1\cdots \quad = \quad \underline{A}B_0\underline{A}B_1\cdots$$

(ii) if $\underline{A}$ is odd

$$\underline{A} * B_0B_1\cdots = \underline{A}\check{B}_0\underline{A}\check{B}_1\cdots$$

where

$$\check{B}_i = \begin{cases} M & (B_i = L) \\ L & (B_i = M) \\ R & (B_i = R) \end{cases}$$

(iii) if $\underline{A}$ is even

$$\underline{A} * *B_0B_1\cdots = \underline{A}B_0\bar{\underline{A}}B_1\cdots$$

(iv) if $\underline{A}$ is odd

$$\underline{A} * *B_0B_1 \cdots = \underline{A}\check{B}_0\underline{\bar{A}}\hat{B}_1 \cdots$$

where

$$\check{B}_i = \begin{cases} L & (B_i = L) \\ R & (B_i = M) \\ M & (B_i = R) \end{cases}$$

$$\hat{B}_i = \begin{cases} M & (B_i = L) \\ L & (B_i = M) \\ R & (B_i = R). \end{cases}$$

Let $\underline{A}$ be maximal. We say $\underline{A}$ is **primary** if it cannot be written as $\underline{B} * \underline{D}$ or $\underline{B} * *\underline{D}$ with $\underline{B} \neq \phi$ and $\underline{D} \neq \phi$.

There are some kneading sequences $K_L(f)$ that contain the symbol C_L. These sequences can be written $(\underline{A}C_L)^\infty$ or $(\underline{A}C_R\underline{\bar{A}}C_L)^\infty$ by using a sequence $\underline{A} \neq \phi$. These sequences satisfy the following inequalities :

$$\begin{array}{ccccc} \underline{A} * L^\infty & < & \underline{A}C_L & < & \underline{A} * ML^\infty \\ \underline{A} * *M^\infty & < & \underline{A}C_R\underline{\bar{A}}C_L & < & \underline{A} * *RM^\infty \end{array}$$

Proposition 2.1 *Let $\underline{A}C_L$ be maximal. If $\underline{A} * L^\infty \leq K_L(f) \leq \underline{A} * ML^\infty$ then there is a $\underline{B}$ such that $K_L(f) = \underline{A} * \underline{B}$. This $\underline{B}$ is maximal.*

Proof. Put $n = |\underline{A}C_L|$. Assume that $\underline{A}$ is even. We first show that

$$\underline{A} * L^\infty \leq \sigma^n(K_L(f)) \leq \underline{A} * ML^\infty. \tag{1}$$

Our assumptions implies

$$(\underline{A}L)^\infty \leq K_L(f) \leq \underline{A}M(\underline{A}L)^\infty.$$

Then we have $K_L(f) = \underline{A}B_0\cdots$, where $B_0 = L, C_L$, or M. If $B_0 = L$ then $\sigma^n((\underline{A}L)^\infty) \leq \sigma^n(K_L(f))$. If $B_0 = M$ then $\sigma^n(\underline{A}M(\underline{A}L)^\infty) \leq \sigma^n(K_L(f))$. In both of the cases we have $(\underline{A}L)^\infty \leq \sigma^n(K_L(f))$. We get $\sigma^n(K_L(f)) \leq K_L(f)$ since $K_L(f)$ is maximal. Therefore we obtain the inequality (1). If $B_0 = C_L$, the inequality (1) holds since $\sigma^n(K_L(f)) = K_L(f)$.

By induction, for all $p \geq 1$

$$\underline{A} * L^\infty \leq \sigma^{np}(K_L(f)) \leq \underline{A} * ML^\infty.$$

Thus $K_L(f)$ is of the form $\underline{A}B_0\underline{X}$. From the reasoning in the preceding paragraph, it follows that X must be again of the same form. Hence

$$K_L(f) = \underline{A}B_0\underline{A}B_1\cdots = \underline{A} * \underline{B}.$$

In the case that $\underline{A}$ is odd, we also have $K_L(f) = \underline{A} * \underline{B}$.

Next, we show that $\underline{B} = B_0B_1\cdots$ is maximal. For that purpose it is enough to prove that $\sigma^k(\underline{B}) \leq \underline{B}$ for any k. Since $K_L(f)$ is maximal, it follows that for any k

$$\sigma^{kn}(\underline{A} * \underline{B}) \leq \underline{A} * \underline{B}. \tag{2}$$

We assume that for some $\tilde{k}$

$$B_kB_{k+1}\cdots B_{k+\tilde{k}-1} = B_0B_1\cdots B_{\tilde{k}-1} \tag{3}$$

and

$$B_{k+\tilde{k}} \neq B_{\tilde{k}}. \tag{4}$$

If $\underline{A}$ is even, then from (2)

$$\underline{A}B_k\underline{A}B_{k+1}\cdots\underline{A}B_{k+\tilde{k}} < \underline{A}B_0\underline{A}B_1\cdots\underline{A}B_{\tilde{k}}$$

