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Uniformization of unbounded invariant Fatou components of transcendental entire functions

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1 Introduction

Let f be a transcendental entire function, and let $F_f \subset \mathbb{C}$ and $J_f \subset \mathbb{C}$ be the Fatou set and Julia set of f respectively. A connected component U of F_f is called a *Fatou component.* Then U is either a *wandering domain* (that is, $f^m(U) \cap f^n(U) = \emptyset$ for all $m, n \in \mathbb{N}$ $(m \neq n)$) or *eventually periodic* (that is, $f^m(U)$ is periodic for an $m \in \mathbb{N}$). If it is periodic, it is well known that there are four possibilities; U is either an attractive basin, a parabolic basin, a Siegel disk, or a Baker domain. Note that U cannot be a Herman ring. This fact follows easily from the maximum principle.

In this paper we consider an unbounded periodic (that is, $f^n(U) \subseteq U$ for some $n \in \mathbb{N}$) Fatou component U. It is known that U is simply connected ([**B**], [**EL**]) and so let $\varphi : \mathbb{D} \to U$ be a uniformization (Riemann map) of U, where $\mathbb{D}$ is a unit disk. The boundary ∂U of U can be very complicated as the following example shows:

Example. Let us consider the exponential family $E_\lambda(z) := \lambda e^z$. If the parameter λ satisfies $\lambda = te^{-t}$ $(|t| < 1)$, then there exists a unique unbounded completely invariant attractive basin U which is equal to the Fatou set F_{E_λ} and ∂U is equal to the Julia set J_{E_λ} which is so called a Cantor bouquet. Moreover,

$$\Theta_\infty := \left\{ e^{i\theta} \;\middle|\; \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty \right\} \subset \partial\mathbb{D}$$

is dense in $\partial\mathbb{D}$ ([**DG**]). This implies that φ is highly discontinuous on $\partial\mathbb{D}$ and hence ∂U ($= J_{E_\lambda}$) has a very complicated structure. In fact the Hausdorff dimension of J_{E_λ} is equal to 2 ([**Mc**]).

Later, Baker and Weinreich investigated the boundary behavior of φ generally in the case of attractive basins, parabolic basins and Siegel disks and showed the following:

Theorem (Baker-Weinrech, [BW]). Let U be an unbounded invariant Fatou component, then either

(i) $f^n \to \infty$ in U (that is, U is a Baker domain) or

(ii) the point ∞ belongs to the impression of every prime end of U. □

From the classical theory of prime end by Carathéodory it is well known that there is a 1 to 1 correspondence between $\partial\mathbb{D}$ and the set of all the prime ends of U. Let us denote $P(e^{i\theta})$ the prime end corresponding to the point $e^{i\theta} \in \partial\mathbb{D}$. The impression $\mathrm{Im}(P(e^{i\theta}))$ of a prime end $P(e^{i\theta})$ is a subset of ∂U which is known to be written as follows:

$$\mathrm{Im}(P(e^{i\theta})) = \left\{ p \in \partial U \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{z_n\}_{n=1}^{\infty} \subset \mathbb{D} \\ \text{whcih satisfies } \lim_{n\to\infty} z_n = e^{i\theta},\ \lim_{n\to\infty} \varphi(z_n) = p \end{array} \right\}$$

For the details of the theory of prime end, see for example, [**CL**]. Define the set $I_\infty \subset \partial\mathbb{D}$ by

$$I_\infty := \{e^{i\theta} \in \partial\mathbb{D} \mid \infty \in \mathrm{Im}(P(e^{i\theta}))\},$$

then the above result asserts that $I_\infty = \partial\mathbb{D}$ in the case of unbounded attractive basins, parabolic basins and Siegel disks. This shows that ∂U is extremely complicated.

On the other hand, ∂U can be very "simple" in the case when U is a Baker domain. For example, the function

$$f(z) := 2 - \log 2 + 2z - e^z$$

has a Baker domain U on which f is univalent and whose boundary ∂U is a Jordan curve (that is, $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is a Jordan curve and $\partial U \subset \mathbb{C}$ is a Jordan arc, [**Ber**, Theorem 2]). In this case I_∞ consists of only a single point.

Then what can we say about the set I_∞ in general when U is a Baker domain? For this problem we obtain the following:

Main Theorem. Let f be a transcendental entire function and suppose that f has an invariant Baker domain U. Let $\varphi : \mathbb{D} \to U$ be a uniformization of U and the set I_∞ as above. Assume that $f|U : U \to U$ is not univalent.

(1) If $f|U$ is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$, then I_∞ contains a perfect set $K \subset \partial\mathbb{D}$.

