数理解析研究所講究録
1043 巻 1998 年 192-197

A Game Theoretic Approach to Cost Allocation in Network System

成瀬 喜則
Yoshinori Naruse
富山商船高等専門学校
Toyama National College of Maritime Technology

前田 隆
Takashi Maeda
金沢大学経済学部
Faculty of Economics
Kanazawa University

1 Introduction

In this article, we describe the problem of sharing the fixed cost to construct a network through the cooperative game theoretical approach, based on several axioms.

There are several agents that have different kinds of information for users. We suppose that all of the agents agree to cooperate and undertake to make new network systems between agents. This construction of a network system enable users who belong to agents get information about other agents. In these networks, the values of them are determined by the utility which agents derive from other agents.

At first we propose the three axioms. They are individual rationality, Pareto optimality and aggregate monotonicity. We set the characteristic function of this problem and considered how the cost should be allocated among the agents. Among the most commonly used of these game theory concept is the Shapley value.

We propose a new method for allocating the joint cost of this project using the Shapley value.

2 A Game Model

Suppose that there are three kinds of systems which are at a distance from each other and all systems agree to cooperate and undertake the investment project on the construction of the network system.

It is assumed that the set of systems (in other words, players) $1, 2$ and 3 are linked to each other in order and make up the network system (Fig.1).

In order to construct a network system, some costs are necessary. Let c_{12} be the cost for constructing the network link between player 1 and 2,and c_{23} be the cost for constructing the network link between player 2 and 3. It is assumed that player 1 is not linked to player 3. By convention,$c(\emptyset) = 0$.

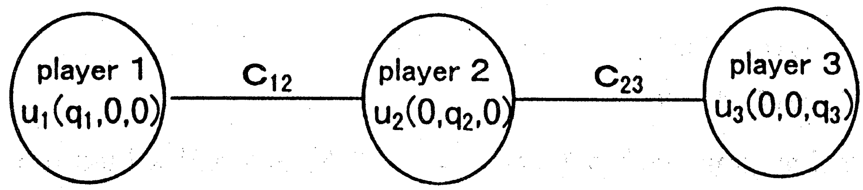

Fig.1 Cost Charge in 3-Network

The amount of information which player 1 , 2 and 3 possess are denoted by q_1 , q_2 , q_3. We assume that the profit to the information which player i possesses is represented by a function $u_i(q_1, q_2, q_3)$. Hence the profits represented by a function $u_1(q_1,0,0)$, $u_2(0,q_2,0)$, $u_3(0,0,q_3)$ change to $u_1(q_1,q_2,q_3)$, $u_2(q_1,q_2,q_3)$, $u_3(q_1,q_2,q_3)$ respectively.

The incentives issue is considered first. Let $x_i, (i = 1,2,3)$ be the cost charged to player i. Then the following inequalities should hold at the time of completion to construct the network system.

$$
\begin{aligned}
z_1 &= u_1(q_1,q_2,q_3) - u_1(q_1,0,0) \geq x_1 \\
z_2 &= u_2(q_1,q_2,q_3) - u_2(0,q_2,0) \geq x_2 \\
z_3 &= u_3(q_1,q_2,q_3) - u_3(0,0,q_3) \geq x_3
\end{aligned}
$$

Furthermore the allocation of x must satisfy $x_1 + x_2 + x_3 = c_{12} + c_{23}$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ simultaneously. Note that z_i, $(i = 1,2,3)$ is $i's$ marginal saving.

$$
\begin{aligned}
u_1(q_1,q_2,q_3) - u_1(q_1,0,0) &= n_1(e_2 + e_3) \\
u_2(q_1,q_2,q_3) - u_2(0,q_2,0) &= n_2(e_1 + e_3) \\
u_3(q_1,q_2,q_3) - u_3(0,0,q_3) &= n_3(e_1 + e_2)
\end{aligned}
$$

In this case, player 2 and 3 might have different value judgements of player 1. It means that it is better to denote these values as e_1^2 , e_1^3. To make the discussion easier,we define that the value judgements by players 2 and 3 of player 1 is the same. In this section,it is assumed that the amount of information which each player possesses is known to other players equally.

Thus we set that $e_1^2 = e_1^3 = e_1$, $e_2^1 = e_2^3 = e_2$, $e_3^1 = e_3^2 = e_3$. Furthermore,the player i consists of the number of users n_i $(i = 1,2,3)$.

Thus we have the following conditions.

$$z_1 = n_1(e_2 + e_3)$$

$$z_2 = n_2(e_1 + e_3)$$
$$z_3 = n_3(e_1 + e_2)$$

A cooperative game with players $N = \{1, 2, 3\}$ is a real valued function $v(S)$ defined on all coalitions $S \subseteq N.v(S)$ is the value of S.