Since $\underline{A}B_k\underline{A}B_{k+1}\cdots B_{k+\tilde{k}-1}\underline{A} = \underline{A}B_0\underline{A}B_1\cdots B_{\tilde{k}-1}\underline{A}$, we obtain $\sigma^k(\underline{B}) < \underline{B}$. We also have the same inequality in the case $\underline{A}$ is odd. ef the assumption (3) (4) does not hold for any $\tilde{k}$, then $\underline{B}$ is periodic with period k, i.e. $\sigma^k(\underline{B}) = \underline{B}$. ∎

Lemma 2.2 *Assume* $\underline{A} * L^\infty \leq K_L(f) \leq \underline{A} * ML^\infty$. *If* $K_L(f) = \underline{A} * \underline{B}$ *then* $L^\infty \leq \underline{B} \leq ML^\infty$ *and* $\sigma(\underline{B}) \leq \sigma^k(\underline{B})$ *for any* $k \geq 1$.

Lemma 2.3 *Assume* $\underline{A} * L^\infty \leq K_L(f) \leq \underline{A} * ML^\infty$. *If* $\sigma^n(K_L(f)) \leq I(x) \leq K_L(f)$, *then* $\sigma^n(K_L(f)) \leq I(f^n(x)) \leq K_L(f)$, *where* $n = |AC_L|$.

Proposition 2.4 *Let* $\underline{AC_R\bar{A}C_L}$ *be maximal. If* $\underline{A}{*}{*}M^\infty \leq K_L(f) \leq \underline{A}{*}{*}RM^\infty$ *then there is a* $\underline{B}$ *such that* $K_L(f) = \underline{A} * *\underline{B}$. *This* $\underline{B}$ *is maximal.*

Lemma 2.5 *Assume* $\underline{A} * *M^\infty \leq K_L(f) \leq \underline{A} * *RM^\infty$ *If* $K_L(f) = \underline{A} * *\underline{B}$ *then* $M^\infty \leq \underline{B} \leq RM^\infty$ *and* $\sigma(\underline{B}) \leq \sigma^k(\underline{B})$ *for any* $k \geq 1$.

Lemma 2.6 *Assume* $\underline{A} * *M^\infty \leq K_L(f) \leq \underline{A} * *RM^\infty$. *If* $\sigma^n(K_L(f)) \leq I(x) \leq K_L(f)$, *then* $\sigma^n(K_L(f)) \leq I(f^n(x)) \leq K_L(f)$, *where* $n = |AC_R|$.

2.3 The properties of kneading sequence for PL bimodal maps

Now we remember Lemma 1.2 that for $(\lambda, \mu) \in D_0$ we have $F^2_{\lambda,\mu}(t_1) > 0$. The map $F_{\lambda,\mu}(x)$ has a fixed point $x = 0$. The itinerary of this point is M^∞. Therefore for $(\lambda, \mu) \in D_0$ $K_L(F_{\lambda,\mu}) > RM^\infty$.

Proposition 2.7 *If* $K_L(F_{\lambda,\mu}) > RM^\infty$, *then* $K_L(F_{\lambda,\mu})$ *is primary.*

Proof. Assume that $K_L(F_{\lambda,\mu}) = \underline{AB}$. Lemma 2.3 implies that there is a interval J such that $F^n_{\lambda,\mu}(J) = J$ for $n = |\underline{A}C_L|$. We can take $\{x|\sigma^n(K_L(F_{\lambda,\mu})) \leq I(x) \leq K_L(F_{\lambda,\mu})\}$ for the above J. Then we find $F^n_{\lambda,\mu}$ on the interval J is unimodal. Let κ and $-\nu$ $(\kappa, \nu > 0)$ be the slopes of $F^n_{\lambda,\mu}(J)$. Let k be the total number of L's and R's in $\underline{A}$, so the number of M's is $n - 1 - k$. If $\underline{A}$ is even, then we get the slopes

$$\begin{aligned}
\kappa &= (-\mu)^{n-1-k}\lambda^k\lambda \geq \lambda^2 \\
(-\nu) &= (-\mu)^{n-1-k}\lambda^k(-\mu) \leq -\lambda\mu
\end{aligned}$$

respectively. If $\underline{A}$ is odd, then

$$\begin{aligned} \kappa &= (-\mu)^{n-1-k}\lambda^k(-\mu) \geq \mu^2 \\ (-\nu) &= (-\mu)^{n-1-k}\lambda^k\lambda \leq -\lambda\mu. \end{aligned}$$

In both of the cases, we get $\frac{1}{\kappa} + \frac{1}{\nu} < 1$, since $K_L(F_{\lambda,\mu}) > RM^\infty$ implies $\frac{1}{\lambda} + \frac{1}{\mu} < 1$. This contradicts the result in Misiurevicz-Visinescu [2] that $\frac{1}{\kappa} + \frac{1}{\nu} > 1$ is necessary for the existence of an F-invariant interval for unimodal maps. ∎

Proposition 2.8 *If $K_L(F_{\lambda,\mu}) > RM^\infty$, then $K_L(F_{\lambda,\mu})$ is primary.*