(2) If $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$, then I_∞ contains a perfect set $K \subset \partial\mathbb{D}$.

(3) If $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ $z \mapsto z+1$, then $I_\infty = \partial\mathbb{D}$.

If $f|U$ is univalent, then $^{\#}I_\infty = 1, 2$ or ∞.

Remark In the Main Theorem we assume that U is an invariant Baker domain for simplicity. Of course, we can obtain the same result when U is a periodic Baker domain of period $p \geq 2$.

This result is based on the classification of Baker domains and an arbitrary Baker domain falls into one of the above three cases. We explain the details in §2. In §3 we show the outline of the proof of the Main Theorem. Baker and Weinreich's result can be also proved by the similar method used in the proof of the Main Theorem. So we briefly show this in §4.

2 Classification of Baker domains

In this section we classify Baker domains from the dynamical point of view. Now let U be an invariant Baker domain. By definition $f^n|U \to \infty$ $(n \to \infty)$ locally uniformly, so put

$$g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D},$$

then g is conjugate to $f|U : U \to U$ and from the dynamics of $f|U$, g has no fixed point in $\mathbb{D}$. By the theorem of Denjoy and Wolff, there exists a unique point $p_0 \in \partial\mathbb{D}$ (which is called Denjoy-Wolff point) and $g^n \to p_0$

locally uniformly. It is known that there exists a radial limit

$$c := \lim_{r \nearrow 1} g'(rp_0) \quad \text{with} \quad 0 < c \leq 1,$$

which means that p_0 is either an attracting or a parabolic fixed point of the boundary map of g. Next put

$$z_n := g^n(0) \quad \text{and} \quad q_n := \frac{z_{n+1} - z_n}{1 - \overline{z_n} z_{n+1}},$$

then by the Schwarz-Pick's lemma $\{|q_n|\}_{n=1}^{\infty}$ turns out to be a decreasing sequence and hence there exists a limit $\lim_{n\to\infty} |q_n|$ ([**P**]). By using this limit and the value c, the dynamics of g on $\mathbb{D}$ can be classified for three different classes as follows. This result is essentially due to Baker and Pommerenke ([**BP**], [**P**]). They treated anlytic functions in the halfplane $\mathbb{H}$ and obtained some results. The following is the translation of their results into the case of analytic functions in $\mathbb{D}$ which is conformally equivalent to $\mathbb{H}$.

Theorem (1) If $c < 1$, then g is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(z) = \dfrac{(1+c)z + 1 - c}{(1-c)z + 1 + c}$.

(2) If $c = 1$ and $\lim_{n\to\infty} |q_n| > 0$, then g is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(z) = \dfrac{(1 \pm 2i)z - 1}{z - 1 \pm 2i}$.

(3) If $c = 1$ and $\lim_{n\to\infty} |q_n| = 0$, then g is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(z) = z + 1$. □

On the other hand, König investigated the relation between the above classification and the dynamics of $f|U : U \to U$ and obtained the following result:

Theorem (König, [K]) For an arbitrary point $w_0 \in U$ define

$$w_n := f^n(w_0) \quad \text{and} \quad d_n := \text{dist}\ (w_n, \partial U),$$

where "dist" is a Euclidean distance. Then

(1) $f|U$ is semi-conjugate to a hyperbolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ if and only if there exists a constant $\beta = \beta(f) > 0$ such that

$$\frac{|w_{n+1} - w_n|}{d_n} \geq \beta \quad (n \in \mathbb{N})$$

holds for any $w_0 \in U$.

(2) $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{D} \to \mathbb{D}$ if and only if

$$\liminf_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} > 0$$

holds for any $w_0 \in U$ but

$$\inf_{w_0 \in U} \limsup_{n\to\infty} \frac{|w_{n+1} - w_n|}{d_n} = 0.$$

(3) $f|U$ is semi-conjugate to a parabolic Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(z) = z + 1$ if and only if

$$\lim_{n\to\infty} \frac{w_{n+1} - w_n}{d_n} = 0$$

holds for any $w_0 \in U$. □

For each cases König also gave concrete examples satisfying the above conditions:

(1) $f(z) = 3z + e^{-z}$,

(2) $f(z) = z + 2\pi i \alpha + e^z$, where $\alpha \in (0, 1)$ satisfies the Diophantine condition,

(3) $f(z) = e^{\frac{2\pi i}{p}}\left(z + \int_0^z e^{-\zeta^p} d\zeta\right)$, where $p \in \mathbb{N},\ p \geq 2$.