Consider the characteristic function v as follows:

$$\begin{aligned}
v(\emptyset) &= 0 \\
v(1) &= u_1(q_1, 0, 0)\ ,\ v(2) = u_2(0, q_2, 0)\ ,\ v(3) = (0, 0, q_3) \\
v(12) &= u_1(q_1, q_2, 0) + u_2(q_1, q_2, 0) - c_{12} \\
v(23) &= u_2(0, q_2, q_3) + u_3(0, q_2, q_3) - c_{23} \\
v(13) &= u_1(q_1, 0, q_3) + u_3(q_1, 0, q_3) - c_{13} \\
v(123) &= u_1(q_1, q_2, q_3) + u_2(q_1, q_2, q_3) + u_3(q_1, q_2, q_3) - c_{12} - c_{23}
\end{aligned}$$

The following game (N, v') is strategically equivalent to the game (N, v) and the equations mentioned above can be rewritten as:

$$\begin{aligned}
v'(\emptyset) &= 0 && (1) \\
v'(1) &= v'(2) = v'(3) = 0 && (2) \\
v'(12) &= n_1e_2 + n_2e_1 - c_{12} && (3) \\
v'(23) &= n_2e_3 + n_3e_2 - c_{23} && (4) \\
v'(13) &= n_1e_3 + n_3e_1 - c_{13} && (5) \\
v'(123) &= n_1(e_2 + e_3) + n_2(e_1 + e_3) + n_3(e_1 + e_2) - c_{12} - c_{23} && (6)
\end{aligned}$$

Hence, the following theorem is given.

Theorem. 1 *If v' is subadditive and $v'(ij) \geq 0$,then the game (N, v') is convex.*

Proof. Consider the 3-player game (N, v') defined on $N = 1, 2, 3$. This cooperative game is convex if and only if the following inequality is held:

$$v'(T \cup \{i\}) - v'(T) \quad \geq v'(S \cup \{i\}) - v'(S)$$

for $i \in N$ and $S \subset T \subset N - \{i\}$.

Since $v'(ij) \geq 0$,for $i = 1$

$$\begin{aligned}
v'(123) - v'(23) - \{v'(12) - v'(2)\} &= n_1e_3 + n_3e_1 \geq 0 \\
v'(123) - v'(23) - \{v'(13) - v'(3)\} &= n_1e_2 + n_2e_1 \geq 0 \\
v'(12) - v'(2) - v'(1) &= n_1e_2 + n_2e_1 - c_{12} \geq 0 \\
v'(13) - v'(3) - v'(1) &= n_1e_3 + n_3e_1 - c_{13} \geq 0
\end{aligned}$$

We can show this for $i = 2$, 3in the same way.

□

Here,$v(S \cup \{i\}) - v(S)$ represents the marginal contribution of i to S.

3 The Shapley value

We consider the problem on how to allocate the benefits of cooperation equitably among players. There are several well-known allocation procedures,which involve distinct ideas. The first rule is that it divides the savings from the grand coalition equally among the players. The second rule is known as the nucleous,which is the allocation that lexicongraphically minimizes the vector of excesses,when these are arranged in the order of dexcending magnitude.

Let us decide which rule to adopt as the allocation rule of this model. Define that the allocation rule must obey the principle of "aggregate monotonicity". Aggregate monotonicity states the following context.

Suppose that all players agree to cooperate and undertake to make a new network system between agents with a specified allocation of estimated costs.

This construction of a network system can make users who belongs to agents get information about other agents. In these networks,the values of them are determined by the utility ,which agents derive from other agents.

If the value for a network system might be changed using the completed system,only $v(N)$ has changed. Since the alternative network systems were not made,the available data are the value of the completed network system and the previously estimated values of those undertaken. This means that the changing amount of value may be allocated, but no one should benefit by having his assessment reduced.