Theorem 2.9 *Let $K_L(F_{\lambda,\mu})$ be a maximal and primary sequence such that $K_L(F_{\lambda,\mu}) > RM^\infty$. There is ν such that $K(g_\nu) = K_L(F_{\lambda,\mu})$, where g_ν is the PL bimodal map with the slopes alternately $\nu, -\nu, \nu (\nu > 1)$.*

Proof. From the maximality and the primarity of $K_L(F_{\lambda,\mu})$ as well as Proposition 2.1, we have one of the inequalities $K_L(F_{\lambda,\mu}) < \underline{A} * L^\infty$ or $K_L(F_{\lambda,\mu}) > \underline{A} * ML^\infty$. We set

$$M_{F_{\lambda,\mu}} = \{\nu : K_L(g_\nu) < K_L(F_{\lambda,\mu})\}$$

and

$$P_{F_{\lambda,\mu}} = \{\nu : K_L(g_\nu) > K_L(F_{\lambda,\mu})\},$$

and we claim these are open. We show only that $M_{F_{\lambda,\mu}}$ is open, and we can prove that $P_{F_{\lambda,\mu}}$ is open in the same way. We put $n = |\underline{A}C_L|$, and assume that $\underline{A}$ is even. We take $\nu \in M_{F_{\lambda,\mu}}$ with $K_L(g_\nu) = \underline{A}D_nD_{n+1}\cdots < K_L(F_{\lambda,\mu})$ such that D_n is not equal to the nth symbol of $K_L(F_{\lambda,\mu})$. If $D_n \neq C_L$ then it is obvious that $M_{K(F_{\lambda,\mu})}$ is open. If $D_n = C_L$ then $K_L(F_{\lambda,\mu}) > \underline{A}*ML^\infty$. In this case there exists s_0 such that $K_L(F_{\lambda,\mu}) > \underline{A}M\underline{A}L^{s_0}\cdots$. Thus the sets $\left\{\tilde{\underline{D}} = \underline{A}M(\underline{A}L)^s : s > s_0\right\}$ and $\left\{\tilde{\underline{D}} = \underline{A}L(\underline{A}L)^s : \text{for any } s\right\}$ are included in $M_{F_{\lambda,\mu}}$, and $M_{F_{\lambda,\mu}}$ is open. If $\underline{A}$ is odd, we can also prove that $M_{F_{\lambda,\mu}}$ is open in the same way. ∎

3 Monotonicity of topological entropy

In this section we consider the monotonicity of kneading sequences and that of topological entropy.

We define an order on a pair of the parameters.

(i) We say $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ if $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$.

(ii) We say $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ if $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ and either $\lambda_1 \neq \lambda_2$ or $\mu_1 \neq \mu_2$.

We denote the pair of $K_L(F_{\lambda,\mu})$ and $K_R(F_{\lambda,\mu})$ by $K(\lambda, \mu)$. We say $K(\lambda_1, \mu_1) < K(\lambda_2, \mu_2)$ if and only if $K_L(F_{\lambda_1,\mu_1}) < K_L(F_{\lambda_2,\mu_2})$ and $K_R(F_{\lambda_1,\mu_1}) > K_R(F_{\lambda_2,\mu_2})$.

Let $h(\lambda, \mu)$ be the topological entropy of $F_{\lambda,\mu}$.

Proposition 3.1 *Let* $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in D \setminus D_0$. *If* $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ *then* $h(\lambda_1, \mu_1) < h(\lambda_2, \mu_2)$.

We can show this proposition applying the result of [2] that proved the monotonicity of the topological entropy for PL unimodal maps.

Let

$$D_1 = \{(\lambda, \mu) \in D_0 : RC_R LC_L < K_L(F_{\lambda,\mu}) < R(RL)^\infty\}.$$

We obtain a proposition about the monotonicity of kneading sequences as follows:

Proposition 3.2 *Let* $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in D_0 \setminus D_1$. *If* $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ *then* $K(\lambda_1, \mu_1) < K(\lambda_2, \mu_2)$.

The proof of this proposition is given by an analytical estimation.

Theorem 3.3 *Let* $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in D_0 \setminus D_1$. *If* $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ *then* $h(\lambda_1, \mu_1) < h(\lambda_2, \mu_2)$.

We can prove this theorem from Theorem 2.9 and Proposition 3.2.

Theorem 3.4 *For a constant c with $0 < c < \log(3)$, the iso-entropy curve given by $h(\lambda, \mu) = c$ is connected.*

We prove this theorem in [3].

References

[1] P.Collet and J.P.Eckman, "Iterated maps on the interval as dynamical systems", Birkhauser, Boston, 1980.

[2] M.Misiurewicz and E.Visinescu, Kneading sequences of skew tent maps, Ann.Inst.Henri Poincaŕe **27**(1991) 125-140.

[3] H.Oka and S.Nakamura, Monotonicity of topological entroppy for symmetric PL bimodal maps, in preparation.