Note that in the case (3), the function f above has a Baker domain of period $p \geq 2$, not an invariant one. Of course, if we consider f^p instead of f, f^p has an invariant Baker domain.

3 Outline of the proof of Main Theorem

With the above classification, we show the outline of the proof of Main Theorem.

Since $U \subset \mathbb{C}$ is unbounded, we have $I_\infty \neq \emptyset$ and it is easy to see that I_∞ is a closed subset of $\partial\mathbb{D}$. Then $\partial\mathbb{D} \setminus I_\infty$ is open in $\partial\mathbb{D}$ and it can be shown that g can be analytically continued over $\partial\mathbb{D} \setminus I_\infty$. So in particular g is analytic on $\partial\mathbb{D} \setminus I_\infty$ and we have

$$g(\partial\mathbb{D} \setminus I_\infty) \subseteq \partial\mathbb{D} \setminus I_\infty.$$

If g is a d to 1 map $(2 \leq d < \infty)$, then g is a finite Blaschke product of degree d and its Julia set J_g is either $\partial\mathbb{D}$ or a Cantor set (in particular, it is a perfect set) in $\partial\mathbb{D}$. Assume that $J_g \cap (\partial\mathbb{D} \setminus I_\infty) \neq \emptyset$, then from the general property of the dynamics of rational maps and the g-invariance of $\partial\mathbb{D} \setminus I_\infty$ we have

$$\partial\mathbb{D} \subset \partial\mathbb{D} \setminus I_\infty,$$

that is, $I_\infty = \emptyset$, which is a contradiction. Therefore we have $J_g \subset I_\infty$. This proves the case (1) and (2) with a further assumption that g is a finite to one map.

If g is an ∞ to 1 map, we can show that

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} \subset I_\infty$$

holds for every $z_0 \in \mathbb{D}$ (there may be some exception) and the set $\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D}$ is either equal to $\partial\mathbb{D}$ or at least contains a certain perfect set $K \subset \partial\mathbb{D}$. This result comes from a property of g as a boundary map $g : \partial\mathbb{D} \to \partial\mathbb{D}$. This completes the proof for the case (1) and (2).

For the case (3), since we have $\lim_{n\to\infty} |q_n| = 0$, we can obtain that

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} = \partial\mathbb{D} \subset I_\infty,$$

and hence $I_\infty = \partial\mathbb{D}$. This fact comes from the ergodic property of g as an inner function. This completes the proof for the case (3).

If g is univalent, then g is either hyperbolic or parabolic Möbius transformation. g has either one or two fixed points and the every orbit of a point other than the fixed points has infinitely many points. On the other hand, we have

$$g(\partial\mathbb{D} \setminus I_\infty) \subseteq \partial\mathbb{D} \setminus I_\infty,$$

so we can conclude that $^{\#}I_\infty = 1, 2$ or ∞. □

4 Another proof of the theorem by Baker and Weinreich

We show only the outline. Put $V := \partial\mathbb{D} \setminus I_\infty$ and suppose that V is not empty. Then $g(V) \subseteq V$ and g is 1 to 1 on each component of V in $\partial\mathbb{D}$. As we mentioned in §3 g can be extended to an analytic function in $\widehat{\mathbb{C}} \setminus I_\infty$ by reflection principle. Now assume that g is not univalent, then we have $^{\#}I_\infty \geq 3$ and hence $\{g^n\}_{n=1}^{\infty}$ is a normal family.

If U is an attractive basin, then there exists a fixed point $p \in \mathbb{D}$ of g and from the dynamics of f in U we have

$$g^n(z) \to p \quad (n \to \infty) \quad z \in \mathbb{D}.$$

On the contrary, we have

$$g^n \to \frac{1}{\bar{p}} \quad (n \to \infty) \quad z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}},$$

which is a contradiction. This show that $V = \emptyset$ and hence $I_\infty = \partial\mathbb{D}$ holds in this case.

If U is a parabolic basin, we can show that

$$\lim_{n\to\infty} |q_n| = 0$$

by using the several theorems and propositions by Doering and Mañé ([**DM**]). Then we have

$$\overline{\bigcup_{n=1}^{\infty} g^{-n}(z_0)} \cap \partial\mathbb{D} = \partial\mathbb{D} \subset I_\infty,$$

which shows that $I_\infty = \partial\mathbb{D}$.

If U is a Siegel disk, then g is an irrational rotation and so we have $g^n(V) = \partial\mathbb{D} \subseteq V$ for an n, which implies that $I_\infty = \emptyset$ and this is a contradiction. Hence in this case we have again that $I_\infty = \partial\mathbb{D}$. This completes the proof. □

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