For each fixed N there exists a unique allocation rule ϕ defined for all characteristic function v on N that is symmetric,charges dummies nothing,additive,and is monotonicity,namely the Shapley value. The Shapley value can be calculated as follows:

$$\begin{aligned}
\phi_1(v') &= \frac{1}{6}v'(12) + \frac{1}{6}v'(13) + \frac{2}{6}\{v'(123) - v'(23)\} \\
&= \frac{1}{2}(n_1e_2 + n_2e_1 + n_1e_3 + n_3e_1) - \frac{3}{6}c_{12} - \frac{1}{6}c_{13} \qquad (7) \\
\phi_2(v') &= \frac{1}{6}v'(12) + \frac{1}{6}v'(23) + \frac{2}{6}\{v'(123) - v'(13)\} \\
&= \frac{1}{2}(n_1e_2 + n_2e_1 + n_2e_3 + n_3e_2) - \frac{3}{6}c_{12} - \frac{3}{6}c_{23} + \frac{2}{6}c_{13} \qquad (8)
\end{aligned}$$

$$\begin{aligned}\phi_3(v') &= \frac{1}{6}v'(13) + \frac{1}{6}v'(23) + \frac{2}{6}\{v'(123) - v'(12)\} \\ &= \frac{1}{2}(n_1e_3 + n_3e_1 + n_2e_3 + n_3e_2) - \frac{3}{6}c_{23} - \frac{1}{6}c_{13} \qquad (9)\end{aligned}$$

It is shown that the Shapley value is a core solution concept since the game (N, v') is convex. Namely,the procedure ϕ which is shown in (7) , (8) and (9) are characterized by the axioms:individual rationality ,Pareto optimality and aggregate monotonicity.

Let $x = (x_1^S, x_2^S, x_3^S)$ be a cost allocation vector. Consider the following equalities for calculating the amounts players should pay.

$$\begin{aligned}n_1(e_2 + n_3) - x_1^S &= \phi_1(v') \\ n_2(e_1 + n_3) - x_2^S &= \phi_2(v') \\ n_3(e_1 + n_2) - x_3^S &= \phi_3(v')\end{aligned}$$

From (7) , (8) and (9) ,we also have the following allocation solution.

$$\begin{aligned}x_1^S &= \frac{1}{2}(n_1e_2 + n_1e_3 - n_2e_1 - n_3e_1) + \frac{3}{6}c_{12} + \frac{1}{6}c_{13} \qquad (10) \\ x_2^S &= \frac{1}{2}(n_2e_1 + n_2e_3 - n_1e_2 - n_3e_2) + \frac{3}{6}c_{12} + \frac{3}{6}c_{23} - \frac{2}{6}c_{13} \qquad (11) \\ x_3^S &= \frac{1}{2}(n_3e_1 + n_3e_2 - n_1e_3 - n_2e_3) + \frac{3}{6}c_{23} + \frac{1}{6}c_{13} \qquad (12)\end{aligned}$$

We can explain this solution shown in (10) , (11) and (12) as follows. Player 1 should pay for construction of a network system with $\frac{3}{6}c_{12} + \frac{1}{6}c_{13}$. Futhermore player 1 should pay a half of $n_1e_2 + n_1e_3$ which means the marginal contribution in addition to this. On the other hand,there is a charge reduction of half of $n_2e_1 + n_3e_1$ for player 1.

We can explain also $\frac{3}{6}c_{12} + \frac{1}{6}c_{13}$ as follows. Player 1 should share the cost of c_{12} and the cost of c_{13} equally among all of players, where c_{12} is charged for linking players 1 and 2, and c_{13} is a dummy charge. Contrary to this sharing,$\frac{1}{3}c_{13}$ should be returned to player 1.

4 Summary and conclusions

In this paper,we examined a fair allocation model using tools of a cooperative game theory. We adopted the Shapley value as a solution of this model since the Shapley value is a unique and efficient solution that has satisfied several axioms, especially individual rationality ,Pareto optimality and aggregate monotonicity. It was also shown that the Shapley value is in the core under the condition that the game is convex.

These results can be extended to the problem of the n network systems case. In this paper,we discussed in detail that the amount of information of each player was known to other players equivalently.

References

[1] M.Aoki:The Significance of Nucleolus in Common Cost Allocation,The Keizai gaku,Annual Report of the Economic Soceity, Tohoku UniversityVol.58 No.3(1996), pp.325-341

[2] Henriet,D.,and Moulin,H.:Traffic-based cost allocation in a network.RAND Journal of Economics,Vol.2 Summer (1996),pp.332-345

[3] Moulin,H.and S.Shenker:Average Cost Pricing versus Serial Cost Sharing:An Axiomatic Comparison.Journal of Economic Theory 64(1994),pp178-201

[4] Sharkey.W.W.:Suggestions for a game-theoretic approach to public utility and cost allocation.The Bell Journal of Ecomonics,pp57-68

[5] Young,H.P.:Monotonic Solutions of Cooperative Games. International Journal of Game Theory.Vol.14 No.2(1985),pp.65-72

[6] Young,H.P.,N.Okada.and T.Hashimoto.:Cost allocation in Water Resources Development. Water Resources Research.Vol.18 No.3(1982),pp.463-475